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A NOTE ON WAVELETS AND S-ASYMPTOTICS

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Abstract. The aim of this paper is to analyze the asymptotic behavior at infinity of the integral wavelet transform of somewhat more general elements than L^2 functions, namely generalized functions from the space of exponential distributions $\mathcal{K}_1'$. We prove both an Abelian and a Tauberian type theorem at infinity for the integral wavelet transform.

1. Introduction

The analogue of the Fourier transform (FT) in the Fourier analysis is the integral wavelet transform (WT) in wavelet analysis. Recall that the integral wavelet transform of $f \in L^2(\mathbf{R})$ is defined by

$$(W_\psi f(x))(a,b) := \frac{1}{\sqrt{|a|}} \int_{\mathbf{R}} f(x) \psi\left(\frac{x-b}{a}\right) dx, \tag{WT1}$$

where the *wavelet* $\psi \in L^2(\mathbf{R})$ has nonzero L^2 norm. The main shortcoming of the FT, namely its bad localization property, has been greatly overcome by the WT, see, e.g., [5], [1], [2]. In view of that, the WT has become a powerful tool in wavelet analysis and its numerous applications.

The aim of this paper is to analyze the asymptotic behavior at infinity of the WT of somewhat more general elements than L^2 functions, namely generalized functions from the space of exponential distributions $\mathcal{K}_1'$. For details on $\mathcal{K}_1'$ see, e.g., [7], [6]. We prove both an Abelian and a Tauberian type theorem at infinity for the WT.

Of course, this study asked for a choice of an appropriate definition of the asymptotic behavior of distributions at infinity. As is well known, there are several such definitions, see, e.g., the monographs [9], [7], where one can also find their several applications. Let us mention here only two, namely the quasiasymptotic behavior and the S-asymptotics.

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The first, introduced by B. I. Zavialov in 1973, turned out to be very useful in mathematical physics and also for Tauberian type theorems for distributional integral transforms. In the previous paper [8], we applied the quasiasymptotic behavior at a finite point in the analysis of behavior of tempered distributions, and obtained a generalization of Walter's result in [10]. At a finite point, being a natural generalization of the Lojasiewicz's value at a point [4], the quasiasymptotic behavior has local nature. However, at infinity, it is of global nature; e.g., the δ-distribution does not "behave" (in the quasiasymptotic sense) at infinity as zero, but simply just as δ itself.

As mentioned at the very beginning, the localization is essential for the WT; hence, for our purposes, we had to choose an asymptotic behavior at infinity with local properties. Such an asymptotic behavior of distributions is the S-asymptotics, which is naturally connected to the space of exponential distributions $\mathcal{K}_1'$. Therefore, we had to adopt our wavelet analysis to this space. Namely, instead of rapidly decreasing wavelets used in [8], we use here wavelets from the space $\mathcal{K}_1$.

2. Notions and notations

An infinitely differentiable ϕ over $\mathbf{R}$ is in $\mathcal{K}_1$ if it satisfies the condition

$$\nu_k(\phi) = \sup_{n\leq k,\, x\in\mathbf{R}} e^{k|x|}|D^n\phi| < \infty.$$

We will use the equivalence of the weak and the strong topology in $\mathcal{K}_1$, and the fact that the convolution of two functions f and g from $\mathcal{K}_1$, defined by

$$f * g(x) = \int f(t)\, g(x-t)\, dt$$

converges for every $x \in \mathbf{R}$ and is still in $\mathcal{K}_1$. The dual space of the space $\mathcal{K}_1$ is denoted by $\mathcal{K}_1'$. It was proved in [3] that $\mathcal{K}_1'$ is a normal space of distributions, i.e., it can be injectively embedded in the space of distributions $\mathcal{D}'$. The often used name for an element of $\mathcal{K}_1'$ is "exponential distribution". Roughly, the last name comes from the fact that any $T \in \mathcal{K}_1'$ is a finite sum of distributional derivatives of continuous functions which do not tend to infinity faster than a function of the form e^{kx}, for some $k \in \mathbf{R}$. The notation $\langle T, \phi\rangle$ denotes the scalar product between $T \in \mathcal{K}_1'$ and $\phi \in \mathcal{K}_1$. One easily shows that $T * \phi$, given by

$$T * \phi(x) = \langle T(t), \phi(x-t)\rangle$$

(i.e., the convolution of T and ϕ), is an infinitely differentiable function from $\mathcal{K}_1$. It holds

$$\langle T * g, \phi\rangle = \langle T, \check{g} * \phi\rangle,$$

where $\check{g}(x) = g(-x)$, provided that T is from $\mathcal{K}_1'$ and g and ϕ are from $\mathcal{K}_1$.

The S-asymptotics was originally defined in the space of tempered distributions. However, following [7] and [6], we rather use its definition in $\mathcal{K}_1'$:

DEFINITION 1. We say that a distribution $T \in \mathcal{K}_1'$ has S-asymptotics if there exists a distribution $g \in \mathcal{K}_1'$ such that

$$\lim_{n\to\infty} \left\langle \frac{T(x+h)}{c(h)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \qquad (\forall \phi \in \mathcal{K}_1).$$

In that case we have $c(h) = e^{\alpha h} L(e^h)$ and $g(x) = Ce^{\alpha x}$, and write $T \overset{s}{\sim} g$.

From now on, we fix a function ψ from $\mathcal{K}_1$ with nonzero L^2 norm; such functions are in wavelet analysis called *analyzing wavelets*, or simply wavelets. The well known examples of the wavelets are the Gaussian function

$$\frac{1}{2\sqrt{\pi\alpha}} \exp(-x^2/(4\alpha)),$$

together with its derivatives and differences.

In view of (WT1), we define the WT in $\mathcal{K}_1$ as follows.

DEFINITION 2. Assume $T \in \mathcal{K}_1'$ and the wavelet $\psi \in \mathcal{K}_1$. Then the wavelet transform of T is given by

$$W_\psi T(a,b) := \frac{1}{\sqrt{|a|}} \left\langle T(x), \psi\left(\frac{x-b}{a}\right) \right\rangle, \tag{WT2}$$

We will also use the following notation

$$W_{\psi_a} T(b) := \langle T(x), \psi_a(x-b) \rangle,$$

where $\psi_a(x) = \dfrac{1}{\sqrt{|a|}} \psi\left(\dfrac{x}{a}\right)$. Note that the transform (WT2) is well defined in $\mathcal{K}_1'$, since $\psi \in \mathcal{K}_1$ implies $\dfrac{1}{\sqrt{|a|}} \psi\left(\dfrac{x-b}{a}\right) \in \mathcal{K}_1$ as well.

3. Abelian and Tauberian theorem for the WT

We first state a theorem of Abelian type. It is a generalization of Proposition 12.5, page 115 in [7].

THEOREM 1. *Assume a distribution $T \in \mathcal{K}_1'$ has S-asymptotics equal to $g(x) = Ce^{\alpha x}$, ($T \overset{s}{\sim} g$). Then its wavelet transform $W_{\psi_a} T$, given by (WT2), also has S-asymptotics, more precisely*

$$W_{\psi_a} T \overset{s}{\sim} M_{\alpha,a} g, \quad \textit{where} \quad M_{\alpha,a} = \int e^{-\alpha t} \check{\psi}_a(t)\, dt.$$

Proof. We have by the definition of the S-asymptotics:

$$\lim_{n\to\infty}\left\langle \frac{W_{\psi_a}T(x+h)}{c(h)},\phi(x)\right\rangle = \lim_{n\to\infty}\left\langle \frac{\langle T(s),\psi_a(s-x-h)}{c(h)},\phi(x)\right\rangle$$

$$= \lim_{n\to\infty}\left\langle \frac{(T*\check{\psi}_a)(x+h)}{c(h)},\phi(x)\right\rangle = \lim_{n\to\infty}\left\langle \frac{T(x+h)}{c(h)},(\psi_a*\phi)(x)\right\rangle$$

$$= \langle g(x),(\psi_a*\phi)(x)\rangle = \langle g*\check{\psi}_a(x),\phi(x)\rangle = C\left\langle \int e^{\alpha s}\psi_a(x-s)\,ds,\phi(x)\right\rangle$$

$$= C\left\langle \int e^{\alpha(x-t)}\check{\psi}_a(t)\,dt,\phi(x)\right\rangle = CM_{\alpha,a}\langle e^{\alpha x},\phi(x)\rangle.$$

The last integral converges since $\phi\in\mathcal{K}_1$. We have also used the fact that the convolution of two functions from $\mathcal{K}_1$ is an element of $\mathcal{K}_1$. ■

From now on we assume that the parameter a is greater than zero with no loss of generality. For obtaining the result of Tauberian type we will use the following lemma.

Lemma. *Let $\mathcal{B}$ be a bounded subset of $\mathcal{K}_1$, i.e., there exists a constant $C_k>0$ such that*

$$\nu_k(\phi)=\sup_{n\le k,\,x\in\mathbf{R}} e^{k|x|}|D^n\phi(x)|\le C_k,$$

for all $\phi\in\mathcal{B}$ and all $k\in\mathbf{N}$. Then the set $\{\,\psi_a*\phi\mid\phi\in\mathcal{B}\,\}$ is bounded in $\mathcal{K}_1$, uniformly in $a>0$.

Proof. We first show that the set is bounded for fixed $a>0$. For $\phi\in\mathcal{B}$ we have $\sup_{j_1\le k_1,\,x\in\mathbf{R}} e^{k_1|x|}|D^{j_1}\phi(x)|<C_{k_1}$. For a seminorm ν_k, $k\in\mathbf{N}$, on the set $\{\,\psi_a*\phi\mid\phi\in\mathcal{B}\,\}$ we have

$$\nu_k(\psi_a*\phi)=\sup_{n<k,\,x\in\mathbf{R}} e^{k|x|}|D^n(\psi_a*\phi)(x)| = \sup_{n<k,\,x\in\mathbf{R}} e^{k|x|}\left|\int D^n_x\psi_a(t)\phi(x-t)\,dt\right|$$

$$\le \sup_{n<k,\,x\in\mathbf{R}} e^{k|x|}\left|\int C_{k_1}e^{-k_1|x-t|}\psi_a(t)\,dt\right|$$

$$\le \sup_{n<k,\,x\in\mathbf{R}} e^{k|x|}C_{k_1}C_{k_2,a}e^{-k_1|x|}\int e^{k_1|t|}e^{-k_2|t|}\,dt\le C_{k,a}<\infty.$$

For a fixed k, we choose $k_1\ge k$, and then $k_2\ge k_1$, which implies that the set $\{\,\psi_a*\phi\mid\phi\in\mathcal{B}\,\}$ is bounded in $\mathcal{K}_1$ for fixed $a>0$.

Let us now assume that $a\in(0,1)$. Since we have

$$|\psi_a(u)|=\left|\frac{1}{\sqrt{a}}\psi\left(\frac{u}{a}\right)\right|\le\frac{1}{\sqrt{a}}C_{k_2}e^{-k_2|u/a|},$$

we obtain

$$\sup_{n<k,\,x\in\mathbf{R}} C_{k_1}C_{k_2}e^{(k-k_1)|x|}\int \exp(k_1|u|)\frac{1}{\sqrt{a}}e^{-k_2|u|/a}du$$

$$\le \sup_{n<k,\,x\in\mathbf{R}} C_{k_1}C_{k_2}e^{(k-k_1)|x|}\int\frac{1}{\sqrt{a}}e^{(ak_1-k_2)|u|/a}\,du.$$

When a tends to zero, for given k we choose $k_1 \geq k$ and $k_2 \geq [ak_1] + 1$. Since for $a < 1$ we have $k_1 \geq [ak_1] + 1$, if we choose $k_2 \geq k_1$, the parameter k_2 will be independent of a ($[\cdot]$ stands for the greatest integer part). Then we have

$$\lim_{a \to 0} \frac{1}{\sqrt{a}} e^{(ak_1 - k_2)|u|/a} = 0,$$

so the set $\{\, \psi_a * \phi \mid \phi \in \mathcal{B} \,\}$ is bounded in $\mathcal{K}_1$ uniformly in $a \in (0,1)$. ∎

Now we are ready to prove

THEOREM 2. *Let the wavelet transform $W_{\psi_a} T$ of a distribution $T \in \mathcal{K}_1'$ has S-asymptotics equal to*

$$C M_{\alpha,a} e^{\alpha x}, \quad \textit{where} \quad M_{\alpha,a} = \int e^{-\alpha t} \check{\psi}_a(t)\, dt.$$

Then the distribution T has S-asymptotics equal to $Ce^{\alpha x}$.

Proof. Let $\phi \in \mathcal{B}$, and $a \in (0,1)$. Then we have

$$\begin{aligned}
\lim_{n \to \infty} \left\langle \frac{T(x+h)}{c(h)}, (\psi_a * \phi)(x) \right\rangle &= \lim_{n \to \infty} \left\langle \frac{W_{\psi_a} T(x+h)}{c(h)}, \phi(x) \right\rangle \\
&= C \left\langle e^{\alpha x} \int e^{-\alpha t} \check{\psi}_a(t)\, dt, \phi(x) \right\rangle \\
&= C \langle e^{\alpha x}, (\psi_a * \phi)(x) \rangle,
\end{aligned}$$

where we have used the previous lemma and the equivalence of the weak and the strong topology in $\mathcal{K}_1$. The theorem is proved. ∎

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