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ESTIMATES FOR DERIVATIVES AND INTEGRALS OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF NONSELF-ADJOINT STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENTS (III)

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Abstract. In this paper we consider derivatives of higher order and certain "double" integrals of the eigenfunctions and associated functions of the formal Sturm-Liouville operator

$$\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x)$$

defined on a finite or infinite interval $G \subseteq \mathbf{R}$. We suppose that the complex-valued potential $q = q(x)$ belongs to the class $L_1^{loc}(G)$ and that piecewise continuously differentiable coefficient $p = p(x)$ has a finite number of the discontinuity points in G.

Order-sharp upper estimates are obtained for the suprema of the moduli of the k-th order derivatives ($k \geqslant 2$) of the eigenfunctions and associated functions $\{\,\overset{i}{u}_\lambda(x) \mid i = 0, 1, \dots\}$ of the operator $\mathcal{L}$ in terms of their norms in metric L_2 on compact subsets of G (on the entire interval G). Also, order-sharp upper estimates are established for the integrals (over closed intervals $[y_1, y_2] \subseteq \overline{G}$)

$$\int_{y_1}^{y_2} \bigg(\int_a^y \overset{i}{u}_\lambda(\xi)\,d\xi \bigg) dy\,, \qquad \int_{y_1}^{y_2} \bigg(\int_y^b \overset{i}{u}_\lambda(\xi)\,d\xi \bigg) dy$$

in terms of L_2-norms of the mentioned functions when G is finite.

The corresponding estimates for derivatives $\overset{i}{u}{}'_\lambda(x)$ and integrals $\int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy$ were proved in [5]–[6].

Introduction

1. Definitions. Consider the formal Sturm-Liouville operator

$$\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x)\,, \tag{1}$$

which is defined on an arbitrary interval $G = (a, b)$ of the real axis $\mathbf{R}$. Let $x_0 \in G$ be a point of discontinuity of the coefficient p. If we suppose that

$$p(x) = \begin{cases} p_1(x)\,, & x \in (a, x_0)\,, \\ p_2(x)\,, & x \in (x_0, b)\,, \end{cases} \qquad q(x) = \begin{cases} q_1(x)\,, & x \in (a, x_0)\,, \\ q_2(x)\,, & x \in (x_0, b)\,, \end{cases}$$

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then the following conditions are imposed on the coefficients:

1) $p_1(x) \in \mathcal{C}^{(1)}(a, x_0]$, and $p_2(x) \in \mathcal{C}^{(1)}[x_0, b)$.

2) $p_1(x) \geqslant \alpha_1 > 0$ everywhere on $(a, x_0]$, and $p_2(x) \geqslant \alpha_2 > 0$ everywhere on $[x_0, b)$.

3) $q(x) \in L_1^{loc}(G)$ is a complex-valued function.

Definition 1. A complex-valued function $\overset{\circ}{u}_\lambda(x) \not\equiv 0$ is called an *eigenfunction of the operator* (1) *corresponding to the* (*complex*) *eigenvalue* λ ($\lambda = \mathcal{R}e\,\lambda + i\,\mathcal{I}m\,\lambda$) if it satisfies the following conditions:

(a) $\overset{\circ}{u}_\lambda(x)$ is absolutely continuous on any finite closed subinterval of G.

(b) $\overset{\circ}{u}{}'_\lambda(x)$ is absolutely continuous on any finite closed subinterval of the half-open intervals $(a, x_0]$ and $[x_0, b)$.

(c) $\overset{\circ}{u}_\lambda(x)$ satisfies the differential equation

$$-\big(p_1(x)\,\overset{\circ}{u}{}'_\lambda(x)\big)' + q_1(x)\,\overset{\circ}{u}_\lambda(x) = \lambda\,\overset{\circ}{u}_\lambda(x) \tag{2}$$

almost everywhere on (a, x_0), and the differential equation

$$-\big(p_2(x)\,\overset{\circ}{u}{}'_\lambda(x)\big)' + q_2(x)\,\overset{\circ}{u}_\lambda(x) = \lambda\,\overset{\circ}{u}_\lambda(x) \tag{3}$$

almost everywhere on (x_0, b).

(d) $p_1(x_0)\,\overset{\circ}{u}{}'_\lambda(x_0 - 0) = p_2(x_0)\,\overset{\circ}{u}{}'_\lambda(x_0 + 0)$.

Definition 2. A complex-valued function $\overset{i}{u}_\lambda(x) \not\equiv 0$ ($i = 1, 2, \dots$) is called an *associated function* (*of the i-th order*) *of the operator* (1) *corresponding to the eigenfunction* $\overset{\circ}{u}_\lambda(x)$ *and the eigenvalue* λ if it satisfies the following conditions:

($a^\star$) Conditions (a), (b) and (d) of Definition 1 hold for $\overset{i}{u}_\lambda(x)$.

($b^\star$) $\overset{i}{u}_\lambda(x)$ satisfies the differential equation

$$-\big(p_1(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + q_1(x)\,\overset{i}{u}_\lambda(x) = \lambda\,\overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{4}$$

almost everywhere on (a, x_0), and the differential equation

$$-\big(p_2(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + q_2(x)\,\overset{i}{u}_\lambda(x) = \lambda\,\overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{5}$$

almost everywhere on (x_0, b).

1.1. Let K be any compact set of positive measure lying strictly within G. We will use the notation $K_R = \{\, x \in G \mid \rho(x, \overline{K}\,) \leqslant R \,\}$, where $R \in (0, \rho(K, \partial G))$, and $\overline{K}$ is the intersection of all closed intervals containing K. (By $\rho(A, B)$ we denote the distance of a set $A \subset \mathbf{R}$ from a set $B \subset \mathbf{R}$.)

If $\lambda = r\,e^{i\varphi}$, then $\sqrt{\lambda} \overset{\text{def}}{=} \sqrt{r}\,e^{i\varphi/2}$, where $\varphi \in (-\pi/2, 3\pi/2]$.

2. Main theorems. We present the following results.

THEOREM 1. *(a) Suppose that* $q_1(x) \in \mathcal{C}^{(k-2)}(a,x_0]\,, q_2(x) \in \mathcal{C}^{(k-2)}[x_0,b)$, $p_1(x) \in \mathcal{C}^{(k-1)}(a,x_0]\,, p_2(x) \in \mathcal{C}^{(k-1)}[x_0,b)$ $(k \geqslant 2)$. *Then the functions* $\overset{i}{u}_\lambda(x)$ $(i=0,1,\dots)$ *have derivatives* $\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)$ $(2 \leqslant j \leqslant k)$, *continuous on the half-open intervals* $(a,x_0]$ *and* $[x_0,b)$, *and for every compact set* $K \subset G$ *there exist a number* $R \in (0,\rho(K,\partial G))$ *and constants* $r(K_R,\mathcal{I}m\sqrt{\lambda})\,, C_{ij}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})$ *such that*

$$\sup_{x\in K}\left|\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)\right| \leqslant C_{ij}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})\,\|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{6}$$

if $0 \leqslant |\,\mathcal{R}e\sqrt{\lambda}\,| \leqslant r(K_R,\mathcal{I}m\sqrt{\lambda})$, *and*

$$\sup_{x\in K}\left|\frac{d^j}{dx^j},\overset{i}{u}_\lambda(x)\right| \leqslant C_{ij}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})\,|\sqrt{\lambda}|^j\,\|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{7}$$

if $|\mathcal{R}e\sqrt{\lambda}| > r(K_R,\mathcal{I}m\sqrt{\lambda})$.

(b) Let $q(x) \in L_1(G)$, *and suppose that* $\overset{i}{u}_\lambda(x) \in L_2(G)$ $(i=0,1,2,\dots)$ *if* G *is an infinite interval. If the functions* $p_1(x)$, $p_2(x)$, $q_1(x)$, $q_2(x)$ *are bounded along with all their derivatives, then derivatives* $\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)$ $(i=0,1,\dots;\,2\leqslant j\leqslant k)$ *are bounded on the half-open intervals* $(a,x_0]$ *and* $[x_0,b)$, *and there exist constants* $r(G,\mathcal{I}m\sqrt{\lambda})\,, C_{ij}(G,p,q,\mathcal{I}m\sqrt{\lambda})$ *such that*

$$\sup_{x\in G}\,|\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)\,| \leqslant C_{ij}(G,p,q,\mathcal{I}m\sqrt{\lambda})\,\|\overset{i}{u}_\lambda\|_{L_2(G)} \tag{8}$$

for $0 \leqslant |\mathcal{R}e\sqrt{\lambda}\,| \leqslant r(G,\mathcal{I}m\sqrt{\lambda})$, *and*

$$\sup_{x\in G}\,|\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)\,| \leqslant C_{ij}(G,p,q,\mathcal{I}m\sqrt{\lambda})\,|\sqrt{\lambda}|^j\,\|\overset{i}{u}_\lambda\|_{L_2(G)} \tag{9}$$

for $|\mathcal{R}e\sqrt{\lambda}| > r(G,\mathcal{I}m\sqrt{\lambda})$.

Note here that talking about continuity of the function $\frac{d^j}{dx^j}\overset{i}{u}_\lambda(x)$ on the set $(a,x_0]$ (on the set $[x_0,b)$) by the value of this function at x_0 we mean $\frac{d^j}{dx^j}(x_0-0)$ $(\frac{d^j}{dx^j}(x_0+0))$.

In the following theorem we will suppose additionally that the functions $\overset{i}{u}_\lambda(x)$ are absolutely continuous on the whole closed interval $\overline{G}$, and that the functions $\overset{i}{u}{}'_\lambda(x)$ are absolutely continuous on the closed intervals $[a,x_0]$ and $[x_0,b]$.

THEOREM 2. *Let* $q(x) \in L_1(G)$, *where* G *is a finite interval, and suppose that* $p_1(x) \in \mathcal{C}^{(1)}[a,x_0]\,, p_2(x) \in \mathcal{C}^{(1)}[x_0,b]$. *Then there exist a closed interval* $K \subset G$ *and constants* $r(G,\mathcal{I}m\sqrt{\lambda})\,, D_{i2}(G,K_R,p,q,\mathcal{I}m\sqrt{\lambda})$ *such that the following estimates hold uniformly with respect to the numbers* $a \leqslant y_1 < y_2 \leqslant b$:

(*a*) *If* $\overset{i}{u}{}'_\lambda(a) = 0 (i = 0, 1, \dots)$, *then*

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant D_{i2}(G, K_R, p, q, \mathcal{I}m\, \sqrt{\lambda}\,)\, \|\, \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{10}$$

for every eigenvalue λ, *and*

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant D_{i2}(G, K_R, p, q, \mathcal{I}m\, \sqrt{\lambda}\,)\, \frac{1}{|\lambda|}\, \|\, \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{11}$$

for $|\mathcal{R}e\, \sqrt{\lambda}| > r(G, \mathcal{I}m\, \sqrt{\lambda}\,)$, *where* $R \in (0, \rho(K, \partial G))$ *is some fixed number.*

(*b*) *If* $\overset{i}{u}{}'(b) = 0$ $(i = 0, 1, \dots)$, *then*

$$\left| \int_{y_1}^{y_2} \left(\int_y^b \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant D_{i2}(G, K_R, p, q, \mathcal{I}m\, \sqrt{\lambda}\,)\, \|\, \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{12}$$

for every eigenvalue λ, *and*

$$\left| \int_{y_1}^{y_2} \left(\int_y^b \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant D_{i2}(G, K_R, p, q, \mathcal{I}m\, \sqrt{\lambda}\,)\, \frac{1}{|\lambda|}\, \|\, \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{13}$$

for $|\mathcal{R}e\, \sqrt{\lambda}| > r(G, \mathcal{I}m \sqrt{\lambda}\,)$, *where* $R \in (0, \rho(K, \partial G))$ *is some fixed number.*

2.1. For the sake of simplicity we have supposed that the coefficient $p(x)$ has only one point of discontinuity. But all stated results remain valid when this function has an arbitrary finite number of such points.

2.2. Let us give a few comments on the theorems.

Remark 1. It is possible to replace $\|\, \overset{i}{u}_\lambda \|_{L_2(K_R)}$ in the estimates (6)-(7) by $\max_{x \in K_{R_0}} |\, \overset{i}{u}_\lambda(x)\,|$, for a fixed number $R_0 \in (0, R)$. If G is a finite interval, then there exists a closed interval $\tilde{K} \subset G$ such that we can replace $\|\, \overset{i}{u}_\lambda \|_{L_2(G)}$ and $C_{ij}(G, \cdot)$ in the estimates (8)-(9) by $\max_{x \in \tilde{K}_{R_0}} |\, \overset{i}{u}_\lambda(x)\,|$ and $C_{ij}(\tilde{K}_{R_0}, \cdot)$ respectively.

Remark 2. The estimates (10) and (12) are actually valid without boundary conditions imposed in the theorem.

It is possible to replace $\|\, \overset{i}{u}_\lambda \|_{L_2(K_R)}$ in estimates (10)–(13) by $\max_{x \in K} |\, \overset{i}{u}_\lambda(x)\,|$, with constants $D_{i2}(\cdot)$ changed correspondingly. As a consequence we obtain, by virtue of estimates (24) and (16), that the estimates (10)–(13) are valid for *every* closed interval $K \subseteq \overline{G}$ (with corresponding constants $D_{i2}(G, K_R, p, q, \mathcal{I}m\, \sqrt{\lambda}\,)$).

The number $r(G, \mathcal{I}m\, \sqrt{\lambda}\,)$ is the same in both Theorems 1 and 2.

Remark 3. Let $\sigma(\mathcal{L})$ be some set of eigenvalues of the operator (1). If there exists a constant A not depending on numbers $\lambda \in \sigma(\mathcal{L})$ and such that

$$|\, \mathcal{I}m \sqrt{\lambda}\, | \leqslant A\,, \qquad \lambda \in \sigma(\mathcal{L})\,, \tag{14}$$

then constants $C_{0j}(\cdot)$, $D_{02}(\cdot)$ and $r(\cdot)$ do not depend on the numbers λ, which means that it is possible to define these constants uniformly with respect to the parameter $\lambda \in \sigma(\mathcal{L})$.

If the numbers $\lambda \in \sigma(\mathcal{L})$ satisfy (14) and zero is not a limit point of the set $\{\, |\mathcal{R}e\sqrt{\lambda}\,| \mid \lambda \in \sigma(\mathcal{L})\,\}$, then the other constants appearing in the estimates (6)–(13) do not depend on these numbers.

REMARK 4. The constants $C_{ij}(\cdot)$ and $D_{i2}(\cdot)(i \geqslant 1)$ actually are the same for all associated functions corresponding to the specific eigenfunction, i.e., these constants do not depend on the order i of the associated function.

REMARK 5. Theorems 1–2 include the case when the function $p(x)$ is continuous at the point x_0 (and has the required differentiability properties at that point). Especially, if $p_1(x) = p_2(x) = 1$ for $x \in G$, then the operator (1) reduces to the formal Schrödinger operator

$$\mathcal{L}(u)(x) = -u''(x) + \tilde{q}(x)\, u(x)\,. \tag{15}$$

In that case the corresponding estimates for derivatives of eigenfunctions of an arbitrary non-negative self-adjoint extension of the operator (15) were first derived in [1]. If the extension is generated by the boundary conditions $\overset{i}{u}{}'_{\lambda}(a) = 0 = \overset{i}{u}{}'_{\lambda}(b)$, then the estimates that correspond to the estimates (10)–(13) were proved and used in [2].

The estimates for derivatives of eigenfunctions and associated functions of nonself-adjoint operator (15) were first announced in [3].

REMARK 6. The example exposed in Remark 6 of [5] shows that the estimates (6)–(9) are best possible with respect to the order of the parameter λ. Also, consider the following example.

Let the operator $\mathcal{L}(u)(x) = -u''(x)$ be defined on the interval $G = (0,1)$, and let the eigenfunctions and associated functions of this operator satisfy the boundary conditions $u(0) = u(1)\,, u'(0) = 0$. Then $\sigma(\mathcal{L}) = \{\,(2n\pi)^2 \mid n = 0,1,\dots\}$ is the set of all eigenvalues, the eigenfunctions have the form $\overset{\circ}{u}_0(x) = 1\,, \overset{\circ}{u}_n(x) = \cos 2n\pi x$ $(n \in \mathbf{N})$; the associated functions corresponding to the eigenfunction $\overset{\circ}{u}_0(x)$ do not exist, and the others have the form

$$\overset{1}{u}_n(x) = -x\,\frac{\sin 2n\pi x}{4n\pi}\,, \qquad n \in \mathbf{N}\,.$$

In this case the order of parameter λ in the corresponding estimates (10)–(13) can not be improved.

3. Auxiliary estimates. In the proofs of Theorems 1–2 we will essentially use a set of estimates for eigenfunctions and associated functions of the operator (1), and for their first derivatives, too. Those estimates are stated in the following lemmas.

LEMMA 1. *(a) If* $q(x) \in L_1^{loc}(G)$*, then for any compact set* $K \subset G$ *there exist a number* $R \in (0, \rho(K, \partial G))$ *and constants* $C_i(K_R, p, q, \mathcal{I}m\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ *such that*

$$\max_{x\in K} |\overset{i}{u}_\lambda(x)| \leqslant C_i(K_R, p, q, \mathcal{I}m\sqrt{\lambda})\, \|\overset{i}{u}_\lambda\|_{L_2(K_R)}\,. \tag{16}$$

(b) Suppose that $q(x) \in L_1(G)$*, and that* $\overset{i}{u}_\lambda(x) \in L_2(G)$ *if* G *is an infinite interval. If* $p_1(x)$ *and* $p_2(x)$ *are bounded along with their first derivatives, then there exist constants* $C_i(G, p, q, \mathcal{I}m\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ *such that*

$$\sup_{x\in G} |\overset{i}{u}_\lambda(x)| \leqslant C_i(G, p, q, \mathcal{I}m\sqrt{\lambda})\, \|\overset{i}{u}_\lambda\|_{L_2(G)}\,. \tag{17}$$

LEMMA 2. *(a) If* $q(x) \in L_1^{loc}(G)$*, then for any compact set* $K \subset G$ *there exist a number* $R \in (0, \rho(K, \partial G))$ *and constants* $A_i(K_R, p, q, \mathcal{I}m\sqrt{\lambda})$, $A_i(K_R, p, q)$ $(i = 1, 2, \dots)$ *such that*

$$\begin{gathered}\max_{x\in K} |\overset{i-1}{u}_\lambda(x)| \leqslant A_i(K_R, p, q, \mathcal{I}m\sqrt{\lambda})\, |\sqrt{\lambda}| \cdot \max_{x\in K_R} |\overset{i}{u}_\lambda(x)| \qquad \text{for} \quad \lambda \neq 0\,,\\ \max_{x\in K} |\overset{i-1}{u}_\lambda(x)| \leqslant A_i(K_R, p, q) \cdot \max_{x\in K_R} |\overset{i}{u}_\lambda(x)| \qquad \text{for} \quad \lambda = 0\,.\end{gathered} \tag{18}$$

(b) Suppose that $q(x) \in L_1(G)$*, and that* $\overset{i}{u}_\lambda(x) \in L_2(G)$ *if* G *is an infinite interval. If* $p_1(x)$ *and* $p_2(x)$ *are bounded along with their first derivatives, then there exist constants* $A_i(G, p, q, \mathcal{I}m\sqrt{\lambda})$, $A_i(G, p, q)$ $(i = 1, 2, \dots)$ *such that*

$$\begin{gathered}\sup_{x\in G} |\overset{i-1}{u}_\lambda(x)| \leqslant A_i(G, p, q, \mathcal{I}m\sqrt{\lambda})\, |\sqrt{\lambda}| \cdot \sup_{x\in G} |\overset{i}{u}_\lambda(x)| \qquad \text{for } \lambda \neq 0\,,\\ \sup_{x\in G} |\overset{i-1}{u}_\lambda(x)| \leqslant A_i(G, p, q) \cdot \sup_{x\in G} |\overset{i}{u}_\lambda(x)| \qquad \text{for} \quad \lambda = 0\,.\end{gathered} \tag{19}$$

LEMMA 3. *(a) If* $q(x) \in L_1^{loc}(G)$*, then for any compact set* $K \subset G$ *there exist a number* $R \in (0, \rho(K, \partial G))$ *and constants* $r(K_R, \mathcal{I}m\sqrt{\lambda})$, $C_{i1}(K_R, p, q, \mathcal{I}m\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ *such that*

$$\sup_{x\in K} |\overset{i}{u}{}'_\lambda(x)| \leqslant C_{i1}(K_R, p, q, \mathcal{I}m\sqrt{\lambda})\, \|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{20}$$

for $0 \leqslant |\mathcal{R}e\sqrt{\lambda}| \leqslant r(K_R, \mathcal{I}m\sqrt{\lambda})$*, and*

$$\sup_{x\in K} |\overset{i}{u}{}'_\lambda(x)| \leqslant C_{i1}(K_R, p, q, \mathcal{I}m\sqrt{\lambda})\, |\sqrt{\lambda}| \cdot \|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{21}$$

for $|\mathcal{R}e\sqrt{\lambda}| > r(K_R, \mathcal{I}m\sqrt{\lambda})$.

(b) Let $q(x) \in L_1(G)$ and (when the interval G is infinite) $\overset{i}{u}_\lambda(x) \in L_2(G)$. If the functions $p_1(x)$ and $p_2(x)$ are bounded together with their first derivatives, then there exist constants $r(G, \mathcal{I}m\sqrt{\lambda})$, $C_{i1}(G, p, q, \mathcal{I}m\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ such that

$$\sup_{x\in G} |\overset{i}{u}{}'_\lambda(x)| \leqslant C_{i1}(G, p, q, \mathcal{I}m\sqrt{\lambda}) \, \| \overset{i}{u}_\lambda \|_{L_2(G)} \tag{22}$$

for $0 \leqslant |\mathcal{R}e\sqrt{\lambda}| \leqslant r(G, \mathcal{I}m\sqrt{\lambda})$, and

$$\sup_{x\in G} |\overset{i}{u}{}'_\lambda(x)| \leqslant C_{i1}(G, p, q, \mathcal{I}m\sqrt{\lambda}) \, |\sqrt{\lambda}| \cdot \| \overset{i}{u}_\lambda \|_{L_2(G)} \tag{23}$$

for $|\mathcal{R}e\sqrt{\lambda}| > r(G, \mathcal{I}m\sqrt{\lambda})$.

3.1. The estimates (16)–(19) were established in [4], and estimates (20)–(23) in [5]–[6]. It was proved there that the constants appearing in those estimates do not depend on numbers $\lambda \in \sigma(\mathcal{L})$ if $\sigma(\mathcal{L})$ satisfies conditions described in Remark 3. Also, the constants $C_i(\cdot)$, $A_i(\cdot)$ and $C_{i1}(\cdot)$ $(i = 1, 2, \dots)$ are independent of the parameter i.

3.2. Global estimate (17) may be sharpened in the following way: If G is a finite interval, then for any closed interval $K \subset G$ there exist constants $C_i(K, p, q, \mathcal{I}m\sqrt{\lambda})$ such that

$$\sup_{x\in G} |\overset{i}{u}_\lambda(x)| \leqslant C_i(K, p, q, \mathcal{I}m\sqrt{\lambda}) \cdot \max_{x\in K} |\overset{i}{u}_\lambda(x)| \, . \tag{24}$$

Further, if $K \subset G$ is an closed interval, then $\max_{x\in K_R} |\overset{i}{u}_\lambda(x)|$ and $A_i(K_R, \cdot)$ in estimates (18) can be replaced by $\max_{x\in K} |\overset{i}{u}_\lambda(x)|$ and $\tilde{A}_i(K, \cdot)$ respectively.

3.3. The statements from Remark 1 are also valid in the case of estimates (20)–(23).

3.4. Theorems 1–2 are results of independent interest; along with Lemas 1–3 and the two theorems proved in [5]–[6] they provide for a complete and definitive picture of estimates of the considered type. But they may also play an important role in study of uniform convergence on G (or on compact subsets of G) of derivatives of partial sum of spectral expansion (for an absolutely continuous function) generated by an arbitrary complete and minimal system of eigenfunctions and associated functions of the operators (1) and (15).

§1. Local estimates of the second derivative

1. On the differential equations (2)–(5). Everything in the proof of Theorem 1 is based on the differential equations (2)–(5) and the estimates (16)–(23). We will first prove the following assertions: If $q(x) \in \mathcal{C}(G \setminus \{x_0\})$, then

1) the eigenfunctions and associated functions of the operator (1) have continuous second derivative on the set $G \setminus \{x_0\}$;

2) there exist the finite one-side derivatives $\overset{i}{u}{}''_\lambda(x_0-0)$, $\overset{i}{u}{}''_\lambda(x_0+0)$ $(i = 0, 1, \dots)$;

3) the eigenfunctions and associated functions satisfy the corresponding differential equation *everywhere* on the interval (a, x_0) or (x_0, b).

1.1. Let $\overset{i}{u}_\lambda(x)$ be an associated function of the operator (1) corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(x)$ and the eigenvalue λ. Let $[c, d]$ be an arbitrary closed subinterval of the interval (a, x_0). The function $f_1(x) \overset{\text{def}}{=} p_1(x)\, \overset{i}{u}{}'_\lambda(x)$ is absolutely continuous on $[c, d]$. Thus for every $x \in (c, d)$ we have $f_1(x) = f_1(c) + \int_c^x f_1'(\xi)\, d\xi$ or, by virtue of the differential equation (2), the equality

$$f_1(x) = f_1(c) + \int_c^x \left[\, q_1(\xi)\, \overset{i}{u}_\lambda(\xi) - \lambda\, \overset{i}{u}_\lambda(\xi) + \overset{i-1}{u}_\lambda(\xi) \,\right] d\xi\, . \tag{25}$$

By the continuity of $q_1(x)$ on (a, x_0) the integral (25) has continuous first derivative on (c, d). Therefore, we obtain from (25) that $f_1(x)$ is continuously differentiable on (c, d) and satisfies the differential equation

$$f_1'(x) = q_1(x)\, \overset{i}{u}_\lambda(x) - \lambda\, \overset{i}{u}_\lambda(x) + \overset{i-1}{u}_\lambda(x) \tag{26}$$

everywhere on the interval (c, d) and, consequently, on the whole interval (a, x_0).

Now, it results from the continuous differentiability of $p_1(x)$ and $f_1(x)$ that the function $\overset{i}{u}{}'_\lambda(x)$ has continuous first derivative on (a, x_0).

1.2. Analogously, using the differential equation (3) instead of (2), we can prove that the function $f_2(x) \overset{\text{def}}{=} p_2(x)\, \overset{i}{u}{}'_\lambda(x)$ is continuously differentiable on (x_0, b) and satisfies the differential equation

$$f_2'(x) = q_2(x)\, \overset{i}{u}_\lambda(x) - \lambda\, \overset{\circ}{u}_\lambda(x) + \overset{i-1}{u}_\lambda(x) \tag{27}$$

everywhere on this interval, and that the function $\overset{i}{u}{}'_\lambda(x)$ has continuous first derivative on the whole interval (x_0, b).

1.3. Let us examine the behaviour of function $\overset{i}{u}{}''_\lambda(x)$ in a neighborhood of the point x_0. We know that $\overset{i}{u}{}'_\lambda(x)$ is (absolutely) continuous on every closed interval $[c, x_0] \subset (a, x_0]$ (on every closed interval $[x_0, d] \subset [x_0, b)$). Hence, using (26)–(27), we obtain that the finite one-side derivatives $\overset{i}{u}{}''_\lambda(x_0 - 0)$ and $\overset{i}{u}{}''_\lambda(x_0 + 0)$ exist;

$$\begin{aligned} &\overset{i}{u}{}''_\lambda(x_0 - 0) = \lim_{x \to x_0 - 0} \overset{i}{u}{}''_\lambda(x) = \\ &= \frac{1}{p_1(x_0)} \left[\, q_1(x_0)\, \overset{i}{u}_\lambda(x_0) - \lambda\, \overset{i}{u}_\lambda(x_0) + \overset{i-1}{u}_\lambda(x_0) - p_1'(x_0 - 0)\, \overset{i}{u}{}'_\lambda(x_0 - 0) \,\right]; \end{aligned} \tag{28}$$

$$\begin{aligned} &\overset{i}{u}{}''_\lambda(x_0 + 0) = \lim_{x \to x_0 + 0} \overset{i}{u}{}''_\lambda(x) = \\ &= \frac{1}{p_2(x_0)} \left[\, q_2(x_0)\, \overset{i}{u}_\lambda(x_0) - \lambda\, \overset{i}{u}_\lambda(x_0) + \overset{i-1}{u}_\lambda(x_0) - p_2'(x_0 + 0)\, \overset{i}{u}{}'_\lambda(x_0 + 0) \,\right]. \end{aligned} \tag{29}$$

In that sense, the function $\overset{i}{u}{}''_\lambda(x)$ is continuous on the half-open intervals $(a, x_0]$ and $[x_0, b)$.

2. Local estimates (6)–(7). Let us prove now estimates (6)–(7) in the case $j=2$. We will first consider the second derivative of an associated function $\overset{i}{u}_\lambda(x)$ corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(x)$ and the eigenvalue λ.

2.1. Let $K\subset G$ be an arbitrary compact set (of strictly positive measure). (The most complicated case is when $x_0\in K$, and it will be considered only.) According to Lemmas 1 and 3, there exist a number $R\in(0,\rho(K,\partial G))$ and constants $r(K_R,\mathcal{I}m\sqrt{\lambda}\,)$, $C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)$, $C_{i1}(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)$ such that estimates (16), (20)–(21) hold with $i>0$. (Without loss of generality we can suppose that the number R is the same in both Lemmas. In fact, that is the case! Furthermore, $r(K_R,\mathcal{I}m\sqrt{\lambda}\,)>1$.) Also, by Lemma 2 and 3.2 in Introduction, there is a constant $\tilde{A}_i(\overline{K},p,q,\mathcal{I}m\sqrt{\lambda}\,)$ such that the first estimate (18) is valid, with $\max_{x\in K_R}|\overset{i}{u}_\lambda(x)|$ replaced by $\max_{x\in\overline{K}}|\overset{i}{u}_\lambda(x)|$. Using that estimate, the corresponding estimates (16) and (21), and the differential equations (26)–(27), we obtain that the following holds for every $x\in K\setminus\{x_0\}$:

$$
\begin{aligned}
|\overset{i}{u}{}''_\lambda(x)| &\leqslant \frac{1}{\alpha^2}\Big[\,\gamma'(K,p)\cdot\sup_{\xi\in K}|\overset{i}{u}{}'_\lambda(\xi)|+\gamma(K,q)\cdot\max_{\xi\in K}|\overset{i}{u}_\lambda(\xi)|+\\
&\qquad\qquad +|\lambda|\cdot\max_{\xi\in K}|\overset{i}{u}_\lambda(\xi)|+\max_{\xi\in\overline{K}}|\overset{i-1}{u}_\lambda(\xi)|\,\Big]\\
&\leqslant \frac{1}{\alpha^2}\Big[\,\gamma'(K,p)C_{i1}(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)|\sqrt{\lambda}|+\big(\gamma(K,q)+|\lambda|\big)C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)+\\
&\qquad +\tilde{A}_i(\overline{K},p,q,\mathcal{I}m\sqrt{\lambda}\,)\,C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)\,|\sqrt{\lambda}|\,\Big]\cdot\|\overset{i}{u}_\lambda\|_{L_2(K_R)}\,. \qquad (30)
\end{aligned}
$$

By these inequalities one can get the estimate

$$
\begin{aligned}
&\sup_{x\in K\setminus\{x_0\}}|\overset{i}{u}{}''_\lambda(x)|\leqslant\\
&\leqslant\frac{1}{\alpha^2}\Big[\,\gamma'(K,p)\,C_{i1}(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)+\big(\gamma(K,q)+1\big)\,C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)+\\
&\qquad +\tilde{A}_i(\overline{K},p,q,\mathcal{I}m\sqrt{\lambda}\,)\,C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda}\,)\Big]\,|\lambda|\cdot\|\overset{i}{u}_\lambda\|_{L_2(K_R)}\,, \qquad (31)
\end{aligned}
$$

where $|\mathcal{R}e\sqrt{\lambda}|>r(K_R,\mathcal{I}m\sqrt{\lambda}\,)$. Here the following notations are used:

$$
\alpha\overset{\text{def}}{=}\min\{\sqrt{\alpha_1},\,\sqrt{\alpha_2}\,\},\qquad \gamma'(K,p)\overset{\text{def}}{=}\max\Big\{\max_{x\in K^-}|p_1'(x)|,\ \max_{x\in K^+}|p_2'(x)|\Big\},
$$
$$
\gamma(K,q)\overset{\text{def}}{=}\max\Big\{\max_{x\in K^-}|q_1(x)|,\ \max_{x\in K^+}|q_2(x)|\Big\},
$$

with $K^-\overset{\text{def}}{=}\{x\in K\,|\,x\leqslant x_0\}$, $K^+\overset{\text{def}}{=}\{x\in K\,|\,x_0\leqslant x\}$.

If $x=x_0$, then by virtue of the mentioned estimates (from Lemmas 1–3) and equalities (28)–(29) we conclude that for $\max\{|\overset{i}{u}{}''_\lambda(x_0-0)|,|\overset{i}{u}{}''_\lambda(x_0+0)|\}$ the first inequality (30) holds, too. Hence, it results that the estimate

$$\sup_{x\in K} |\overset{i}{u}{}''_\lambda(x)| \leqslant C_{i2}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})\,|\lambda|\cdot\|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{32}$$

is valid if $|\mathcal{R}e\sqrt{\lambda}| > r(K_R,\mathcal{I}m\sqrt{\lambda})$, where $C_{i2}(\cdot)$ denotes the constant from estimate (31).

2.2. If $0 \leqslant |\mathcal{R}e\sqrt{\lambda}| \leqslant r(K_R,\mathcal{I}m\sqrt{\lambda})$, then we have the estimate

$$\sup_{x\in K} |\overset{i}{u}{}''_\lambda(x)| \leqslant \tilde{C}_{i2}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})\cdot\|\overset{i}{u}_\lambda\|_{L_2(K_R)}, \tag{33}$$

with the constant

$$\tilde{C}_{i2}(K_R,p,q,\mathcal{I}m\sqrt{\lambda}) \overset{\text{def}}{=} \frac{1}{\alpha^2}\Big[\,\gamma'(K,p)\,C_{i1}(K_R,p,q,\mathcal{I}m\sqrt{\lambda})+$$
$$+\big[\gamma(K,q)+\big(r(K_R,\mathcal{I}m\sqrt{\lambda})\big)^2+\big(\mathcal{I}m\sqrt{\lambda}\big)^2\big]\,C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda})+$$
$$+\tilde{A}_i(\overline{K},p,q,\mathcal{I}m\sqrt{\lambda})\,C_i(K_R,p,q,\mathcal{I}m\sqrt{\lambda})\sqrt{\big(r(K_R,\mathcal{I}m\sqrt{\lambda})\big)^2+\big(\mathcal{I}m\sqrt{\lambda}\big)^2}\,\Big].$$

This assertion directly follows from the first inequality (30).

Note that using max if necessary we can obtain the same constant in both estimates (6) and (7), as it is stated in the proposition (*a*) of Theorem 1.

2.3. By the above considerations we can conclude that the estimates (32)–(33) are also valid for the function $\overset{\circ}{u}{}''_\lambda(x)$, with constants $C_{02}(K_R,\cdot)$, $\tilde{C}_{02}(K_R,\cdot)$ obtained from the constants $C_{i2}(\cdot)$ and $\tilde{C}_{i2}(\cdot)$ by replacement $i \longmapsto 0$ and removing the $\tilde{A}_i(\cdot)$-term.

§2. Global estimates of the second derivative

1. Global estimates (8)–(9). Let us prove now the estimates (8)–(9) in case $j=2$. We will first consider the second derivative of an associated function $\overset{i}{u}_\lambda(x)$ corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(x)$ and the eigenvalue λ.

1.1. According to Lemmas 1–3, there exist a number $r(G,\mathcal{I}m\sqrt{\lambda}) > 1$ and constants $C_i(G,p,q,\mathcal{I}m\sqrt{\lambda})$, $A_i(G,p,q,\mathcal{I}m\sqrt{\lambda})$, $C_{i1}(G,p,q,\mathcal{I}m\sqrt{\lambda})$ such that the estimates (17), (19) and (22)–(23) hold. Using those estimates and differential equations (4)–(5), for every $x \in G\setminus\{x_0\}$ we obtain the inequalities

$$|\overset{i}{u}{}''_\lambda(x)| \leqslant \frac{1}{\alpha^2}\Big[\max\Big\{\sup_{\xi\in(a,x_0]}|p_1'(\xi)|,\ \sup_{\xi\in[x_0,b)}|p_2'(\xi)|\Big\}\cdot\sup_{\xi\in G}|\overset{i}{u}{}'_\lambda(\xi)|+$$
$$+\Big(\max\Big\{\sup_{\xi\in(a,x_0]}|q_1(\xi)|,\ \sup_{\xi\in[x_0,b)}|q_2(\xi)|\Big\}+|\lambda|\Big)\cdot\sup_{\xi\in G}|\overset{i}{u}_\lambda(\xi)|+\sup_{\xi\in G}|\overset{i-1}{u}_\lambda(\xi)|\Big]$$
$$\leqslant \frac{1}{\alpha^2}\Big[\,\gamma'(G,p)\,C_{i1}(G,p,q,\mathcal{I}m\sqrt{\lambda})\,|\sqrt{\lambda}|+\big(\gamma(G,q)+|\lambda|\big)\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda})+$$
$$+A_i(G,p,q,\mathcal{I}m\sqrt{\lambda})\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda})\,|\sqrt{\lambda}|\,\Big]\cdot\|\overset{i}{u}_\lambda\|_{L_2(G)}, \tag{34}$$

wherefrom it follows the estimate

$$\sup_{x\in G\setminus\{x_0\}} |\overset{i}{u}{}''_\lambda(x)| \leqslant$$

$$\leqslant \frac{1}{\alpha^2}\Big[\,\gamma'(G,p)\,C_{i1}(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)+\big(\,\gamma(G,q)+1\,\big)\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)+$$

$$+\,A_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\Big]\,|\,\lambda\,|\cdot\|\,\overset{i}{u}_\lambda\,\|_{L_2(G)} \qquad (35)$$

if $|\,\mathcal{R}e\sqrt{\lambda}\,|>r(G,\mathcal{I}m\sqrt{\lambda}\,)$; here $\gamma'(G,p)$, $\gamma(G,q)$ have the obvious meaning.

By (28)–(29) it follows that $\max\big\{\,|\,\overset{i}{u}{}''_\lambda(x_0-0)\,|,\ |\,\overset{i}{u}{}''_\lambda(x_0+0)\,|\,\big\}$ satisfies the estimate (35), too. Thus, we get the estimate

$$\sup_{x\in G}|\,\overset{i}{u}{}''_\lambda(x)\,| \leqslant C_{i2}(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\,|\,\lambda\,|\cdot\|\,\overset{i}{u}_\lambda\,\|_{L_2(G)} \qquad (36)$$

if $|\,\mathcal{R}e\sqrt{\lambda}\,|>r(G,\mathcal{I}m\sqrt{\lambda}\,)$, where $C_{i2}(\cdot)$ is the constant from estimate (35).

1.2. If $0\leqslant|\,\mathcal{R}e\sqrt{\lambda}\,|\leqslant r(G,\mathcal{I}m\sqrt{\lambda}\,)$, then using equations (4)–(5), equalities (28)–(29) and applying the mentioned above estimates (from Lemmas 1–3) to the right-hand side of the first inequality (34), we obtaine the estimate

$$\sup_{x\in G}|\,\overset{i}{u}{}''_\lambda(x)\,| \leqslant \tilde{C}_{i2}(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\cdot\|\,\overset{i}{u}_\lambda\,\|_{L_2(G)}\,, \qquad (37)$$

where the constant has the form

$$\tilde{C}_{i2}(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\overset{\mathrm{def}}{=}\frac{1}{\alpha^2}\Big[\,\gamma'(G,p)\,C_{i1}(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)+$$

$$+\big[\,\gamma(G,q)+\big(\,r(G,\mathcal{I}m\sqrt{\lambda}\,)\,\big)^2+\big(\,\mathcal{I}m\sqrt{\lambda}\,\big)^2\,\big]\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)+$$

$$+A_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\,C_i(G,p,q,\mathcal{I}m\sqrt{\lambda}\,)\sqrt{\big(\,r(G,\mathcal{I}m\sqrt{\lambda}\,)\,\big)^2+\big(\,\mathcal{I}m\sqrt{\lambda}\,\big)^2}\ \Big].$$

Using max if necessary, we can get *the same constant* in both estimates (36) and (37), as it is stated in the proposition (b) of Theorem 1.

1.3. The global estimates for $\overset{\circ}{u}{}''_\lambda(x)$ can be obtained in the same way as in the case of local estimates (see 2.3 § 1).

2. On Remarks 1 and 3–4. Analysing the first inequalities (31) and (34), and having in mind Remark 2 from [5], we see that Remark 1 holds true in the case $j=2$.

2.1. If $\sigma(\mathcal{L})$ is a set of eigenvalues of the operator (1) satisfying conditions described in Remark 3, then every constant from Lemmas 1–3 appearing in the "local" and "global" constants $C_{i2}(\cdot)$, $\tilde{C}_{i2}(\cdot)$ $(i=0,1,\dots)$ does not depend on the numbers $\lambda\in\sigma(\mathcal{L})$ (see 3.1 in Introduction). Replace $\mathcal{I}m\sqrt{\lambda}$ by A in all the terms in constants $C_{i2}(\cdot)$, $\tilde{C}_{i2}(\cdot)$ containing $\mathcal{I}m\sqrt{\lambda}$ explicitly. Therefore we obtain constants $C_{i2}(\cdot,p,q,A)$, $\tilde{C}_{i2}(\cdot,p,q,A)$ not depending on $\lambda\in\sigma(\mathcal{L})$.

2.2. The assertion stated in Remark 4 follows from the fact that all the constants from Lemmas 1–3 appearing in the constants $C_{i2}(\cdot)$, $\tilde{C}_{i2}(\cdot)$ $(i=1,2,\cdots)$ do not depend on the order i of the associated function (see 3.1 in Introduction).

§3. Estimates for derivatives of higher order

1. Existence of the derivatives. We continue the proof of Theorem 1 by considering those parts of propositions (a)–(b) concerning the existence and continuity of higher order derivatives of eigenfunctions and associated functions. The starting point is the fact that the functions $\overset{i}{u}_\lambda(x)$ $(i = 0, 1, 2, \dots)$ are solutions of the corresponding differential equations (2)–(5) everywhere on (a, x_0) or (x_0, b) if $q(x) \in \mathcal{C}(G \setminus \{x_0\})$. Therefore, in that case those equalities are identities with respect to x.

1.1. Suppose $j = 3$ and consider, for example, the identity (4):

$$p_1(x)\,\overset{i}{u}{}''_\lambda(x) = -p_1'(x)\,\overset{i}{u}{}'_\lambda(x) + q_1(x)\,\overset{i}{u}_\lambda(x) - \lambda\,\overset{i}{u}_\lambda(x) + \overset{i-1}{u}_\lambda(x)$$

where $x \in (a, x_0)$ and $i \geqslant 1$. By virtue of condition $p_1(x) \geqslant \alpha_1 > 0$ it follows from this identity that the function $\overset{i}{u}{}''_\lambda(x)$ has continuous first derivative on (a, x_0), and for every $x \in (a, x_0)$ we have

$$\begin{aligned} p_1(x)\,\overset{i}{u}{}^{(3)}_\lambda(x) = -p_1'(x)\,\overset{i}{u}{}''_\lambda(x) - \big(p_1'(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + {}\\ + \big(q_1(x)\,\overset{i}{u}_\lambda(x)\big)' - \lambda\,\overset{i}{u}{}'_\lambda(x) + \overset{i-1}{u}_\lambda{}'(x)\,. \end{aligned} \tag{38}$$

Analogously, using identity (5), one can prove existence of continuous third derivative of $\overset{i}{u}_\lambda(x)$ on the interval (x_0, b), and establish the identity

$$\begin{aligned} p_2(x)\,\overset{i}{u}{}^{(3)}_\lambda(x) = -p_2'(x)\,\overset{i}{u}{}''_\lambda(x) - \big(p_2'(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + {}\\ + \big(q_2(x)\,\overset{i}{u}_\lambda(x)\big)' - \lambda\,\overset{i}{u}{}'_\lambda(x) + \overset{i-1}{u}_\lambda{}'(x) \end{aligned} \tag{39}$$

on this interval.

By continuity of $\overset{i}{u}{}''_\lambda(x)$ on the half-open intervals $(a, x_0]$ and $[x_0, b)$ (see 1.3 § 1), and by existence of the finite limits $\lim_{x\to x_0-0} \overset{i}{u}{}^{(3)}_\lambda(x)$ and $\lim_{x\to x_0+0} \overset{i}{u}{}^{(3)}_\lambda(x)$, which follows from (38)–(39), we can conclude that the finite one-side derivatives $\overset{i}{u}{}^{(3)}_\lambda(x_0 - 0)\,, \overset{i}{u}{}^{(3)}_\lambda(x_0 + 0)$ exist and that $\overset{i}{u}{}^{(3)}_\lambda(x)$ is continuous on the half-open intervals mentioned above.

1.2. The case of the eigenfunction function $\overset{\circ}{u}_\lambda(x)$ is simpler; instead of identities (38)–(39) we have to use two identities of analogous form: $\overset{i-1}{u}_\lambda(x)$ is removed, and $\overset{i}{u}_\lambda(x)$ is replaced by $\overset{\circ}{u}_\lambda(x)$.

1.3. Now, we have everything necessary for completion of the proof of the first statements in propositions (a)–(b) of Theorem 1. This should be done by the mathematical induction. We omit the details.

2. Estimates of the derivatives. The proof of estimates (6)–(7) and (8)–(9) in general case should be completed by the mathematical induction, too. The first step is getting identities of form (38)–(39) for the functions

$$p_k(x)\,\frac{d^j}{dx^j}\,\overset{i}{u}_\lambda(x) \qquad (\,k = 1, 2\,;\ i = 0, 1, 2, \dots)$$

in case of arbitrary $j \in \mathbf{N}$; this can be done by starting from identities (38)–(39) and their analogues for $i = 0$. Then, by the obtained identities, one can derive the desired estimates, as it was done in the case of the second derivatives.

3. On Remarks 1 and 3–4. It is not difficult to verify that the assertions from Remarks 1, 3 and 4 concerning the estimates of derivatives and the corresponding constants are valid in general case. The proof is based directly on the mentioned above identities.

§4. Proof of theorem 2.

1. The estimate (11). Let $\overset{i}{u}_\lambda(x)$ be an associated function of the operator (1) corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(x)$ and the eigenvalue $\lambda \neq 0$. Suppose $\overset{i}{u}{}'_\lambda(a) = 0$ and establish the estimate (11).

1.1. If $y \in [a, x_0)$, then the following equality holds:

$$
\begin{aligned}
\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi &= \sqrt{p_1(a)}\ \overset{i}{u}_\lambda(a)\, \frac{\sin\sqrt{\lambda}\,\overline{\rho}_2(a, y-a)}{\sqrt{\lambda}} - \\
&- \frac{1}{\sqrt{\lambda}} \int_a^y \frac{p_1'(\xi)}{2\sqrt{p_1(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi - a) - \overline{\rho}_2(a, y-a))\, d\xi + \\
&+ \frac{1}{\sqrt{\lambda}} \int_0^{\overline{\rho}_2(a, y-a)} \left(\int_a^{a+\rho_2(a,t)} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi-a) - t)\, d\xi \right) dt - \\
&- \frac{1}{\sqrt{\lambda}} \int_0^{\overline{\rho}_2(a, y-a)} \left(\int_a^{a+\rho_2(a,t)} \overset{i-1}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi-a) - t)\, d\xi \right) dt\,.
\end{aligned} \tag{40}
$$

This equality formally follows from the equalities (20) and (29) in [6], by putting there $y_1 = a\,, y_2 = y\,, j_1 = 1$. Actually, it can be derived by applying the procedure from 1.2–1.4 § 1 and 1.1–1.2 § 2 in [6] to the closed interval $[a, y]$, instead of $[y_1, y_2]$.

1.2. If $y \in (x_0, b]$, then we have that

$$
\begin{aligned}
\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi &= \sqrt{p_1(a)}\ \overset{i}{u}_\lambda(a)\, \frac{\sin\sqrt{\lambda}\,\overline{\rho}_2(a, y-a)}{\sqrt{\lambda}} - \\
&-\left(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\right) \overset{i}{u}_\lambda(x_0)\, \frac{\sin\sqrt{\lambda}\,(\overline{\rho}_2(a, x_0 - a) - \overline{\rho}_2(a, y-a))}{\sqrt{\lambda}} - \\
&- \frac{1}{\sqrt{\lambda}} \int_a^y \frac{p_j'(\xi)}{2\sqrt{p_j(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi - a) - \overline{\rho}_2(a, y-a))\, d\xi + \\
&+ \frac{1}{\sqrt{\lambda}} \int_0^{\overline{\rho}_2(a, y-a)} \left(\int_a^{a+\rho_2(a,t)} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi-a) - t)\, d\xi \right) dt - \\
&- \frac{1}{\sqrt{\lambda}} \int_0^{\overline{\rho}_2(a, y-a)} \left(\int_a^{a+\rho_2(a,t)} \overset{i-1}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(a, \xi-a) - t)\, d\xi \right) dt
\end{aligned} \tag{41}
$$

(see equalities (19) and (28) in [6]).

Note that all terms appearing in (40)–(41) are continuous functions with respect to the variable $y \in [a,b]$, and, consequently, integrable on any closed interval $[y_1, y_2] \subseteq [a,b]$.

1.3. Let $[y_1, y_2] \subseteq [a,b]$ be an arbitrary closed interval. In order to prove estimate (11) we have to integrate equality (40) (with respect to variable y) if $[y_1, y_2] \subseteq [a, x_0]$, and the equality (41) if $x_0 < y_2$. It follows from (40)–(41) that the second case is the more complicated one; we will consider in detail that case only. Thus, we have

$$\begin{aligned}
&\int_{y_1}^{y_2} \Big(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \Big)\, dy = \frac{\sqrt{p_1(a)}\, \overset{i}{u}_\lambda(a)}{\sqrt{\lambda}} \int_{y_1}^{y_2} \sin \sqrt{\lambda}\, \overline{\rho}_2(a, y-a)\, dy - \\
&- \big(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\, \big)\, \frac{\overset{i}{u}_\lambda(x_0)}{\sqrt{\lambda}} \int_{y_1}^{y_2} \sin \sqrt{\lambda}\, (\overline{\rho}_2(a, x_0 - a) - \overline{\rho}_2(a, y-a))\, dy - \\
&- \frac{1}{\sqrt{\lambda}} \int_{y_1}^{y_2} \Big(\int_a^y \frac{p_j'(\xi)}{2\sqrt{p_j(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \sin \sqrt{\lambda}\, (\overline{\rho}_2(a, \xi - a) - \overline{\rho}_2(a, y-a))\, d\xi \Big)\, dy + \\
&+ \frac{1}{\sqrt{\lambda}} \int_{y_1}^{y_2} \Big(\int_0^{\overline{\rho}_2(a,y-a)} \Big(\int_a^{a+\rho_2(a,t)} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \sin \sqrt{\lambda}\, (\overline{\rho}_2(a, \xi - a) - t)\, d\xi \Big)\, dt \Big)\, dy - \\
&- \frac{1}{\sqrt{\lambda}} \int_{y_1}^{y_2} \Big(\int_0^{\overline{\rho}_2(a,y-a)} \Big(\int_a^{a+\rho_2(a,t)} \overset{i-1}{u}_\lambda(\xi)\, \sin \sqrt{\lambda}\, (\overline{\rho}_2(a, \xi - a) - t)\, d\xi \Big)\, dt \Big)\, dy\, .
\end{aligned} \tag{42}$$

Consider the integral $\int_{y_1}^{y_2} \sin \sqrt{\lambda}\, \overline{\rho}_2(a, y-a)\, dy$. Introducing a new variable $t = \overline{\rho}_2(a, y-a)$, and using then the integration by parts, we obtain an expression for that integral, wherefrom it results the following estimate:

$$\begin{aligned}
\Big| \int_{y_1}^{y_2} \sin \sqrt{\lambda}\, \overline{\rho}_2(a, y-a)\, dy \Big| \leqslant \Big(4\,\gamma(G,p) \sqrt{1 + \mathrm{sh}^2\big(\frac{b-a}{\alpha}\, \mathcal{I}m \sqrt{\lambda}\big)} + \\
+ \frac{b-a}{\alpha}\, \gamma'(G,p) \sqrt{1 + \mathrm{sh}^2(\mathcal{I}m \sqrt{\lambda})}\, \Big)\, \frac{1}{|\sqrt{\lambda}\,|}\, .
\end{aligned}$$

Here constants $\gamma(G,p)$, $\gamma'(G,p)$ have the meaning analogous to the meaning of constants $\gamma(K,q)$ and $\gamma'(K,p)$ introduced in 2.1 § 1. Denote by $\tilde{C}(G,p,\mathcal{I}m \sqrt{\lambda}\,)$ the constant from the above estimate. (That estimate is valid also if sin is replaced by cos.) That is why we have the estimate

$$\begin{aligned}
\Big| \frac{\sqrt{p_1(a)}\, \overset{i}{u}_\lambda(a)}{\sqrt{\lambda}} \int_{y_1}^{y_2} \sin \sqrt{\lambda}\, \overline{\rho}_2(a, y-a)\, dy \Big| \leqslant \\
\leqslant \gamma(G,p)\, \tilde{C}(G,p,\mathcal{I}m \sqrt{\lambda}\,)\, \frac{1}{|\lambda|} \cdot \sup_{x \in G} |\, \overset{i}{u}_\lambda(x)\,|\, .
\end{aligned} \tag{43}$$

1.4. Transforming the second integral on the right-hand side of (42) by the corresponding trigonometrical identity, we can obtain, analogously to the previous case, the estimate

$$\Big| \int_{y_1}^{y_2} \sin\sqrt{\lambda}\,(\overline{\rho}_2(a,x_0-a)-\overline{\rho}_2(a,y-a))\,dy \Big| \leqslant$$
$$\leqslant 2\,\tilde{C}(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)\sqrt{1+\mathrm{sh}^2\big(\,\frac{b-a}{\alpha}\,\mathcal{I}m\,\sqrt{\lambda}\,\big)}\;\frac{1}{|\,\sqrt{\lambda}\,|}\;. \quad (44)$$

Denote by $C(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)$ the constant from this estimate. Now, using (44), we get that

$$\Big|\,(\,\sqrt{p_2(x_0)}-\sqrt{p_1(x_0)}\,)\,\frac{\overset{i}{u}_\lambda(x_0)}{\sqrt{\lambda}}\int_{y_1}^{y_2}\sin\sqrt{\lambda}\,(\overline{\rho}_2(a,x_0-a)-\rho_2(a,y-a))\,dy\,\Big| \leqslant$$
$$\leqslant 2\,\gamma(G,p)\,C(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)\,\frac{1}{|\,\lambda\,|}\cdot\sup_{x\in G}|\,\overset{i}{u}_\lambda(x)\,|\,. \quad (45)$$

1.5. Applying first the Fubini's theorem to the third integral on the right-hand side of (42), and then the estimate (44) to the obtained interior integrals, we have the estimate

$$\begin{aligned}&\Big|\int_{y_1}^{y_2}\Big(\int_a^y\frac{p_j'(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{i}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-\overline{\rho}_2(a,y-a))\,d\xi\Big)\,dy\,\Big|\leqslant\\ &\quad\leqslant\frac{2\,(b-a)}{\alpha}\,\gamma'(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)\,C(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)\,\frac{1}{|\,\sqrt{\lambda}\,|}\cdot\sup_{x\in G}|\,\overset{i}{u}_\lambda(x)\,|\,.\end{aligned} \quad (46)$$

1.6. It remains to estimate the last two integrals on the right-hand side of (42). For the first one we use the Fubini's theorem again:

$$\int_0^{\overline{\rho}_2(a,y-a)}\Big(\int_a^{a+\rho_2(a,t)}q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-t)\,d\xi\Big)\,dt=$$
$$=\frac{1}{\sqrt{\lambda}}\int_a^y q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\big[\cos\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-\overline{\rho}_2(a,y-a))-1\,\big]\,d\xi\,.$$

By virtue of this equality we get the estimate

$$\Big|\int_{y_1}^{y_2}\Big(\int_0^{\overline{\rho}_2(a,y-a)}\Big(\int_a^{a+\rho_2(a,t)}q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-t)\,d\xi\Big)\,dt\Big)\,dy\,\Big|\leqslant$$
$$\leqslant\frac{(b-a)^2}{\alpha}\,\|\,q\,\|_{L_1(G)}\cdot\sqrt{2+\mathrm{sh}^2\big(\,\frac{b-a}{\alpha}\,\mathcal{I}m\,\sqrt{\lambda}\,\big)}\;\frac{1}{|\,\sqrt{\lambda}\,|}\cdot\sup_{x\in G}|\,\overset{i}{u}_\lambda(x)\,|\,. \quad (47)$$

1.7. Denote by $R_{(40)}(a;y;\lambda;\overset{i}{u}_\lambda)$ (by $R_{(41)}(a;y;\lambda;\overset{i}{u}_\lambda)$) the sum of all the terms (with corresponding signs) on the right-hand side of equality (40) (of equality (41)), excluding the last integral. Also, denote by $D_{i2}'(G,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)$ the biggest of the constants appearing in estimates (43), (45), (46) and (47). Then it results from those estimates and (42) that the following estimate is valid:

$$\max\left\{ \left| \int_{y_1}^{y_2} R_{(40)}(a;y;\lambda;\overset{i}{u}_\lambda)\,dy \right|, \left| \int_{y_1}^{y_2} R_{(41)}(a;y;\lambda;\overset{i}{u}_\lambda)\,dy \right| \right\} \leqslant$$
$$\leqslant 4\,D'_{02}(G,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\,\frac{1}{|\,\lambda\,|}\cdot\sup_{x\in G}|\,\overset{i}{u}_\lambda(x)\,|\,. \quad (48)$$

According to the content of 3.2 in Introduction, for any closed interval $K \subset G$ there exists a constant $C_i(K,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)$ such that estimate (24) holds. By the proposition (a) of Lemma 1 there exist a number $R \in (0,\rho(K,\partial G))$ and a constant $C_i(K_R,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)$ such that estimate (16) is valid. Now, using the mentioned estimates, we obtain from (48) the estimate

$$\max\left\{ \left| \int_{y_1}^{y_2} R_{(40)}(a;y;\lambda;\overset{i}{u}_\lambda)\,dy \right|, \left| \int_{y_1}^{y_2} R_{(41)}(a;y;\lambda;\overset{i}{u}_\lambda)\,dy \right| \right\} \leqslant$$
$$\leqslant 4D'_{i2}(G,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)C_i(K,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)C_i(K_R,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\frac{1}{|\,\lambda\,|}\|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}. \quad (49)$$

1.8. Let us finally estimate the last integral on the right-hand side of (42). By the Fubini's theorem that integral can be transformed in the following one:

$$\int_{y_1}^{y_2}\left(\int_a^y \overset{i-1}{u}_\lambda(\xi)\,\big[\cos\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-\overline{\rho}_2(a,y-a))-1\,\big]\,d\xi\right)dy =$$
$$= \int_{y_1}^{y_2}\left(\int_a^y \overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-\rho_2(a,y-a))\,d\xi\right)dy - \int_{y_1}^{y_2}\left(\int_a^y \overset{i-1}{u}_\lambda(\xi)\,d\xi\right)dy\,. \quad (50)$$

Using the Fubini's theorem again, and applying then the estimate (44) (with sin replaced by cos), the estimate (24) and the estimate (16), we obtain that for every closed interval $K \subset G$ the following estimate holds:

$$\left|\int_{y_1}^{y_2}\left(\int_a^y \overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(a,\xi-a)-\overline{\rho}_2(a,y-a))\,d\xi\right)dy\right| \leqslant$$
$$\leqslant 2\,(b-a)\,C(G,p,\mathcal{I}m\,\sqrt{\lambda}\,)\,A_i(G,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\,C_i(K,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\times$$
$$\times\,C_i(K_R,p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\cdot\|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}\,. \quad (51)$$

1.9. In order to obtain the appropriate estimate for the second integral on the right-hand side of (50), we have to refer to the paper [6]. A careful analysis of 1.4 § 2 and 1.7 § 1 in that paper shows that the following estimate is valid for every number $y \in [y_1, y_2]$:

$$\left|\int_a^y \overset{i-1}{u}_\lambda(\xi)\,d\xi\right| \leqslant \tilde{D}_{i1}(G,\tilde{K}_{R_0},p,q,\mathcal{I}m\,\sqrt{\lambda}\,)\frac{1}{|\sqrt{\lambda}\,|}\cdot\max_{x\in\tilde{K}_{R_0}}|\,\overset{i-1}{u}_\lambda(x)\,|\,,$$

where $\tilde{K} \subset G$ is some fixed closed interval, and $R_0 \in (0, \rho(\tilde{K}, \partial G))$ is a fixed number. This estimate is valid if $|\mathcal{R}e\sqrt{\lambda}| > r(G, \mathcal{I}m\sqrt{\lambda})$, with $r(G, \mathcal{I}m\sqrt{\lambda})$ defined in 1.2 § 3 in [5].

By virtue of the modified first estimate (18) (see 3.2 in Introduction), and by estimate (16) we conclude that

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i-1}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant (b-a)\, \tilde{D}_{i1}(G, \tilde{K}_{R_0}, p, q, \mathcal{I}m\sqrt{\lambda}\,)\times$$
$$\times\, \tilde{A}_i(\tilde{K}_{R_0}, p, q, \mathcal{I}m\sqrt{\lambda}\,)\, C_i(K, p, q, \mathcal{I}m\sqrt{\lambda}\,) \cdot \|\, \overset{i}{u}_\lambda \,\|_{L_2(K_R)}\,, \qquad (52)$$

where $K \overset{\text{def}}{=} \tilde{K}_{R_0}$, and $R \in (R_0, \rho(\tilde{K}, \partial G))$ is a fixed number.

1.10. Let us now summarize the content of 1.1–1.9: There exist a closed interval $K \subset G$ and a constant $r(G, \mathcal{I}m\sqrt{\lambda}\,)$ such that the estimate

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant D_{i2}(G, K_R, p, q, \mathcal{I}m\sqrt{\lambda}\,)\, \frac{1}{|\lambda|} \cdot \|\, \overset{i}{u}_\lambda \,\|_{L_2(K_R)} \qquad (53)$$

holds uniformly with respect to the numbers $a \leqslant y_1 < y_2 \leqslant b$ if λ satisfies condition $|\mathcal{R}e\sqrt{\lambda}| > r(G, \mathcal{I}m\sqrt{\lambda}\,)$. This estimate follows from (42) and (49)–(52); the constant $D_{i2}(\cdot)$ is defined as sum of the constant from (49) and the maximum of the constants appearing in estimates (51)–(52).

1.11. The above proof "works" in the case of the eigenfunction $\overset{\circ}{u}_\lambda(x)$, too. One should start from the equalities (40)–(41) in which $\overset{i}{u}_\lambda(x)$ is replaced by $\overset{\circ}{u}_\lambda(x)$ and the last integral (containing $\overset{i-1}{u}_\lambda(\cdot)$) is removed.

2. The estimate (10). Proof of the estimate (10) is simpler. By virtue of estimates (24) and (16) we get directly the inequalities

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant (b-a)^2 \cdot \sup_{x \in G} |\overset{i}{u}_\lambda(x)\,| \leqslant$$
$$\leqslant (b-a)^2\, C_0(K, p, q, \mathcal{I}m\sqrt{\lambda}\,)\, C_0(K_R, p, q, \mathcal{I}m\sqrt{\lambda}\,) \cdot \|\, \overset{i}{u}_\lambda \,\|_{L_2(K_R)}\,,$$

or the estimate

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \overset{i}{u}_\lambda(\xi)\, d\xi \right) dy \right| \leqslant \tilde{D}_{02}(G, K_R, p, q, \mathcal{I}m\sqrt{\lambda}\,)\, \|\, \overset{i}{u}_\lambda \,\|_{L_2(K_R)}\,, \qquad (54)$$

where $K \subset G$ is an arbitrary closed interval, $R \in (0, \rho(K, \partial G))$ is a fixed number, and $i = 0, 1, \ldots$.

2.1. We see that this estimate is actually valid independently of the boundary condition $\overset{i}{u}{}'_\lambda(a) = 0$; also, λ is an arbitrary eigenvalue.

2.2. Using max if necessary, we can obtain *the same constant* in both estimates (53) and (54), as it is stated in the proposition (a) of Theorem 2.

3. The estimates (12)–(13). The proof of these estimates is completely analogous to the one of estimates (10)–(11).

4. On Remarks 2–3. Analysing the content of 1.7–1.9 and 2, we see that the second statement in Remark 2 holds true.

4.1. The constants $D_{i2}(\cdot)$ and $\tilde{D}_{i2}(\cdot)$ can be defined independently of the spectral parameter $\lambda \in \sigma(\mathcal{L})$ if the set $\sigma(\mathcal{L})$ satisfies the conditions described in Remark 3. The general principle is the same as in the case of constants $C_{ij}(\cdot)$; under the mentioned conditions the constants from Lemmas 1–3 have the property of independency, whereas $\mathcal{I}m\,\sqrt{\lambda}$ should be replaced by A in other terms appearing in $D_{i2}(\cdot)$ (and $\tilde{D}_{i2}(\cdot)$) and containing $\mathcal{I}m\,\sqrt{\lambda}$.

4.2. The independency of constants $D_{i2}(\cdot)$ and $\tilde{D}_{i2}(\cdot)$ of the parameter i is based on the same property of constants (from Lemmas 1–3) entering into the structure of $D_{i2}(\cdot)$ and $\tilde{D}_{i2}(\cdot)$.

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