МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 77–79

UDC 515.126.4
оригинални научни рад
research paper

A FIXED POINT THEOREM IN BANACH SPACES OVER TOPOLOGICAL SEMIFIELDS

Slobodan Č. Nešić

Abstract. Let X be a Banach space over a topological semifield and T_1, $T_2\colon X \to X$ two maps satisfying the condition (1). Then T_1 and T_2 have a common fixed point.

1. Introduction

The notion of topological semifield has been introduced by M. Antonovskiĭ, V. Boltyanskiĭ and T. Sarymsakov in [1]. Let E be a topological semifield and K the set of all its positive elements. Take any two elements x, y in E. If $y - x$ is in $\overline{K}$ (in K), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield, so called the axis of E, isomorphic to the field $\mathbf{R}$ of real numbers. Consequently by identifying the axis and $\mathbf{R}$, each topological semifield can be regarded as a topological linear space over the field $\mathbf{R}$.

The ordered triple (X, d, E) is called a metric space over the topological semifield if there exists a mapping $d\colon X \times X \to E$ satisfying the usual axioms for a metric (see [1], [2] and [4]).

Linear spaces considered in this paper are defined on the field $\mathbf{R}$. Let X be a linear space. The ordered triple $(X, \|\ \|, E)$ is called a feeble normed space over the topological semifield if there exists a mapping $\|\ \|\colon X \to E$ satisfying the usual axioms for a norm (see [1] and [3]).

2. Main result

We shall use the following definition.

Definition 1. Let $(X, \|\ \|, E)$ be a feeble normed space over a topological semifield E and let $d(x, y) = \|x - y\|$ for all x, y in X. A space $(X, \|\ \|, E)$ is said to be a Banach space over the topological semifield E if (X, d, E) is sequentially complete metric space over the topological semifield E.

Key words: Banach space over a topological semifield, common fixed point, Cauchy sequence, sequentially complete metric space.

Now we shall prove the following result.

THEOREM 1. *Let X be a Banach space over a topological semifield E and T_1, $T_2\colon X \to X$ two maps satisfying the condition*

$$\|x - T_1x\|^m + \|y - T_2y\|^m \ll p\|x - y\|^m \tag{1}$$

for all x, y in X, where p, t are in $\mathbf{R}$, $0 < t < 1$, $1 \leqslant pt^m < 2$ and $m = 1, 2, \ldots$. Then the sequence $\{x_n\}_{n=0}^{\infty}$, the members of which are

$$x_{2n+1} = (1-t)x_{2n} + tT_1x_{2n}, \quad x_{2n+2} = (1-t)x_{2n+1} + tT_2x_{2n+1}, \quad x_0 \in X, \tag{2}$$

converges to the common fixed point of T_1 and T_2 in X.

Proof. Let x_0 in X be an arbitrary point . From (2) we get

$$\|x_{2n+1} - x_{2n}\| = t\|T_1x_{2n} - x_{2n}\|, \quad \|x_{2n+2} - x_{2n+1}\| = t\|T_2x_{2n+1} - x_{2n+1}\|. \tag{3}$$

If in (1) we put $x = x_{2n}$ and $y = x_{2n+1}$, then by (3) we have

$$t^{-m}(\|x_{2n+1} - x_{2n}\|^m + \|x_{2n+2} - x_{2n+1}\|^m) \ll p\|x_{2n} - x_{2n+1}\|^m$$

and hence

$$\|x_{2n+2} - x_{2n+1}\| \ll (pt^m - 1)^{1/m}\|x_{2n} - x_{2n+1}\| \tag{4}$$

for all n. Now, if we put in (1) $x = x_{2n+2}$ and $y = x_{2n+1}$, and use (3), we get

$$t^{-m}(\|x_{2n+3} - x_{2n+2}\|^m + \|x_{2n+2} - x_{2n+1}\|^m) \ll p\|x_{2n+2} - x_{2n+1}\|^m$$

and hence

$$\|x_{2n+3} - x_{2n+2}\| \ll (pt^m - 1)^{1/m}\|x_{2n+2} - x_{2n+1}\| \tag{5}$$

for all n. From (4) and (5) we then obtain

$$\|x_n - x_{n+1}\| \ll (pt^m - 1)^{1/m}\|x_{n-1} - x_n\|$$

which implies

$$\|x_n - x_{n+1}\| \ll (pt^m - 1)^{1/m}\|x_0 - x_1\|.$$

Since $0 \leqslant pt^m - 1 < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. As X is a Banach space over the topological semifield E, we deduce that $\{x_n\}$ converges to a point u in X.

Now putting $x = u$ and $y = x_{2n+1}$ in (1) we have

$$\|u - T_1u\|^m + \|x_{2n+1} - T_2x_{2n+1}\|^m \ll p\|u - x_{2n+1}\|^m,$$

i.e.

$$\|u - T_1u\|^m + t^{-m}\|x_{2n+2} - x_{2n+1}\|^m \ll p\|u - x_{2n+1}\|^m.$$

If now n tends to infinity one has $\|u - T_1u\|^m \ll 0$, which implies $T_1u = u$. Hence, u is a fixed point for T_1. Similarly, $T_2u = u$. So u is a common fixed point of T_1 and T_2. This completes the proof. ∎

REFERENCES

[1] Antonovskiĭ, M., Boltyanskiĭ, V. and Sarymsakov, T., *Topological semifields*, Tashkent 1960.

[2] Antonovskiĭ, M., Boltyanskiĭ, V. and Sarymsakov, T., *Metric spaces over semifields*, Tashkent 1961.

[3] Kasahara, S., *On formulations of topological linear spaces by topological semifields*, Math. Seminar Notes, Vol. 1 (1973), 11–29

[4] Mamuzić, Z., *Some remarks on abstract distance in general topology*, $E\Lambda EY\Theta EPIA$ 2 (1979), 433–446, Athens, Greece

(received 14.06.1994.)

Studentska 14, 11000 Belgrade, Yugoslavia