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On Congruences of Super-Associative Algebras With $n-$quasigroup Operations, $n \geq 3$

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Dedicated to professor M. Tasković on his 60^{th} birthday

Abstract. In this paper $Con(Q,\sum)$, where $(Q,\sum)$ is a super-associative algebras with $n-$quasigroup operations, $n \geq 3$, is described.

1. Preliminaries

Definition 1.1 ([2]). Let $n \geq 2$ and let $(Q;A)$ be an $n-$groupoid. Then:

1) we say that $(Q;A)$ is an $n-$semigroup iff for every $i,j \in \{1,\dots,n\}$, $i<j$, the following law holds:
$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$
(:$< i,j > -$associative law);
2) we say that $(Q;A)$ is an $n-$quasigroup iff for every $i \in \{1,\dots,n\}$ and for all $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the following equality holds
$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n;$$
3) we say that $(Q;A)$ is a Dörnte $n-$group (briefly: $n-$group) iff $(Q;A)$ is an $n-$semigroup and $n-$quasigroup as well.

Remark 1.1. A notion of an $n-$group was introduced by W. Dörnte (inspired by E. Noether) in [2] as a generalization of the notion of a group. See, also [12].

Proposition 1.1 ([9]). *Let $n \geq 2$ and let $(Q;A)$ be an $n-$groupoid. Then the following statements are equivalent: (i) $(Q;A)$ is an $n-$group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q;A,^{-1},\mathbf{e})$ [of the type $< n, n-1, n-2 >$]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,
(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*
(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$*; and*

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(iii) there are mappings $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A,^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

($\overline{a}$) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,
($\bar{b}$) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}) = x$ *and*
($\overline{c}$) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

Remark 1.2. $\mathbf{e}$ is an $\{1, n\}-$neutral operation of $n-$groupoid $(Q; A)$ iff algebra $(Q; A, \mathbf{e})$ [of the type $< n, n-2 >$] satisfies the laws (b) and $(\bar{b})$ from 1.2 [6]. Operation $^{-1}$ from 1.2 [$(c), (\overline{c})$] is a generalization of the inverse operation in a group [7]. Cf. Chapter II and Chapter III in [12].

2. Auxiliary part

Definition 2.1 ([8])**.** We say that an algebra $(Q; \cdot, \varphi, b)$ [of the type $< 2, 1, 0 >$] is a Hosszú-Gluskin algebra of order n $(n \geq 3)$ [briefly: $nHG-$algebra] iff the following statements hold:

(1) $(Q; \cdot)$ is a group;
(2) $\varphi \in Aut(Q; \cdot)$;
(3) $\varphi(b) = b$; and
(4) for all $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$.

(Cf. IV-2.1 in [12].)

Proposition 2.1. *Let* $(Q; \cdot, \varphi, b)$ *be an* $bHG-$*algebra. Also let* $A(x_1^n) \stackrel{def}{=} x_1 \cdot \varphi(x_2) \cdots \varphi^{n-1}(x_n) \cdot b$ *for all* $x_1^n \in Q$*. Then* $(Q; A)$ *is an* $n-$*group.*

Definition 2.2 ([8])**.** We say that an $nHG-$algebra $(Q; \cdot, \varphi, b)$ is associated (or corresponds) to the $n-$group $(Q; A)$ iff for all $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdots \varphi^{n-1}(x_n) \cdot b$.

Theorem 2.1 (Hosszú-Gluskin Theorem [3, 4])**.** *Let* $(Q; A)$ *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* $n \geq 3$*. Let also* c_1^{n-2} *be arbitrary sequence over* Q*, and let:*

a) $x \cdot y \stackrel{def}{=} A(x, c_1^{n-2}, y)$,
b) $\varphi(x) \stackrel{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$ *and*
c) $b \stackrel{def}{=} A\Big(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}\Big)$ *for all* $x, y \in Q$*. Then, the following statements hold:*
  1) $(Q; \cdot, \varphi, b)$ *is an* $nHG-$*algebra, and*
  2) $(Q; \cdot, \varphi, b)$ *is associated to the* $n-$*group* $(Q; A)$ *[cf. 2.3].*

Remark 2.1. The formulation of the theorem is from [8]. Cf. IV-3.1 in [12].

Proposition 2.2 ([8])**.** *Let* $(Q; A)$ *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation, and* $n \geq 3$*. Further on, let* c_1^{n-2} *be an arbitrary sequence over* Q*, and let for every*

$x, y \in Q$

$$B_{(c_1^{n-2})}(x,y) \stackrel{def}{=} A(x, c_1^{n-2}, y),$$

$$\varphi_{(c_1^{n-2})}(x) \stackrel{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \quad and$$

$$b_{(c_1^{n-2})} \stackrel{def}{=} A\Big(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}\Big)$$

Also let

$$\mathcal{L}_A \stackrel{def}{=} \{(Q; B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}) | c_1^{n-2} \in Q\}.$$

Then for every $nHG-$*algebra* $(Q; \cdot, \varphi, b)$ *the following equivalence holds*

$$(Q; \cdot, \varphi, b) \in \mathcal{L}_A \Leftrightarrow (\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdots \varphi^{n-1}(x_n) \cdot b.$$

Cf. IV-4.1 in [12].

Proposition 2.3 ([10])**.** *Let* $(Q; A)$ *be an* $n-$*group,* $n \geq 3$ *and let* $(Q; \cdot, \varphi, b)$ *be an arbitrary* $nHG-$*algebra associated to the* $n-$*group* $(Q; A)$*. Then, the following equality holds:* $Con(Q; A) = Con(Q; \cdot) \cap Con(Q; \varphi)$*.*

3. Main part

Let $x_1, \dots, x_{2n-1}$ be **subject symbols**, $n \in N\backslash\{1\}$, and let $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \dots, n\}$, be $n-$**ary operational symbols**. Then, we say that

$$(1) \qquad X_1(X_2(x_1^n), x_{n+1}^{2n-1}) = X_{2n-1}(x_1^{i-1}, X_{2i}(x_i^{i+n-1}), x_{i+n}^{2n-1})$$

is a **general** $< 1, i > -$**associative law**. Some of operational simbols in (1) can be equal.

Definition 3.1. Let $(Q; \sum)$ be an algebra in which the following holds: $(Q; Z)$ is an $n-$quasigroup for all $Z \in \sum$. Also let $n \geq 2$ and $|\sum| \geq 2$. Further on, let $x_1, \dots, x_{2n-1}$ be subject symbols, let $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \dots, n\}$, be $n-$ary operational symbols, and let for all $i \in \{2, \dots, n\}$ is $|\{X_1, X_2, X_{2i-1}, X_{2i}\}| \geq 2$. Then, we say that $(Q; \sum)$ is a super-associative algebra with $n-$quasigroup operations (briefly: $SAAnQ$) iff for every substitution of the subject symbols $x_1, \dots, x_{2n-1}$ in (1) by elements $\overline{x}_1, \dots, \overline{x}_{2n-1}$ of Q and for every substitution of the operational symbols $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \dots, n\}$, in (1) by elements $X_1, X_2, X_{2i-1}, X_{2i}$, $i \in \{2, \dots, n\}$, of $\sum$ for all $i \in \{2, \dots, n\}$ the following equality holds:

$$(\overline{1}) \qquad \overline{X}_1(\overline{X}_2(\overline{x}_1^n), \overline{x}_{n+1}^{2n-1}) = \overline{X}_{2i-1}(\overline{x}_1^{i-1}, \overline{X}_{2i}(\overline{x}_i^{i+n-1}), \overline{x}_{i+n}^{2n-1}).$$

A immediate consequence of Def. 3.1 and Def. 1.1, is the following proposition:

Proposition 3.1. *If* $(Q; \sum)$ *is a* $SAAnQ$*,* $n \in N\backslash\{1\}$*, then* $(Q; Z)$ *is an* $n-$*group for all* $Z \in \sum$*.*

Proposition 3.2. *Let* $(Q; \sum)$ *be an* $SAAnQ$ *and* $n \in N\backslash\{1\}$*. Then the following statements hold:*

1° $X_1 \neq X_2 \Rightarrow \{X_{2i-1}, X_{2i}\} = \{X_1, X_2\}$ *and*
2° $X_1 = X_2 \Rightarrow X_{2i-1} = X_{2i}$ *for all* $i \in \{2, \dots, n\}$, *where* $X_1, X_2, X_{2i-1}, X_{2i}$ *from* (1).

Cf. XI-2 in [12].

Remark 3.1. a) Case $n = 2$ is described in [1].
b) Case $n = 3$ Yu. Movsisyan was described in 1984 (cf. [5]).
c) Case $n \geq 3$ was described in [11].

Proposition 3.3 ([11])**.** *Let* $(Q; \sum)$ *be an* $SAAnQ$ *and* $n \geq 3$. *Also, let* A *be an arbitrary operation from* $\sum$ *and* $(Q; \cdot, \varphi, b)$ *be an arbitrary* $nHG-$*algebra associated to the* $n-$*group* $(Q; A)$. *Then, for every* $B \in \sum$ *there is exactly one* $a \in Q$ *such that for every* $x, x_1^n \in Q$ *the following equalities hold:*

°1 $B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \dots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n$,
°2 $(a \cdot b) \cdot x = x \cdot (a \cdot b)$ *and*
°3 $\varphi(a) = a$.

Cf. XI-6.1 and XI-6.2 in [12].

Theorem 3.1. *Let* $(Q; \sum)$ *be an super-associative algebra with* $n-$*quasigroup operations,* $n \geq 3$ *and let* A *be an arbitrary element of* $\sum$. *Then, the following equality holds:*

$$Con(Q; \sum) = Con(Q; A).$$

Proof. Let $(Q; \sum)$ be an super-associative algebra with $n-$quasigroup operations, $n \geq 3$ and let A be an arbitrary element of $\sum$. By Prop. 3.2, $(Q; A)$ is an $n-$group. Further on, let $(Q; \cdot, \varphi, b)$ be an arbitrary $nHG-$algebra associated to the $n-$group $(Q; A)$.

Then, the following statements hold:

°4 For all $B \in \sum$, $(Q; B)$ is an $n-$group (by 3.2); and
°5 For all $B \in \sum$ there is exactly one $a \in Q$ such that the algebra $(Q; \cdot, \varphi, b \cdot a \cdot b)$ is an $nHG-$algebra associated to the $n-$group $(Q; B)$.

Sketch of the proof of °5:

a) $(Q; \cdot, \varphi, b)$ is an $nHG-$algebra *[*associated to the $n-$group $(Q; A)$*]*.

b) $$\varphi(b \cdot a \cdot b) \overset{a)}{=} \varphi(b) \cdot \varphi(a) \cdot \varphi(b) \overset{a)°3}{=} b \cdot a \cdot b;$$

°3 is from 3.5.

c) $$\begin{aligned} \varphi^{n-1}(x)(b \cdot a \cdot b) &= (\varphi^{n-1}(x) \cdot b)(a \cdot b) \\ &\overset{a)}{=} (b \cdot x) \cdot (a \cdot b) \\ &= b \cdot (x \cdot (a \cdot b)) \\ &\overset{°2}{=} b \cdot ((a \cdot b) \cdot x) \\ &= (b \cdot a \cdot b) \cdot x; \end{aligned}$$

°2 is from 3.5.

d) By $a) - c)$, by $°1$ (from 3.5) and by Def. 2.3, we conclude that the algebra $(Q; \cdot, \varphi, b \cdot a \cdot b)$ is a $nHG-$algebra associated to the $n-$group $(Q; B)$.

Finally, by $a)$, by $°5$ and by Prop. 2.6, we conclude that the proposition is satisfied. □

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