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Some Results on Commutativity of Rings

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Abstract. Some new results on commutativity theorems of torsion free unital rings have been obtained.

Throughout the paper, R will denote an associative ring, and $Z(R)$ the center of R. As usual, for any $x, y, \in R$, the commutator $[x, y] = xy - yx$ and the anticommutator $xoy = xy + yx$.

A ring R is said to be commutative or anticommutative according as $[x, y] = 0$ or $x \circ y = 0$, for all $x, y \in R$.

An element $x \in R$ is said to be m-torsion free if $mx = 0$ implies $x = 0$, where m is a positive integer. It is logically interesting to investigate how far a ring is commutative or anticommutative if $[xy, yx] = 0$ or $xyoyx = 0$. Motivated by these observations, Gupta [2] proved that a division ring R is commutative if and only if $[xy, yx] = 0$. A number of authors [2, 3] have extended this result in several ways. Awtar [1] established that a semi prime ring R in which $[xy, yx] \in Z(R)$ is necessarily commutative. In the same paper the possibility of extending the result for arbitrary rings has been ruled out in view of the readily available non-commutative ring of 3×3 strictly, upper triangular matrices over the ring Z of integers which satisfies the above condition.

The following example shows that the above result is not valid for arbitrary rings even if it is unital.

Example 1. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d, \in Z \right\}.$$

Then R is a non-commutative unital ring for which $[xy, yx] \in Z(R)$ for all x, y in R.

Therefore, if one replaces Z by $(GF(p))_2$ in the above example, then R satisfies both the properties $[[xy, yx], x] = 0$ and $[xyoyx, x] = 0$, but R is not commutative.

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One can observe that the ring $(GF(p))_2$ in the above example is of characteristic 2 and some appropriate conditions on the characteristic of the ring implies commutativity.

In this note, we prove two new results on commutativity of unital rings.

Theorem 1. *Let R be an unital ring with $[xyoyx, x] = 0$ for all $x, y \in R$. If R is a 2–torsion free ring, then R is commutative.*

Proof. By the hypothesis, we have

$$x(xy^2x + yx^2y) = (xy^2x + yx^2y)x, \quad \text{for all} \quad x, y \in R.$$

So

$$x[y^2, x]x + [yx^2y, x] = 0. \tag{1}$$

Replacing x by $x + 1$ in (1), we get

$$[y^2, x] + x[y^2, x] + [y^2, x]x + x[y^2, x]x + [y^2, x] + 2[yxy, x] + [y^2xy, x] = 0. \tag{2}$$

Using (1), equation (2) becomes

$$x[y^2, x] + [y^2, x]x + 2[y^2, x] + 2[yxy, x] = 0. \tag{3}$$

Now, replacing x by $x + 1$ in (3) and using (3), one gets $4[y^2, x] = 0$.

Since R is 2–torsion free, this gives

$$[y^2, x] = 0. \tag{4}$$

Finally, replacing y by $y + 1$ in (4) and using (4), we obtain $2[y, x] = 0$.

This implies $[y, x] = 0$ and yields the required result. This completes the proof. □

Now, we shall prove the following result in a more general setting.

Theorem 2. *Let R be an unital ring for which $[xy^mx - yx^my, x] = 0$ for all x, y in R. If R is $m!$–torsion-free ring, then R is commutative.*

Proof. By our assumptions, we have

$$x(xy^mx - yx^my) = (xy^mx - yx^my)x \quad \forall x, y \in R.$$

This implies that

$$x[y^m, x]x = [yx^my, x] \tag{5}$$

Replacing x by $x + 1$ in (4), we get

$$\begin{aligned} &[y^m, x] + x[y^m, x] + [y^m, x]x + x[y^m, x]x = \\ &[y(1 +^m C_1x +^m C_2x^2 + \cdots +^m C_mx^m)y, x] \end{aligned} \tag{6}$$

Using (4) in (6), we obtain

$$[y^m, x] + x[y^m, x] + [y^m, x]x = [y(1 +^m C_1x +^m C_2x^2 + \cdots +^m C_{m-1}x^{m-1})y, x] \tag{7}$$

Replacing x by $1+x$ in (7) and combining with the result thus obtained, we get

$$
\begin{aligned}
2[y^m, x] = \Big[y\Big\{m + {}^mC_2(1+{}^2C_1x) + {}^mC_3(1+{}^3C_1x+{}^3C_2x^2) \\
+{}^mC_4(1+{}^4C_1x+{}^4C_2x^2+4C_3x^3)\cdots \\
\cdots + {}^mC_{m-1}(1+{}^{m-1}C_1x+{}^{m-1}C_2x^2+\cdots \\
\cdots + {}^{m-1}C_{m-2}x^{m-2})\Big\}y, x\Big]
\end{aligned} \tag{8}
$$

Repeating the same arguments 3rd and 4th times, one gets

$$
\begin{aligned}
0 = \Big[y\Big\{ & {}^mC_2{}^2C_1 + {}^mC_3({}^3C_1 + {}^3C_2(1+{}^2C_1x)) + \\
& + {}^mC_4\Big({}^4C_1 + {}^4C_2(1+{}^2C_1x) + {}^4C_3(1+{}^3C_1x+{}^3C_2x^2) + \cdots + \\
& + {}^mC_{m-1}\Big({}^{m-1}C_1 + {}^{m-1}C_2(1+{}^2C_1x) + {}^{m-1}C_3(1+{}^3C_1x+{}^3C_2x^2) + \cdots + \\
& + {}^{m-1}C_{m-2}(1+{}^{m-2}C_1x+{}^{m-2}C_2x^2+\cdots+{}^{m-2}C_{m-3}x^{m-3})\Big)\Big)\Big\}y, x\Big]
\end{aligned} \tag{9}
$$

This gives

$$
\begin{aligned}
0 = \Big[y\Big\{ & {}^mC_3{}^3C_2{}^2C_1 + {}^mC_4({}^4C_2{}^2C_1 + {}^4C_3({}^3C_1 + {}^3C_2(1+{}^2C_1x))) + \\
& + \cdots + {}^mC_{m-1}\Big({}^{m-1}C_2{}^2C_1 + {}^{m-1}C_3({}^3C_1 + {}^3C_2(1+{}^2C_1x)) + \cdots \\
& \cdots + \cdots + {}^{m-1}C_{m-2}\Big({}^{m-2}C_1 + {}^{m-2}C_2(1+{}^2C_1x) + \cdots \\
& \cdots + {}^{m-2}C_{m-3}\big(1+{}^{m-3}C_1x+{}^{m-3}C_2x^2 + \\
& \cdots + {}^{m-3}C_{m-4}x^{m-4})\Big)\Big)\Big\}y, x\Big]
\end{aligned} \tag{10}
$$

Hence, repeating the process of replacing x by $x+1$ m times, and using the previously obtained results at each stage, equation (10) yields

$$0 = [y({}^mC_{m-1}{}^{m-1}C_{m-2}{}^{m-2}C_{m-3}\cdots{}^2C_1)y, x].$$

This implies that

$$m![y^2, x] = 0.$$

Since R is a $m!$ torsion free ring, we obtain

$$[y^2, x] = 0. \tag{11}$$

Now as in the proof of theorem 1, equation (11) can be used to show that R is commutative. This completes the proof. □

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