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Some Comments on Near–P–polyagroups

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ABSTRACT. In this article several propositions of near–P–polyagroups are proved.

1. PRELIMINARIES

Definition 1.1 ([1]). Let $n \geq 2$ and let $(Q;A)$ be an n–groupoid. We say that $(Q;A)$ is a Dörnte n–group [briefly: n–group] iff is an n–semigroup and n–guasigroup as well (See, also [8]).

Definition 1.2 (Cf. [3]). Let $k > 1, s \geq 1, \quad n = k \cdot s + 1$ and let $(Q;A)$ be an n–groupoid. Then: we say that $(Q;A)$ is a **polyagroup of the type** $(s, n-1)$ iff the following statements hold:

1° For all $i, j \in \{1, \dots, n\}(i < j)$ if $i \equiv j \ (mods)$, then $< i, j >$–associative law holds in $(Q;A)$;
2° $(Q;A)$ is an n–quasigroup.

Definition 1.3 ([6]). Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let $(Q;A)$ be an n–groupoid. Then: we say that $(Q;A)$ is a **near–P–polyagroup** / briefly: NP–polyagroup/ **of the type** $(s, n-1)$ iff the following statements hold:

°1 For all $i, j \in \{1, \dots, n\}$ $(i < j)$ if $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \dots, k\}\}$, then the $< i, j >$–associative law holds in $(Q;A)$;
°2 For all $i \in \{t \cdot s + 1 | t \in \{0, 1, \dots, k\}\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds.

Remark 1.1. For $s = 1$ $(Q;A)$ is a $(k+1)$–group, where $k + 1 \geq 3$ $k > 1$.

Proposition 1.1. *Every polyagroup of the type $(s, s-1)$ is an NP–polyagroup of the type $(s, n-1)$. [By definition 1.2 and by definition refdef1.3.]*

Proposition 1.2 ([6]). *Every NP–polyagroup of the type $(s, s-1)$ has an $\{1, n\}$–neutral operation (Cf. II-2 in* [8]*).*

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2. Auxiliary proposition

Proposition 2.1 ([4])**.** *Let $n \geq 2$ and let $(Q;A)$ be an n–groupoid. Further on, let the $< 1, n >$–associative law holds in $(Q;A)$, and let for every $a_1^n \in Q$ there* ***at least one*** *$x \in Q$ and* ***at least one*** *$y \in Q$ such that the following equalities*

$$A(a_1^{n-1}, x) = a_n$$

and

$$A(y, a_1^{n-1}) = a_n$$

hold. Then there is a mapping $\mathbf{e}$ of the set Q^{n-2} into the set Q such that the following laws $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ and $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ hold in the algebra $(Q; A, \mathbf{e})$.

Remark 2.1. $\mathbf{e}$ is an $\{1, n\}$–neutral operation of the n–groupoid $(Q;A)$ [4] (Cf. Chapter II in [8]).

Proposition 2.2. *Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$, let $(Q;A)$ be an near–P–polyagroup of the type $(s, n-1)$*[1] *and $\mathbf{e}$ its $\{1, n\}$–neutral operation. Then the following laws*

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) = x$$

and

$$A(x, a_1^{s-1}, \mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, a) = x$$

hold in the algebra $(Q; A, \mathbf{e})$.

Remark 2.2. For $s = 1$ see proposition 1.1–IV in [8]. In [9] the special case, with condition

$$A\left(\overline{\overset{(j)}{x_j}, y_1^{s-1}} \Big|_{j=1}^{k}, x_{k+1} \right) = A\left(x_1, \overline{\overset{(j)}{y_1^{s-1}}, x_j} \Big|_{j=2}^{k}, \overset{(1)}{y_1^{s-1}}, x_{k+1} \right)$$

is described.

Sketch of a part of the proof.

$$F(x, a_1^{s-1}, a, c_1^{n-2-s}) \overset{\text{def}}{=} A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) \quad \Rightarrow$$

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$$

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x)) \overset{1.3}{\Rightarrow}$$

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$$

$$A(a, c_1^{n-2-s}, A(\mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) \overset{1.5}{\Rightarrow}$$

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$$

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) \overset{1.3}{\Rightarrow}$$

$$F(x, a_1^{s-1}, a, c_1^{n-2-s}) = x \overset{1.5,2.1}{\Longrightarrow}$$

[1] Polyagroup of the type $(s, n-1)$. Cf. 1.4.

$$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) =$$
$$A(x, b_1^{n-2}, \mathbf{e}(b_1^{n-2})).$$

□

Proposition 2.3. *Let $k > 1, s \geq 1, n = k \cdot s + 1$, and let $(Q; A)$ be an $n-$groupoid. Also, let the following statements hold:*

(1) *For all $i, j \in \{1, \ldots, n\}$ $(i < j)$ if $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$, then the $< i, j > -$associative law holds in $(Q; A)$;*

(2) *For every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds*
$$A(a_1^{n-2}, x) = a_n;$$

(3) *For every $a_1^n \in Q$ there is exactly one $y \in Q$ such that the following equality holds*
$$A(y, a_1^{n-2}) = a_n.$$

Then $(Q; A)$ is an near$-P-$polyagroup of the type $(s, n-1)$.

Sketch of the proof. $t \in \{1, \ldots, k-1\}$:

a) $A(a_1^{t\cdot s}, x, b_1^{(k-t)\cdot s}) = A(a_1^{t\cdot s}, y, b_1^{(k-t)\cdot s}) \Rightarrow$
$A(c_1^{(k-t)\cdot s}, A(a_1^{t\cdot s}, x, b_1^{(k-t)\cdot s}), d_1^{t\cdot s}) =$
$A(c_1^{(k-t)\cdot s}, A(a_1^{t\cdot s}, y, b_1^{(k-t)\cdot s}), d_1^{t\cdot s}) \overset{(1)}{\Rightarrow}$
$A(A(c_1^{(k-t)\cdot s}, a_1^{t\cdot s}, x), b_1^{(k-t)\cdot s}, d_1^{t\cdot s}) =$
$A(A(c_1^{(k-t)\cdot s}, a_1^{t\cdot s}, y), b_1^{(k-t)\cdot s}, d_1^{t\cdot s}) \overset{(3)}{\Rightarrow}$
$A(c_1^{(k-t)\cdot s}, a_1^{t\cdot s}, x) = A(c_1^{(k-t)\cdots}, a_1^{t\cdot s}, y) \overset{(2)}{\Rightarrow} x = y.$

b) $A(b_1^{(k-t)\cdot s}, x, a_1^{t\cdot s}) = A(b_1^{(k-t)\cdot s}, y, a_1^{t\cdot s}) \Rightarrow$
$A(d_1^{t\cdot s}, A(b_1^{(k-t)\cdot s}, x, a_1^{t\cdot s}), c_1^{(k-t)\cdot s}) =$
$A(d_1^{t\cdot s}, A(b_1^{(k-t)\cdot s}, y, a_1^{t\cdot s}), c_1^{(k-t)\cdot s}) \overset{(1)}{\Rightarrow}$
$A(d_1^{t\cdot s}, b_1^{(k-t)\cdot s}, A(x, a_1^{t\cdot s}, c_1^{(k-t)\cdot s})) =$
$A(d_1^{t\cdot s}, b_1^{(k-t)\cdot s}, A(y, a_1^{t\cdot s}, c_1^{(k-t)\cdot s})) \overset{(2)}{\Rightarrow}$
$A(x, a_1^{t\cdot s}, c_1^{(k-t)\cdot s}) = A(y, a_1^{t\cdot s}, c_1^{(k-t)\cdot s}) \overset{(3)}{\Rightarrow} x = y.$

c) $A(a_1^{t\cdot s}, x, b_1^{(k-t)\cdot s}) = c \overset{b)}{\Longleftrightarrow}$
$A(c_1^{(k-t)\cdot s}, A(a_1^{t\cdot s}, x, b_1^{(k-t)\cdot s}), d_1^{t\cdot s}) =$
$A(c_1^{(k-t)\cdot s}, c, d_1^{t\cdot s}) \overset{(1)}{\Longleftrightarrow}$
$A(A(c_1^{(k-t)\cdot s}, a_1^{t\cdot s}, x), b_1^{(k-t)\cdot s}, d_1^{t\cdot s}) =$
$A(c_1^{(k-t)\cdot s}, c, d_1^{t\cdot s}).$

□

Proposition 2.4 ([6]). *Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Also let*

(a) *The* $<1, s+1>$ *−associative [*$<(k-1)\cdot s+1, k\cdot s+1>$*-associative] law holds in the* $(Q;A)$*;*

(b) *For every* $x, y, a_1^{n-1} \in Q$ *the following implications holds*

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y$$

$$[A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y].$$

Then the statement $^\circ 1$ *from 1.3 holds.*

Remark 2.3. For $s = 1$ ([5]) see proposition 2.1-III in [8].

Proposition 2.5 ([7])**.** *Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q;A)$ *be an* $n-$*groupoid. Then the following statements are equivalent:*

(i) $(Q;A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$*;*

(ii) *There is at least one* $i \in \{t\cdot s+1 | t \in \{1, \dots, k-1\}\}$ *such that the following conditions hold:*

  (a) *the* $<i-s, i>$ *−associative law holds in* $(Q;A)$*;*

  (b) *the* $<i, i+s>$ *−associative law holds in* $(Q;A)$*;*

  (c) *for every* $a_1^n \in Q$ *there is exactly one* $x \in Q$ *such that the following equality holds* $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

3. Results

Theorem 3.1. *Let* $k > 1, s \geq 1, n = k\cdot s+1$*, let* $(Q;A)$ *be an near−*P*−polyagroup and let* $\mathbf{e}$ *be an* $(n-2)-$*ary operation in* Q*. Also, let the following laws*

(1) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,

(2) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(3) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$

hold in the algebra $(Q;A,\mathbf{e})$*. Then* $(Q;A)$ *is an near−*P*−polyagroup of the type* $(s, n-1)$*.*

Remark 3.1. For $s = 1$([5]) see proposition 2.2-IX in [8].

Proof. Firstly, we prove that under the assumptions the following statements hold:

$\overset{\circ}{1}$ For all $x, y, a_1^{n-1} \in Q$ the implication holds

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y;$$

$\overset{\circ}{2}$ Statement $^\circ 1$ from Def. 1.3 holds;

$\overset{\circ}{3}$ For all $x, y, a_1^{n-1} \in Q$ the implication holds

$$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y;$$

$\overset{\circ}{4}$ For every $a_1^n \in Q$ there is exactly one x and exactly one $y \in Q$ such that the following equalities hold

$$A(a_1^{n-1}, x) = a_n \qquad \text{and} \qquad A(y, a_1^{n-1}) = a_n.$$

Sketch of the proof of $\overset{\circ}{1}$.

$$A(x,a_1^{s-1},a,a_s^{n-2})=A(y,a_1^{s-1},a,a_s^{n-2})\Rightarrow$$
$$A(A(x,a_1^{s-1},a,a_s^{n-2}),a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))=$$
$$A(A(y,a_1^{s-1},a,a_s^{n-2}),a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))\overset{(1)}{\Longrightarrow}$$
$$A(x,a_1^{s-1},A(a,a_s^{n-2},a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1})),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))=$$
$$A(y,a_1^{s-1},A(a,a_s^{n-2},a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1})),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))\overset{(3)}{\Longrightarrow}$$
$$A(x,a_1^{s-1},a,\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))=$$
$$A(y,a_1^{s-1},a,\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))\overset{(3)}{\Longrightarrow}x=y.$$

The proof of $\overset{\circ}{2}$. By $\overset{\circ}{1}$ and by Prop. 2.4.

Sketch of the proof of $\overset{\circ}{3}$.

$$A(a_s^{n-2},a,a_1^{s-1},x)=A(a_s^{n-2},a,a_1^{s-1},y)\Rightarrow$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{n-2}),a_1^{s-1},A(a_s^{n-2},a,a_1^{s-1},x))=$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{n-2}),a_1^{s-1},A(a_s^{n-2},a,a_1^{s-1},y))\overset{\overset{\circ}{2}}{\Rightarrow}$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},A(\mathbf{e}(a_1^{n-2}),a_1^{s-1},a_s^{n-2},a),a_1^{s-1},x)=$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},A(\mathbf{e}(a_1^{n-2}),a_1^{s-1},a_s^{n-2},a),a_1^{s-1},y)\overset{(2)}{\Rightarrow}$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},a,a_1^{s-1},x)=$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},a,a_1^{s-1},y)\overset{(2)}{\Rightarrow}x=y$$

Sketch of the proof of $\overset{\circ}{4}$.

a)

$$A(x,a_1^{s-1},a,a_s^{n-2})=b\overset{\overset{\circ}{1}}{\Leftrightarrow}$$
$$A(A(x,a_1^{s-1},a,a_s^{n-2}),a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))=$$
$$A(b,a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))\overset{(1)}{\Leftrightarrow}$$
$$A(x,a_1^{s-1},A(a,a_s^{n-2},a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1})),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))=$$
$$A(b,a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a}))\overset{(3)}{\Leftrightarrow}$$
$$x=A(b,a_1^{s-1},\mathbf{e}(a_s^{n-2},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{s-1},\overset{n-2-s+1}{a})).$$

b)

$$A(a_s^{n-2},a,a_1^{s-1},x)=b \overset{\overset{\circ}{3}}{\Leftrightarrow}$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{n-2}),a_1^{s-1},A(a_s^{n-2},a,a_1^{s-1},x))=$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{n-2}),a_1^{s-1},b) \overset{\overset{\circ}{2},(2)}{\Leftrightarrow}$$
$$x=A(\mathbf{e}(\overset{n-2-s+1}{a},a_1^{s-1}),\overset{n-2-s}{a},\mathbf{e}(a_1^{n-2}),a_1^{s-1},b).$$

Finally, by $\overset{\circ}{1}-\overset{\circ}{4}$ and by Prop. 2.3, we conclude that $(Q;A)$ is an near$-P-$polyagroup of the type $(s,n-1)$. □

Similarly, one could prove also the following proposition:

Theorem 3.2. *Let $k>1,s\geq 1,n=k\cdot s+1$, let $(Q;A)$ be an $n-$groupoid and let $\mathbf{e}$ be an $(n-2)-$ary operation in Q. Also, let the following laws*

$(\overline{1})$ $A(x_1^{(k-1)\cdot s},A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}),x_{(k-1)\cdot s+n+1}^{2n-1})=A(x_1^{k\cdot s},A(x_{k\cdot s+1}^{2n-1}))$,

(2) $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$,

(3) $A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))=x$

hold in the algebra $(Q;A,\mathbf{e})$. Then $(Q;A)$ is an near$-P-$polyagroup of the type $(s,n-1)$.

Remark 3.2. For $s=1$ ([5]) see Chapter IX-2 in [8].

Corollary 3.1. *Let $k>1,s\geq 1,n=k\cdot s+1$ and let $(Q;A)$ be an $n-$groupoid. Then: $(Q;A)$ is an near$-P-$polyagroup of the type $(s,n-1)$ iff there is a mapping $\mathbf{e}$ of the set Q^{n-2} into the set Q such that the laws (1)-(3) from theorem 3.1 hold in the algebra $(Q;A,\mathbf{e})$ of the type $<n,n-2>$.*

Proof. By Def. 1.3, proposition 1.5 and by theorem 3.1. □

Corollary 3.2. *Let $k>1,s\geq 1,n=k\cdot s+1$ and let $(Q;A)$ be an $n-$groupoid. Then: $(Q;A)$ is an near$-P-$polyagroup of the type $(s,n-1)$ iff there is a mapping $\mathbf{e}$ of the set Q^{n-2} into the set Q such that the laws $(\overline{1}),(2),(3)$ from theorem 3.2 hold in the algebra $(Q;A,\mathbf{e})$ of the type $<n,n-2>$.*

Proof. By definition 1.3, proposition 1.5 and by theorem 3.2. □

Theorem 3.3. *Let $k>1,s\geq 1,n=k\cdot s+1$, let $(Q;A)$ be an $n-$groupoid and let E be an $(n-2)-$ary operation in Q. Also, let the following laws*

(o) $\mathsf{E}(c_1^{n-2-s},b,a_1^{s-1})=\mathsf{E}(a_1^{s-1},c_1^{n-2-s},b)$,

(i) $A(A(x_1^n),x_{n+1}^{2n-1})=A(x_1^s,A(x_{s+1}^{s+n}),x_{s+n+1}^{2n-1})$,

(ii) $A(x,a_1^{n-2},\mathsf{E}(a_1^{n-2}))=x$,

(iii) $A(a,c_1^{n-2-s},\mathsf{E}(a_1^{s-1},a,c_1^{n-2-s}),a_1^{s-1},x)=x$

hold in the algebra $(Q;A,\mathsf{E})$. Then $(Q;A)$ is an near$-P-$polyagroup of the type $(s,n-1)$.

Remark 3.3. For $s=1$ ([2]) see proposition 1.1-XII in [8].

Proof. Firstly, we prove that under the assumption the following statements hold:

$\overline{1}$ For all $x, y, a_1^{n-1} \in Q$ the implication holds

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y;$$

$\overline{2}$ Statement $^\circ 1$ from definition 1.3 holds;

$\overline{3}$ For all $a_1^{s-1}, a, c_1^{n-2-s} \in Q$ the following equality holds

$$a = \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1});$$

$\overline{4}$ For every $a_1^{s-1}, a, a_1^{n-2-s+1}, x, y \in Q$ the following implication holds

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = A(a, a_1^{s-1}, y, c_1^{n-2-s+1}) \Rightarrow x = y;$$

$\overline{5}$ For every $a_1^{s-1}, a, a_1^{n-2-s+1}, x, y \in Q$ the following implication holds

$$A(c_1^{n-2-s+1}, x, a_1^{s-1}, a) = A(c_1^{n-2-s+1}, y, a_1^{s-1}, a) \Rightarrow x = y;$$

$\overline{6}$ For every $x, a, a_1^{s-1}, a_1^{n-2-s+1} \in Q$

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = b \Leftrightarrow$$

$$x = A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})).$$

Sketch of the proof of $\overline{1}$. Sketch of the proof of $\overset{\circ}{1}$.
The proof of $\overline{2}$. By $\overline{1}$ and proposition 2.4.
Sketch of the proof of $\overline{3}$.

$$A(a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})) \overset{(iii)}{=}$$

$$\mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})),$$

$$A(a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})) \overset{(ii)}{=} a.$$

Sketch of the proof of $\overline{4}$.

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = A(a, a_1^{s-1}, y, c_1^{n-2-s+1}) \Rightarrow$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, x, c_1^{n-2-s+1}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, y, c_1^{n-2-s+1}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{2}}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(x, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(y, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))) \overset{(ii)}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, y) \overset{\overline{3}}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, x) =$$

$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, y) \overset{(o)}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, y) \overset{(iii)}{\Rightarrow}$$
$$x = y.$$

Sketch of the proof of $\overline{5}$.

$$A(c_1^{n-1-s}, x, a_1^{s-1}, a) = A(c_1^{n-1-s}, y, a_1^{s-1}, a) \Rightarrow$$
$$A(d_1^{2s}, A(c_1^{n-1-s}, x, a_1^{s-1}, a), d_{2s+1}^{n-1}) =$$
$$A(d_1^{2s}, A(c_1^{n-1-s}, y, a_1^{s-1}, a), d_{2s+1}^{n-1}) \overset{\overline{2}}{\Rightarrow}$$
$$A(A(d_1^{2s}, c_1^{n-2s}), c_{n-2s+1}^{n-1-s}, x, a_1^{s-1}, a, d_{2s+1}^{n-1}) =$$
$$A(A(d_1^{2s}, c_1^{n-2s}), c_{n-2s+1}^{n-1-s}, y, a_1^{s-1}, a, d_{2s+1}^{n-1}) \overset{\overline{4}}{\Rightarrow}$$
$$x = y.$$

Sketch of the proof of $\overline{6}$.

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = b \overset{\overline{5}}{\Leftrightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, x, c_1^{n-2-s+1}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{2}}{\Leftrightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(x, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(ii)}{\Leftrightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{3}}{\Leftrightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(o)}{\Leftrightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(iii)}{\Leftrightarrow}$$
$$x = A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})).$$

Finally, considering $\overline{2}, \overline{4}, \overline{6}$ and by proposition 2.5, we conclude that $(Q; A)$ is an near$-P-$polyagroup of the type $(s, n-1)$. □

Proposition 3.1. *Let $(Q; \cdot)$ be a group, let α be a mapping of the set Q^{s-1} into $Q, k > 1, s > 1$ and let $n = k \cdot s + 1$. Also, let*

$$A(x_1, \overset{(j)}{y_1^{s-1}}, \ldots, x_k, \overset{(k)}{y_1^{s-1}}, x_{k+1}) \overset{def}{=} x_1 \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdots x_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x_{k+1}$$

for all $x_1^{k+1}, \overset{(1)}{y_1^{s-1}} \ldots, \overset{(k)}{y_1^{s-1}} \in Q$. *Further on, let*

$$\mathsf{E}(\overset{(1)}{y_1^{s-1}}, b_1, \ldots, b_{k-1}, \overset{(k)}{y_1^{s-1}}) \overset{def}{=} (\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1},$$

where $^{-1}$ *is an inverse operation in* $(Q;\cdot)$. *Then the following statements hold:*

(*a*) $(Q;A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$*;*
(*b*) E *is an* $\{1,n\}-$*neutral operation of the* $(Q;A)$*;*
(*c*) *If* $(Q;\cdot)$ *commutative group, then* (o) *holds in* $(Q;A)$*;*
(*d*) *If* $(Q;\cdot)$ *is no commutative and* $(Q;\alpha)$ *is a* $(s-1)-$*quasigroup, then the condition* (o) *in* $(Q;A)$ *does not holds.*

Proof. Firstly, we observe that under the assumptions the following statements hold:

$\widehat{1}$ The $<1, s+1>-$associative law holds in the $(Q;A)$;
$\widehat{2}$ E is an $\{1,n\}-$neutral operation of the $(Q;A)$;

Sketch of the proof of $\widehat{1}$.

$$A(A(x_1, \overset{(1)}{y_1^{s-1}}, x_2, \overset{(2)}{y_1^{s-1}}, \ldots, x_k, \overset{(k)}{y_1^{s-1}}, x_{k+1}), \overset{(k+1)}{y_1^{s-1}}, x_{k+2}, \ldots, \overset{(2k)}{y_1^{s-1}}, x_{2k+1}) =$$

$$= (x_1 \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot x_2 \cdot \alpha(\overset{(2)}{y_1^{s-1}}) \cdots x_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x_{k+1}) \cdot$$

$$\alpha(\overset{(k+1)}{y_1^{s-1}}) \cdot x_{k+2} \cdot \alpha(\overset{(k+2)}{y_1^{s-1}}) \cdots \alpha(\overset{(2k)}{y_1^{s-1}}) \cdot x_{2k+1}) =$$

$$= x_1 \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot (x_2 \cdot \alpha(\overset{(2)}{y_1^{s-1}}) \cdots x_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x_{k+1} \cdot$$

$$\alpha(\overset{(k+1)}{y_1^{s-1}}) \cdot x_{k+2}) \cdot \alpha(\overset{(k+2)}{y_1^{s-1}}) \cdots \alpha(\overset{(2k)}{y_1^{s-1}}) \cdot x_{2k+1}) =$$

$$= A(x_1, \overset{(1)}{y_1^{s-1}}, A(x_2, \overset{(2)}{y_1^{s-1}}, \ldots, \overset{(k+1)}{y_1^{s-1}}, x_{k+2}), \overset{(k+2)}{y_1^{s-1}}, \ldots, \overset{(2k)}{y_1^{s-1}}, x_{2k+1}).$$

Sketch of the proof of $\widehat{2}$.

$$x \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot (\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} =$$

$$(\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x = x.$$

By $\widehat{1}$, $\widehat{2}$ and by theorem 3.1, we conclude that the statement (a) holds.

Sketch of the proof of (c).

$$(\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} = (\alpha(\overset{(k)}{y_1^{s-1}}) \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdots b_k)^{-1}.$$

Sketch of the proof of (d). By definition of no commutative group and by definition of $m-$ary quasigroup. □

Corollary 3.3. *Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Also, let E be an $(n-2)-$ary operation in Q such that the following law*

(o) $\mathsf{E}(c_1^{n-2-s}, b, a_1^{s-1}) = \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, b)$

holds in the $(n-2)-$groupoid $(Q; \mathsf{E})$. Then, $(Q; A)$ is an $NP-$polyagroup iff the laws $(i) - (iii)$ from theorem 3.5 hold in the algebra $(Q; A, \mathsf{E})$.

Remark 3.4. For $s = 1$ law (o) holds. In addition, for $s = 1$ $(Q; A)$ is a characterization of $n-$group [2]. See, also Chapter XII-1 in [8].

Proof. By proposition 2.2 and by theorem 3.5. □

Remark 3.5. Similarly, we obtain generalization the following proposition [2]: Let $(Q; A)$ be an $n-$groupoid and let $n \geq 3$. Then: $(Q; A)$ is an $n-$group iff there is a mapping E of the set Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q; A, \mathsf{E})$ *[*of the type $< n, n-2 >$*]*

$$A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1})),$$

$$A(\mathsf{E}(a_1^{n-2}), a_1^{n-2}, x) = x$$

and

$$A(x, \mathsf{E}(a_1^{n-2}), a_1^{n-2}) = x.$$

(Cf. Chapter XII-1 in [8]).

Theorem 3.4. *Let $k > 1, s > 1$[2], $n = k \cdot s + 1, (Q; A)$ be an near$-P-$polyagroup of the type $(s, n-1)$, $\mathbf{e}$ its $\{1, n\}-$neutral operation and let*

$$(\widehat{o}) \qquad A\left(\overline{x_j, \overset{(j)}{y}{}_1^{s-1}}\,\Big|_{j=1}^{k}, x_{k+1}\right) = A\left(x_1, \overline{\overset{(j)}{y}{}_1^{s-1}, x_j}\,\Big|_{j=2}^{k}, \overset{(1)}{y}{}_1^{s-1}, x_{k+1}\right)$$

for every $x_1^{k+1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$. Also, let

$$c_1^{k-1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1}$$

arbitrary sequence over Q.

$$Y \overset{def}{=} \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1},$$

and let

(a) $B_Y(x, y) \overset{def}{=} A\left(x, \overset{(1)}{y}{}_1^{s-1}, c_1, \ldots, c_{k-1}, \overset{(k)}{y}{}_1^{s-1}, y\right),$

(b) $\varphi_Y(x) \overset{def}{=} A\left(\mathbf{e}\left(\overline{\overset{(i)}{y}{}_1^{s-1}, c_i}\,\Big|_{i=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}\right), \overset{(k)}{y}{}_1^{s-1}, x, \overset{(1)}{y}{}_1^{s-1}, c_1, \ldots, \overset{(k-1)}{y}{}_1^{s-1}, c_{k-1}\right),$

[2] For $s = 1$ $(Q; A)$ is a $(k+1)-$group.

(c) $b_Y \overset{def}{=} A\left(\mathbf{e}(a_1^{n-2})^{3)}, \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y_1^{s-1}}, \dots, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right)$

for all $x, y \in Q$. Then the following statements hold:

(1) $(Q; B_y)$ *is a group;*
(2) $\varphi_Y \in Aut(Q; B_Y)$*;*
(3) $\varphi_Y(b_Y) = b_Y$*;*
(4) *For all* $x \in Q,\ B_Y(b_Y, x) = B_Y(\varphi_Y^k(x), b_Y)$*;*
(5) $A\left(x_1, \overset{(1)}{y_1^{s-1}}, \dots, x_k, \overset{(k)}{y_1 s - 1}, x_{k+1}\right) = \overset{k+1}{B_Y}(x_1, \varphi_Y(x_2), \dots, \varphi_Y^k(x_{k+1}), b_Y)$

for all $x_1^{k+1} \in Q$ and for every sequence Y over Q.

Remark 3.6. For $s = 1$ see Chapter IV-3 in [8]. Also, see Prop. 3.6.

Proof. Firstly, let

$$x \cdot y \overset{def}{=} B_Y(x, y), \qquad \varphi(x) \overset{def}{=} \varphi_Y(x), \qquad b \overset{def}{=} b_Y.$$

The proof of (1). By (a) and by Def. 1.3.
Sketch of the proof of (2).

$$\varphi(x \cdot y) = A\left(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, A\left(x, \overset{(1)}{y_1^{s-1}}, c_1, \dots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, y\right),\right.$$
$$\left. \overset{(1)}{y_1^{s-1}}, c_1, \dots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$
$$\overset{\circ 1}{=} A\left(A\left(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, x, \overset{(1)}{y_1^{s-1}}, c_1, \dots, c_{k-1}\right),\right.$$
$$\left. \overset{(k)}{y_1^{s-1}}, y, \overset{(1)}{y_1^{s-1}}, c_1, \dots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$
$$\overset{(b)}{=} A\left(\varphi(x), \overset{(k-1)}{y_1^{s-1}}, y, \overset{(1)}{y_1^{s-1}}, c_1, \dots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$
$$\overset{1.5}{=} A\left(A\left(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \dots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right),\right.$$
$$\left. \overset{(k)}{y_1^{s-1}}, y, \overset{(1)}{y_1^{s-1}}, c_1, \dots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$

[3] $a_1^{n-2} \overset{def}{=} \overset{(1)}{y_1^{s-1}}, c_1, \dots, c_{k-1}, \overset{(k)}{y_1^{s-1}}.$

$$\overset{\circ 1}{=} A\left(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, \right.$$

$$\left. A\left(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, y, \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)\right)$$

$$\overset{(b)}{=} A\left(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, \varphi(y)\right)$$

$$\overset{(a)}{=} \varphi(x) \cdot \varphi(y).$$

Sketch of the proof of (3).

$$\varphi(b) \overset{(b),(c)}{=} A\left(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, A\left(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \ldots, \overset{(k-1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right), \right.$$

$$\left. \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$

$$\overset{\circ 1}{=} A\left(A\left(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \ldots, \overset{(k-1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right), \right.$$

$$\left. \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$

$$\overset{(\widehat{o})}{=} A\left(A\left(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \ldots, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right), \right.$$

$$\left. \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}\right)$$

$$\overset{2.2}{=} A\left(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \ldots, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})\right)$$

$$\overset{(c)}{=} b.$$

Sketch of the proof of (4) [for the case $k = 3,\ s > 1$].

$$b \cdot x \overset{(a)}{=} A(b, a_1^{n-2}, x)$$

$$\overset{(c)}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})), a_1^{n-2}, x)$$

$$\overset{\circ 1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y_1^{s-1}}, A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x))$$

$$\overset{1.5}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y_1^{s-1}}, A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})))$$

$$\overset{fn3}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, A(x, \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})))$$

$$\overset{\circ 1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, x, \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})))$$

$$\overset{(b)}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, \varphi(x), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{1.3,2.2}{=} A(\overline{\mathbf{e}(a_1^{n-2}), \overset{(i)}{y}{}_1^{s-1}}\Big|_{i=1}^{2}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})\overset{(1)}{y}{}_1^{s-1}, A(\varphi(x), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})))), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{(\hat{o})}{=} A(\overline{\mathbf{e}(a_1^{n-2}), \overset{(i)}{y}{}_1^{s-1}}\Big|_{i=1}^{2}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, A(\varphi(x), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})))), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{\circ 1}{=} A(\overline{\mathbf{e}(a_1^{n-2}), \overset{(i)}{y}{}_1^{s-1}}\Big|_{i=1}^{2}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \varphi(x), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{(b)}{=} A(\overline{\mathbf{e}(a_1^{n-2}), \overset{(i)}{y}{}_1^{s-1}}\Big|_{i=1}^{2}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, \varphi(\varphi(x)), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{(\hat{o})}{=} A(\overline{\mathbf{e}(a_1^{n-2}), \overset{(i)}{y}{}_1^{s-1}}\Big|_{i=1}^{2}, A(c_1, \overset{(3)}{y}{}_1^{s-1}, c_2, \overset{(1)}{y}{}_1^{s-1}, \varphi(\varphi(x)), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{\circ 1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, c_1, \overset{(3)}{y}{}_1^{s-1}, c_2, \overset{(1)}{y}{}_1^{s-1}, \varphi(\varphi(x))), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{(\hat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(x))), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$\overset{1.5}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \varphi(\varphi(x)), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$$

$$
\begin{aligned}
&\overset{1.3,2.2}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1},\\
&\qquad A(\varphi(\varphi(x)), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(\widehat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, A(\varphi(\varphi(x)),\\
&\qquad \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{\circ 1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(x)),\\
&\qquad \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(\widehat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(x)),\\
&\qquad \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(b)}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(\varphi(x))), \overset{(1)}{y}{}_1^{s-1},\\
&\qquad\qquad\qquad \mathbf{e}(a_1^{n-2})), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{\circ 1}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(\varphi(x)))), \overset{(1)}{y}{}_1^{s-1},\\
&\qquad\qquad\qquad \mathbf{e}(a_1^{n-2})), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{1.5}{=} A(A(\varphi(\varphi(\varphi(x))), a_1^{n-2}, \mathbf{e}(a_1^{n-2})), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1},\\
&\qquad\qquad\qquad\qquad \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{\circ 1}{=} A(\varphi(\varphi(\varphi(x))), a_1^{n-2}, A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1},\\
&\qquad\qquad\qquad\qquad \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})))\\
&\overset{(a),(c)}{=} \varphi(\varphi(\varphi(x))) \cdot b.
\end{aligned}
$$

The proof of (5). By 1.5, 2.2, $^\circ 1, (\widehat{o}), (a), (b)$ and (c). Cf. sketch of the proof of (4) and Chapter IV-3 in [8]. □

Remark 3.7. $(Q; \mathbf{A})$ from [10] is a $(k+1)-$group if the condition $(\widehat{o})$ holds in $(Q; A)$.

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