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Note on Rapidly Varying Sequences

Dragan Djurčić and Mališa Žižović

Dedicated to professor M. Tasković on his 60^{th} *birthday*

Abstract. In this paper relation between rapidly varying sequence (c_n) and its generated function $f(x) = c_{[c]}, x \geq 1$ is considered. That relation is expressed by the concept of the Bojanić–Seneta proposition type for rapidly variability.

1. Introduction

The function $f : [a, +\infty) \to (0, +\infty), \quad a > 0$ is slowly varying in the sense of Karamata [1], if it is measurable and satisfies the asymptotic relation

$$\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = 1, \quad \lambda > 0. \tag{1}$$

The class of slowly varying functions is denoted by SV_f. The sequence (c_n) of positive numbers is slowly varying in the sense of Karamata [2], if it satisfies the asymptotic relation

$$\lim_{n \to +\infty} \frac{c_{[\lambda n]}}{c_n} = 1, \quad \lambda > 0. \tag{2}$$

The class of slowly varying sequences is denoted by SV_s.

Slow variability in the sense of Karamata is an important asymptotic behavior in the analysis of divergent processes [1].

Ranko Bojanić and Eugen Seneta [2] (see also [6]) introduced a quality relation between sequential property (2) and functional property (1), and founded a unique concept of theory of slow variability in the sense of Karamata.

Theorem 1 (BS). *Let (c_n) be a sequence of positive numbers. Then the following statements are equivalent:*

(a) *(c_n) belongs to the class SV_s;*
(b) *$f(x) = c_{[x]}, \quad x \geq 1$ belongs to the class SV_f.*

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Analog results based on the concept of theorem **BS**, which treat O-regular variability, expanded regular variability and SO-regular variability, can be found in [3, 4, 5].

Milan Tasković [8] proved significant generalization of theorem **BS** for translational slow variavility.

The function $f : [a, +\infty) \to (0, +\infty), \quad a > 0$ is rapidly varying in the sense of de Haan [7], if it is measurable and satisfies the asymptotic property

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = 0, \quad 0 < \lambda < 1. \tag{3}$$

The class of rapidly varying functions is denoted by $R_{\infty,f}$.

The sequence of positive numbers (c_n) is rapidly varying if it satisfies the asymptotic property

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = 0, \quad 0 < \lambda < 1. \tag{4}$$

The class of rapidly varying sequences is denoted by $R_{\infty,s}$.

Rapid variability in the seqential (4) and functional form (3), is rapid variability in the sense of de Haan with index $+\infty$, and in the case of monotonous and unbounded mappings it is related to the slow variability in the sense of Karamata by using generalized inverse [1].

Rapid variability given in the forms (3) and (4), as a duality to asymptotic property of slow variability given in the forms (1) and (2), is an important property in the asymptotic analysis. (see [1]).

2. Results

The following theorem represents the proposition of Bojanić–Seneta type for rapid variability given in the forms (3) and (4).

Theorem 2. *Let (c_n) be a sequence of positive numbers. Then the following statments are equivalent:*

(a) *(c_n) belongs to the class $R_{\infty,s}$;*
(b) *$f(x) = c_{[x]}, \quad x \geq 1$ belongs to the class $R_{\infty,f}$.*

Sketch of the proof.

$(a) \Rightarrow (b)$ Let $\lambda \in (0, 1)$. If $\varepsilon > 0$, then there exists an interval $[a, b] \leq (\lambda, 1)$, such that for $n \geq n_0(\varepsilon)$ and every $\alpha \in [a, b]$

$$\frac{c_{[\alpha n]}}{c_n} < \varepsilon$$

holds. Since

$$\frac{c_{[\lambda x]}}{c_{[x]}} = \frac{c_{[t[p[x]]]}}{c_{[p[x]]}} \cdot \frac{c_{[p[x]]}}{c_{[x]}},$$

where $t = t(x) \in [a, b]$ and $p = \frac{2\lambda}{a+b}$, it follows

$$\overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[x]}} \leq \varepsilon^2.$$

Thus, $f(x) = c_{[x]}, \quad x \geq 1$ belongs to the class $R_{\infty,f}$.

$(b) \Rightarrow (a)$ Trivial case.

□

This theorem provides a unique development of rapidly varying sequences theory and theory of varying functions given in the forms (3) and (4), analogous as theorem **BS** does in the theory of slow variability (see [2]).

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TECHNICAL FACULTY
SVETOG SAVE 65
32000 ČAČAK
SERBIA AND MONTENGERO
E-mail address: `dragandj@tfc.kg.ac.yu`

TECHNICAL FACULTY
SVETOG SAVE 65
32000 ČAČAK
SERBIA AND MONTENGERO
E-mail address: `zizo@tfc.kg.ac.yu`