МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 67–69

UDC 514.144.1
оригинални научни рад
research paper

SOME DISCUSSIONS RELATED TO DISJOINT BER'S SUBPLANES

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Abstract. A projective plane $\mathcal{P}_m$ is a Ber's subplane of a finite projective plane $\mathcal{P}_n$ if every point and line of $\mathcal{P}_n \setminus \mathcal{P}_m$ is incident to some line and some point, respectively, of $\mathcal{P}_n$. It is known that the order of the plane $\mathcal{P}_n$ and its Ber's subplane $\mathcal{P}_m$ satisfy the equation $n = m^2$. In this article we prove some properties of finite projective planes $\mathcal{P}_n$ having disjoint Ber's subplanes covering it.

Let $\mathcal{P}_n$, $n = m^2$, $n \in \mathbf{N}$ be a projective plane. Since the number of points of $\mathcal{P}_n$ equals $n^2+n+1 = (m^2+m+1)(m^2-m+1)$, then, as long as the number of points is concerned, every such a projective plane $\mathcal{P}_n$ could be covered by $m^2 - m + 1$ disjoint Ber's subplanes. We shall give an example (Theorem 1) of such a covering of the projective plane $\mathcal{P}_4$, but it is still an open problem whether every projective plane of order $n = m^2$, $m \in \mathbf{N}$ has a covering by its disjoint Ber's subplanes.

THEOREM 1. *The plane $\mathcal{P}_4$ has a covering by its disjoint Ber's subplanes.*

Proof. Since $4 = 2^2$ and $2^2-2+1 = 3$, Ber's subplanes of $\mathcal{P}_4$ are of order 2 and if disjoint Ber's subplanes cover $\mathcal{P}_4$, their number is 3. Let p_i^j and P_i^j, $i = 1, 2, \dots, 7$, $j = 1, 2, 3$ be the lines and the points of $\mathcal{P}_4$ with the incidences given by:

$$\begin{aligned}
&p_1^1 = \{P_2^1, P_3^1, P_5^1, P_1^2, P_1^3\},\ p_1^2 = \{P_2^2, P_3^2, P_5^2, P_1^1, P_1^3\},\ p_1^3 = \{P_2^3, P_3^3, P_5^3, P_1^1, P_1^2\},\\
&p_2^1 = \{P_1^1, P_4^1, P_5^1, P_6^2, P_6^3\},\ p_2^2 = \{P_1^2, P_4^2, P_5^2, P_6^1, P_6^3\},\ p_2^3 = \{P_1^3, P_4^3, P_5^3, P_6^1, P_6^2\},\\
&p_3^1 = \{P_1^1, P_3^1, P_6^1, P_7^2, P_7^3\},\ p_3^2 = \{P_7^1, P_1^2, P_3^2, P_6^2, P_7^3\},\ p_3^3 = \{P_7^1, P_7^2, P_1^3, P_3^3, P_6^3\},\\
&p_4^1 = \{P_2^1, P_4^1, P_6^1, P_3^2, P_3^3\},\ p_4^2 = \{P_3^1, P_2^2, P_4^2, P_6^2, P_3^3\},\ p_4^3 = \{P_3^1, P_3^2, P_2^3, P_4^3, P_6^3\},\\
&p_5^1 = \{P_1^1, P_2^1, P_7^1, P_4^2, P_4^3\},\ p_5^2 = \{P_4^1, P_1^2, P_2^2, P_7^2, P_4^3\},\ p_5^3 = \{P_4^1, P_4^2, P_1^3, P_2^3, P_7^3\},\\
&p_6^1 = \{P_3^1, P_4^1, P_7^1, P_5^2, P_5^3\},\ p_6^2 = \{P_5^1, P_3^2, P_4^2, P_7^2, P_5^3\},\ p_6^3 = \{P_5^1, P_5^2, P_3^3, P_4^3, P_7^3\},\\
&p_7^1 = \{P_5^1, P_6^1, P_7^1, P_2^2, P_2^3\},\ p_7^2 = \{P_2^1, P_5^2, P_6^2, P_7^2, P_2^3\},\ p_7^3 = \{P_2^1, P_2^2, P_5^3, P_6^3, P_7^3\}.
\end{aligned}$$

It is easy to check that $\mathcal{P}_2^1 = \{P_1^1, \dots, P_7^1\}$, $\mathcal{P}_2^2 = \{P_1^2, \dots, P_7^2\}$ and $\mathcal{P}_2^3 = \{P_1^3, \dots, P_7^3\}$ are disjoint Ber's subplanes which cover $\mathcal{P}_4$. The incidences are probably more transparent in the figure 1. ∎

Figure 1. A covering of $\mathcal{P}_4$ by disjoint Ber's subplanes

Now we are going to prove some properties of a covering of $\mathcal{P}_n$ by disjoint Ber's subplanes. We say that *a line p is incident to some subset $\mathcal{P}$ of $\mathcal{P}_n$* if the set $\mathcal{P}$ is incident to at least two points of p.

THEOREM 2. *Let $\mathcal{P}_m^i$, $i = 1, \dots, s$ be disjoint Ber's subplanes of a projective plane $\mathcal{P}_n$. We denote by $\mathcal{P}^s$ the set $\bigcup_{i=1}^s \mathcal{P}_m^i$ and by $\mathcal{Q}^s$ the set $\mathcal{P}_n \setminus \mathcal{P}^s$. Each point of $\mathcal{P}^s$ is incident to exactly $m+s$ lines of the Ber's subplanes $\mathcal{P}_m^i$, $i = 1, \dots, s$. Each point of $\mathcal{Q}^s$ is incident to exactly s lines of the Ber's subplanes $\mathcal{P}_m^i$, $i = 1, \dots, s$.*

Proof. Every point of $\mathcal{P}_n$ is incident to at least one line of each subplane $\mathcal{P}_m^i$, $i \leqslant s$ by the definition of a Ber's subplane. If $P \in \mathcal{P}_m^i$, then P is incident to $m+1$ lines of $\mathcal{P}_m^i$ and to a single line of each $\mathcal{P}_m^j$, $j \neq i$. (If a point is incident to two lines of a subplane it is incident to that subplane.) Therefore, if P is a point of $\mathcal{P}^s$, it is incident to $m+1+s-1 = m+s$ lines of the subplanes $\mathcal{P}_m^i$, $i \leqslant s$. On the other hand, the points of $\mathcal{Q}^s$ are incident only to a single line of each subplane $\mathcal{P}_m^i$, $i \leqslant s$, which proves the theorem. ■

We shall call the property given by the previous theorem *the homogenity of finite projective planes.* Let us prove one more property of disjoint Ber's subplanes.

THEOREM 3. *If a plane $\mathcal{P}_n$, $n = m^2$, $m \in \mathbf{N}$ contains $m^2 - m$ disjoint Ber's subplanes $\mathcal{P}_m^i$, $i = 1, \dots, m^2 - m$, of order m, then $\mathcal{Q} = \mathcal{P}_n \setminus \bigcup_{i=1}^{m^2-m} \mathcal{P}_m^i$ is a Ber's subplane.*

Proof. Since every line p is incident to a single point or to $m+1$ points of each $\mathcal{P}_m^i$, $i \leqslant m^2 - m$, then it has $k(m+1) + (m^2 - m - k) + l = m^2 + 1$ points, where k is the number of the subplanes $\mathcal{P}_m^i$, $i \leqslant m^2 - m$ to which it is incident and l the number of the points of $p \cap \mathcal{Q}$. Hence, $k > 1$ is impossible, $k = 1$ implies $l = 1$, and $k = 0$ implies $l = m+1$, which proves that a line intersects $\mathcal{Q}$ in a single point if it is not incident to $\mathcal{Q}$, and in $m+1$ points otherwise.

Every point of $\mathcal{P}_n$ is incident to a single line of each Ber's subplane to which it does not belong and to $m+1$ lines of the subplane to which it belongs. This implies that every point P of $\mathcal{Q}$ is incident to $m^2 - m$ lines of the subplanes $\mathcal{P}_m^i$, $i \leqslant m^2 - m$ and $m^2 + 1 - (m^2 - m) = m + 1$ lines not incident to the subplanes $\mathcal{P}_m^i$, $i \leqslant m^2 - m$. Hence, every point P of $\mathcal{Q}$ is incident to $m+1$ lines of $\mathcal{Q}$. By the Theorem 2, every point of $\mathcal{P}_n \setminus \mathcal{Q}$ is incident to $m + (m^2 - m)$ lines of the subplanes $\mathcal{P}_m^i$, $i \leqslant m^2 - m$, and therefore, to a single line of $\mathcal{Q}$.

Since $\mathcal{Q}$ contains $m^2+m+1 = (n^2+n+1) - (m^2+m+1)(m^2-m)$ points lying on lines with $m+1$ such points (points of $\mathcal{Q}$), and since each point of $\mathcal{Q}$ is incident to $m+1$ such lines, as it is well known, $\mathcal{Q}$ is a projective subplane of order m. Finally, $\mathcal{Q}$ is a Ber's subplane according to the properties proved above. ■

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(received 22.02.1993, in revised form 23.10.1994.)