МАТЕМАТИЧКИ ВЕСНИК
48 (1996), 47–50

UDK 511.338
оригинални научни рад
research paper

SOLUTION OF ONE PROBLEM OF G. PÓLYA

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Abstract. We present the complete solution of the following problem of G. Pólya: Circular forest has a center at the origin and radius $R \geqslant 1$. A person is staying at the center and the trees of the radius r are planted at all other lattice points of the forest. Determine the maximal value ρ of the radius r for which the person can see out of the forest and, in the case $r = \rho$, determine the directions in which he/she should look in order to see out of the forest.

1. Formulation of the problem

Circular forest has a center at the origin and radius $R \geqslant 1$. A person is staying at the center and the trees of the radius r are planted at all other lattice points (points with integer coordinates) of the forest.

1) Determine the maximal value ρ of the radius r for which the person staying at the center can see out of the forest.

2) In the case $r = \rho$, determine the directions in which he/she should look in order to see out of the forest. (We assume that the ray tangent to a tree is not blocked by that tree.)

The above problem was studied for the first time by G. Pólya in [1], where he proved that

$$\lim_{R\to\infty} \rho R = 1.$$

In [2] (problem 239) the following estimate

$$\frac{1}{\sqrt{R^2+1}} \leqslant \rho < \frac{1}{R}$$

in the case when R is integer is obtained by method of A. Speiser.

In some more recent books (see [3], [4]) the same estimate is obtained using Minkowski theorem concerned with the existence of the lattice point in the set which is convex and symmetric about the origin.

AMS Subject Classification: 52 C 05
Keywords and phrases: Lattice point, Pick's theorem

We give here the complete solution of Pólya problem: we determine the exact value of ρ and directions in which to look in order to see out of the forest in the case when $r = \rho$.

2. Solution of the problem

Let m be the smallest integer gretaer than R^2 which can be represented as the sum of the squares of two relatively prime integers and let k be a circle which has center at the origin and radius $\sqrt{m}$.

In the sequel we shall prove the following two statements and thus our problem will be comletely solved:

a) If $r > 1/\sqrt{m}$, one cannot see out of the forest no matter in what direction one looks.

b) If $r = 1/\sqrt{m}$, the points from the circle k which can be seen from the origin are those whose coordinates are relatively prime integers.

Hence, $\rho = 1/\sqrt{m}$. When R is integer we have that $\rho = 1/\sqrt{R^2+1}$.

3. Proofs

Denote by A and B points with coordinates $(R, 0)$ and $(0, R)$. Let l be a half-line starting from the origin and avoiding every lattice point from the forest. Without loss of generality we can suppose that l belongs to the first quadrant. Denote by L the intersection point between l and the forest border. Suppose U is the lattice point from the region OAL with the least positive distance from the half-line l and V is the lattice point from the region OBL having the least positive distance from l. Let W be the lattice point symmetric to the origin about the midpoint of the segment UV. Having in mind the extremal properties of the points U and V, we shall prove the following facts:

1) Parallelogram $OUVW$ has unit area.

2) The point W lies outside of the forest.

3) Coordinates of W are relatively prime.

We can prove 1) using the following theorem:

Pick's theorem. *The area of a polygom whose vertices are lattice points is given by $S = i + b/2 - 1$, where i is the number of lattice points lying inside the polygon and b is the number of lattice points lying on the boundary of the polygon.*

The proof of Pick's theorem can be found in [5], [6] and in many other books as well.

Suppose that the triangle OUV has a lattice point M, different from its vertices. M does not lie on l, because the only lattice point from the forest which lies on l is the origin. If M belonged to the region OAL, it would be closer to l than the point U, contrary to the assumption that U is the lattice point from the region OAL with the least positive distance from l. Therefore M does not belong to the region OAL. In the similar way we conclude that M does not belong to the region OBL either. Since the starting assumption brings us to a contradiction, we conclude that it is not correct, i.e. we conclude that the triangle OUV has no lattice points except its vertices. An application of the Pick's theorem yields that the area of OUV equals $1/2$ and therefore the area of the parallelogram $OUVW$ equals 1.

Suppose that the point W belongs to the forest. W does not lie on l, because the only lattice point from the forest which lies on l is the origin. If W belonged to the region OAL, it would be closer to l than the point U (because W lies between U and the intersection point of rays l and UW), contrary to the assumption that U is the lattice point from the region OAL with the least positive distance from l. Therefore W does not belong to the region OAL. In the similar way we conclude that W does not belong to the region OBL either. Since the starting assumption brings us to a contradiction, we conclude that it is not correct, i.e. we cocnlude that the point W does not belong to the forest.

Suppose the coordinates a and b of the point W are not relatively prime. Let $d > 2$ be their greatest common divisor. The point with coordinates $(a/d, b/d)$ is a lattice point belonging to the triangle OUV, different from its vertices. This contradicts the previously proved fact. Hence, coordinates of W must be relatively prime.

It follows from 2) and 3) that $OW \geqslant \sqrt{m}$. Having in mind that the area of $OUWV$ equals 1, we conclude that the distances of the points U and V from the line OW do not exceed $1/\sqrt{m}$. The equality holds if W is lying on the circle k. If the half-line l does not contain the point W, then one of the points U and V is closer to l than to the line OW. Hence,

– when $r > 1/\sqrt{m}$, each ray will be blocked by a tree;

– when $r = 1/\sqrt{m}$, each ray whose direction intersects the circle k at the point with not relatively prime integer coordinates, will be blocked by a tree.

Let C be a point of the circle k whose coordinates are relatively prime integers and let l be a half-line starting from the origin and passing through C. Suppose M is a lattice point from the forest with coordinates $x, y \geqslant 0$. Since the coordinates of the point C are relatively prime integers, there are no lattice points lying inside the segment OC. So, points O, C and M are vertices of a triangle. According to the Pick's theorem, the area of the triangle OCM is greater or equal to $1/2$. Since $OC = \sqrt{m}$, the distance between the point M and the half-line l is graeter or equal to $1/\sqrt{m}$. Therefore, for $r = 1/\sqrt{m}$, the ray whose direction is determine by l will not be blocked by any tree. Thus we have proved the statements a) and b).

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(received 28.11.1995.)

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