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TRANSVERSAL INTERVALLY SPACES

Milan R. Tasković*

Abstract. In this paper we formulate a new structure of spaces which we call it transversal (upper or lower) intervally spaces. We introduce this concept as a natural extension of transversal probabilistic and Menger's spaces. Transversal intervally spaces are a new concept of spaces in the fixed point theory and further a new way in the nonlinear analysis. In this sense, we introduce notions of the intervally contractions on upper and lower transversal intervally spaces and prove some fixed point statements.

1. Introduction, history and main facts

Concept of transversal spaces were introduced in 1998 by Tasković [33] as a nature extension of Fréchet's, Kurepa's, and Menger's spaces in the following sense.

Let X be a nonempty set and let $P := (P, \preccurlyeq)$ be a partially ordered set. The function $\rho : X \times X \to P$ is called an **upper transverse** on X (or *upper transversal*) iff: $\rho[x, y] = \rho[y, x]$, and if there is an **upper bisection function** $g : P \times P \to P$ such that

$$\text{(A)} \qquad \rho[x, y] \preccurlyeq \sup \Big\{ \rho[x, z], \rho[z, y], g\Big(\rho[x, z], \rho[z, y]\Big) \Big\}$$

for all $x, y, z \in X$. A **transversal upper space** is a set X together with a given upper transverse on X.

Let $k = \aleph_\alpha$ $(\alpha \geq 0)$ be a regular cardinal. Call a topological space X an **upper k-transversal space** or a **$g(D_\alpha)$-space** if there exists $\rho : X \times X \to$

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$\omega_\alpha \cup \{\omega_\alpha\} := W$ such that $\rho[x,y] = \omega_\alpha$ if and only if $x = y$, $\rho[x,y] = \rho[y,x]$, and if there is $g : W \times W \to W$ such that (A) for all $x, y, z \in X$.

We notice, Fréchet's spaces are important examples of upper k-transversal spaces.

Open problem 1. *Does for every regular cardinal $k \geq \aleph_0$ there exists an upper k-transversal (i.e., an $g(D_\alpha)$-space) nonlinearly orderable topological space? Does some of the transversal upper spaces have the fixed point property?*

Let X be a nonempty set and we chosen an upper bisection function $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+ := [0, +\infty)$ defined by

$$g(s,t) = \Psi(s) + \tau t \quad (\tau \geq 1,\ \Psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+)$$

for a self-map Ψ with the property $\Psi(x) \to 0$ $(x \to 0)$, then X is an example of transversal upper space, which were introduced in 1974 by M. Cicchese [6].

In connection with the preceding, let $P := (P, \preccurlyeq)$ be a partially ordered set. The function $\rho : X \times X \to P$ is called a **lower transverse** on X (or *lower transversal*) iff: $\rho[x,y] = \rho[y,x]$ and if there is a **lower bisection function** $d : P \times P \to P$ such that

$$\text{(B)} \qquad \inf\Big\{\rho[x,z], \rho[z,y],\ d\Big(\rho[x,z], \rho[z,y]\Big)\Big\} \preccurlyeq \rho[x,y]$$

for all $x, y, z \in X$. A **lower transversal space** is a set X together with a given lower transverse on X.

Let $k = \aleph_\alpha$ $(\alpha \geq 0)$ be a regular cardinal. Call a topological space X a **lower k-transversal space** or $d(D_\alpha)$**-space** if there exists the function $\rho : X \times X \to \omega_\alpha \cup \{\omega_\alpha\} := W$ such that: $\rho[x,y] = \omega_\alpha$ if and only if $x = y$, $\rho[x,y] = \rho[y,x]$, and if there is $d : W \times W \to W$ such that (B) for all $x, y, z \in X$.

Open problem 2. *Does for every regular cardinal $k \geq \aleph_0$ there exists a lower k-transversal (i.e., an $d(D_\alpha)$-space) nonlinearly orderable topological space? Does some of the transversal lower spaces have the fixed point property?*

We notice, in connection with this problem, that work of Đ. Kurepa in 1963 is very important, where there is result that for every regular cardinal $k \geq \aleph_0$ there exists a k-metrizable (i.e., an D_α-space) nonlinearly orderable topological space. A proof of this result was exhibit by S. Todorčević in 1981.

Karl Menger initiated the study of probabilistic metric spaces in 1942. A probabilistic metric space is a space in which the "distance" between any two points is a probability distribution function. Every Menger's space is a lower transversal space (see: Tasković [33]).

The notion of distance $\rho(x,y)$ between points x and y is very old and is connected with measurments.

The possibility of defining such notions as limit and continuity in an arbitrary set is an idea which undoubtedly was first put forward by Maurice Fréchet in 1904, and developed by him in his famous doctoral dissertation 1905.

In 1934 Đuro Kurepa introduced the notion of a pseudo-metric space; and in 1936 also Đ. Kurepa introduced, for a given ordinal α, the notion of $(\triangle^\alpha)$ or (D_α) as the class of pseudo-metric spaces. The case $\alpha = 0$ coincides with the class of metric spaces.

A special feature in the former notions (of Fréchet and Kurepa) is the "triangular relation" occuring in the elementary geometry and in many other cases.

At the same time, Fréchet considered instead of triangular relation, apparently weaker, **regularity condition**: There exists a self-map f of $\mathbb{R}_+ := (0, +\infty)$ into itself such that $f(x) \to 0$ $(x \to 0)$ and that for any triple (a, b, c) of elements of X one has $\rho(a, b) < x$ and $\rho(b, c) < x$ implies $\rho(a, c) < f(x)$.

Fréchet remarked that metric spaces (X, ρ) and the preceding spaces (X, ρ, f) with the regularity condition have similar properties. In 1910 he asked whether this two classes of spaces should be the same. Chittenden in 1917 confirmed this conjecture. A simple proof was exibited by Frink in 1937.

We remarked that an important example of upper transversal spaces is also and every Fréchet's space with the regularity condition. For this an upper bisection function $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ can be defined by $g(s, t) = \max\{x, f(x)\}$.

On the other hand, let $\tau = \omega_\mu$ be a regular cardinal number, X a set, and $(G, +, \preccurlyeq)$ a linerarly ordered abelian group with cofinality cof $(G) = \omega_\mu$ at the identity element $\mathbf{0} \in G$ (which means that $\mathbf{0}$ is the infimum of a strictly decreasing τ-sequence $\{x_\alpha : \alpha \in \tau\} \subset G \backslash \{\mathbf{0}\}$). An τ-**metric** on X is a function $\rho : X \times X \to G$ which satisfies all the metric axioms (i.e., $\rho[x, y] = \mathbf{0}$ if and only if $x = y$, $\rho[x, y] = \rho[y, x]$ and $\rho[x, y] \preccurlyeq \rho[x, z] + \rho[z, y]$).

This definition of space X was given by R. Sikorski in 1950 using the name ω_μ-**metrizable topological space** (if its topology can be induced by some ω_μ-metric on X).

Call, for $k = \aleph_\alpha$ $(\alpha \geq 0)$, a topological space X a k-**metrizable space** or a D_α-**space** if there exist $\rho : X \times X \to \omega_\alpha \cup \{\omega_\alpha\}$ and $\phi : \omega_\alpha \to \omega_\alpha$ such that: $\rho(x, y) = \omega_\alpha$ if and only if $x = y$, $\rho(x, y) = \rho(y, x)$, and if $\rho(x, y) > \phi(\xi)$ and $\rho(y, z) > \phi(\xi)$ implies $\rho(x, z) > \xi$. This definition of space X was given by Đ. Kurepa in 1934.

Obviously, ω_μ-metrizable topological spaces are fundamental examples of upper transversal spaces with the upper bisection function $g : G \times G \to G$ defined by $g(s, t) := s + t$.

Also, D_α-spaces of Đ. Kurepa for $\alpha \geq 0$ are fundamental examples of lower transversal spaces with the lower bisection function $d : P \times P \to P$ defined by $d(s,t) = \inf\{\xi, \phi(\xi)\}$ for some function $\phi : \omega_\alpha \to \omega_\alpha$ and $\xi < \omega_\alpha$.

2. Transversal upper intervally spaces

A fundamental first example of upper transversal spaces for the upper bisection function $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+ := [0, \infty)$ defined by $g(s,t) = s + t$ is a *metric space*.

In connection with the preceding, the function $\rho : X \times X \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ is called an **upper (intervally) transverse** on X (or *upper intervally transversal*) iff: $\rho[x,y] = \rho[y,x]$ and if there is an **upper (intervally) bisection function** $g : [a,b] \times [a,b] \to [a,b]$ such that

$$\text{(Aa)} \qquad \rho[x,y] \leq \max\Big\{\rho[x,z], \rho[z,y], g\Big(\rho[x,z], \rho[z,y]\Big)\Big\}$$

for all $x, y, z \in X$. A **transversal upper intervally space** is a set X together with a given upper intervally transverse $\rho : X \times X \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ on X.

In further, a mapping $M : \mathbb{R} \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ is called an **upper (distribution) function** if it is nonincreasing, left-continuous with $\inf M = a$ and $\sup M = b$. We will denote by $\mathfrak{D}$ the set of all upper (distribution) functions.

Next two spaces are very interesting examples of transversal upper spaces.

First, an **upper statistical space** is a pair $(X, \mathcal{R})$, where X is an abstract set and $\mathcal{R}$ is a mapping of $X \times X$ into the set of all upper (distribution) functions $\mathfrak{D}$. We shall denote the upper (distribution) function $\mathcal{R}(u,v)$ by $M_{u,v}(x)$ or $M_{u,v}$, whence the symbol $M_{u,v}(x)$ will denote the value of $M_{u,v}$ at $x \in \mathbb{R}$. The functions $M_{u,v}$ are assumed to satisfy the following conditions: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$, and

$$\text{(Eq)} \qquad M_{u,v}(x) = a \quad \text{for } x > c \text{ if and only if } u = v,$$

and if $M_{u,r}(x) = a$ and $M_{r,v}(y) = a$ implies $M_{u,v}(x+y) = a$ for all $u, v, r \in X$ and for all $x, y \in \mathbb{R}$.

In view of the condition $M_{u,v}(c) = b$, which evidently, implies that $M_{u,v}(x) = b$ for every $x \leq c$. Thus condition (Eq) is equivalent to the statement: $u = v$ if and only if $M_{u,v}(x) = A(x)$, where $A(x) = b$ if $x \leq c$ and $A(x) = a$ if $x > c$.

Obviously, every metric space may be regarded as an upper statistical space of a special kind. One has only to set $M_{u,v}(x) = A(x - d(u,v))$ for every pair of points (u,v) in the metric space (X,d). Also, $M_{u,v}(x)$ may be interpreted as the "measure" that the distance between u and v is less than x.

Second example of transversal upper space, an **upper intervally space** (or Tasković's intervally space from [33]) is a nonempty set X together with the functions $M_{u,v}(x)$ with the following properties: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$, (Eq), and if there is a nondecreasing function $f : [a,b]\times[a,b] \to [a,b]$ with the property $f(t,t) \leq t$ for all $t \in [a,b]$ such that

$$\text{(Nt)} \qquad M_{u,v}(x+y) \leq f\Big(M_{u,r}(x), M_{r,v}(y)\Big)$$

for all $u, v, r \in X$ and for all $x, y \geq c$. (Namely, the function $f : [a,b] \times [a,b] \to [a,b]$ is **nondecreasing** if $a_i, b_i \in [a,b]$ and $a_i \leq b_i$ $(i = 1,2)$ implies $f(a_1,a_2) \leq f(b_1,b_2)$.).

We notice, if we chosen an upper bisection (intervally) function $g : [a,b] \times [a,b] \to [a,b]$ such that $g = f$ (from (Nt)), then we immediate obtain that every upper intervally space, for $\rho[u,v] = M_{u,v}$, is a transversal upper intervally space; because in this case from (Nt) the following inequalities hold:

$$\rho[u,v] = M_{u,v}(x) \leq f\Big(M_{u,r}(x-y), M_{r,v}(y)\Big) :=$$
$$:= g\Big(\rho[u,r], \rho[r,v]\Big) \leq \max\Big\{\rho[u,r], \rho[r,v], g(\rho[u,r], \rho[r,v])\Big\}.$$

On the other hand, if: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$, (Eq), and if there is a function $\Psi : [a,b] \times [a,b] \to [a,b]$ such that

$$M_{u,v}(x) \leq \Psi\Big(M_{u,r}(x), M_{r,v}(x)\Big)$$

for all $u, v, r \in X$ and for every $x \geq c$, then it is an example of transversal upper intervally space also.

A mapping $\triangle : [0,1]\times[0,1] \to [0,1]$ is a **$\triangle$-norm** if it satisfies: $\triangle(a,1) = a$, $\triangle(0,0) = 0$, $\triangle(a,b) = \triangle(b,a)$, $\triangle(c,d) \geq \triangle(a,b)$ for $c \geq a$, $d \geq b$ and $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Let $\mathcal{B}$ denote the set of all $\triangle$-norms, partially ordered by $\triangle_1 \leq \triangle_2$ if and only if $\triangle_1(a,b) \leq \triangle_2(a,b)$ for all $a, b \in [0,1]$ and $\triangle_1, \triangle_2 \in \mathcal{B}$.

In connection with this, an **upper probabilistic MT-space** is a triplet $(X, \mathcal{R}, \triangle)$, where $(X, \mathcal{R})$ is an upper statistical space and $f \in \mathcal{B}$ satisfies the preceding triangle inequality (Nt).

A very characteristic example, *for further work*, of the transversal upper intervally spaces is the following space in the following form.

A **transversal upper intervally T-space** is a pair (X, ρ), where X is a transversal upper intervally space and where the upper (intervally) transverse $\rho[u,v] = M_{u,v}(x)$ satisfying: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$, and (Eq).

In further, the concept of a neighborhood can be introduced and defined with the aid of the upper intervally transverse. In fact, neighborhoods in transversal upper intervally spaces may be defined in several nonequivalent ways. Here we shall consider only one of these.

If $p \in X$, $\mu > c$ for some $c \in \mathbb{R}$ and r a positive real, then an (μ, r)-**neighborhood** of p, denoted by $U_p(\mu, r)$, is defined by

$$U_p(\mu, r) = \Big\{ q \in X : \rho[p, q] = M_{p,q}(\mu) < a + r \Big\}.$$

Lemma 1. *Let (X, ρ) be a transversal upper intervally space, where the upper transverse $\rho[p, q] = M_{p,q}(x)$. If $\varepsilon_1 \leq \varepsilon_2$ and $r_1 \leq r_2$, then $U_p(\varepsilon_1, r_1) \subset U_p(\varepsilon_2, r_2)$.*

Proof. Suppose $q \in U_p(\varepsilon_1, r_1)$ so that $M_{p,q}(\varepsilon_1) < a+r_1$. Then $M_{p,q}(\varepsilon_2) \leq M_{p,q}(\varepsilon_1) < a + r_1 < a + r_2$, whence, by definition, $q \in U_p(\varepsilon_2, r_2)$. The proof is complete.

Lemma 2. *Let (X, ρ) be a transversal upper intervally T-space, where the upper transverse $\rho[u, v] = M_{u,v}(x)$ and the upper bisection function $g : [a, b] \times [a, b] \to [a, b]$ is nondecreasing such that $g(t, t) \leq t$ for all $t \in [a, b]$. Then (X, ρ) is a Hausdorff space in topology induced by the familly*

$$\{U_p(\mu, r) : p \in X, \ \ \mu > c \text{ for some } c \in \mathbb{R} \text{ and } r > 0\}$$

of neighborhoods.

The proof of this statement is a totally analogous with the corresponding proofs in general topology; and thus we omit it.

From the preceding facts, the above topology satisfies the first axiom of countability. In this topology a sequence $\{p_n\}_{n\in\mathbb{N}}$ in X **converges to a point** $p \in X$ (in notation $p_n \to p$) if for some $c \in \mathbb{R}$ and for every $\mu > c$ and every $\sigma > 0$, there exists an integer $\mathcal{M}(\mu, \sigma)$ such that $p_n \in U_p(\mu, \sigma)$, i.e., $\rho[p, p_n] = M_{p,p_n}(\mu) < a + \sigma$ whenever $n \geq \mathcal{M}(\mu, \sigma)$.

Lemma 3. *Let (X, ρ) be a transversal upper intervally T-space, where the upper transverse $\rho[u, v] = M_{u,v}(x)$. If $p_n \to p$, then $M_{p,p_n} \to M_{p,p} = A$, i.e., for every $x \in \mathbb{R}$, $M_{p,p_n}(x) \to M_{p,p}(x) = A(x)$, and conversely.*

Proof. If $x > c$, then for every $\varepsilon > 0$, there exists an integer $\mathcal{M}(x, \varepsilon)$ such that $\mathcal{M}_{p,p_n}(x) < a + \varepsilon$ whenever $n \geq M(x, \varepsilon)$. But this means that $\lim_{n\to\infty} M_{p,p_n}(x) = a = M_{p,p}(x)$. If $x = c$, then for every n, $M_{p,p_n}(c) = b$ and hence $\lim_{n\to\infty} M_{p,p_n}(c) = b = M_{p,p}(c)$. The converse is immediate. The proof is complete.

Lemma 4. *Let (X, ρ) be a transversal upper intervally T-space, where the upper transverse $\rho[u, v] = M_{u,v}(x)$ and the upper bisection function $g : [a, b] \times [a, b] \to [a, b]$ is nondecreasing such that $g(t, t) \le t$ for all $t \in [a, b]$. Then the upper (distribution) function $\mathcal{R}$ is a lower semicontinuous function of points, i.e., for every fixed x, if $p_n \to p$ and $q_n \to q$, then*

$$\liminf_{n\to\infty} M_{p_n,q_n}(x) = M_{p,q}(x).$$

The proof of this statement is a totally analogous with the corresponding proofs some statements in analysis and general topology; and thus we omit it.

In connection with the preceding, the sequence $\{p_n\}_{n\in\mathbb{N}}$ will be called **fundamental** in X if for some $c \in \mathbb{R}$ and for each $\mu > c$, $\sigma > 0$ there is an integer $\mathcal{M}(\mu, \sigma)$ such that $\rho[p_n, p_m] = M_{p_n,p_m}(\mu) < a + \sigma$ whenever $n, m \ge \mathcal{M}(\mu, \sigma)$. In analogy with the completion concept of metric space, a transversal upper intervally space X will be called **complete** if each fundamental sequence in X converges to an element in X.

In further we introduce a notion of an intervally upper contraction on a transversal upper intervally space and prove a fixed point statement.

A mapping T of a transversal upper intervally space (X, ρ) into itself for $\rho[u, v] = M_{u,v}(x)$ will be called an **intervally upper contraction** iff there exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some $c \in \mathbb{R}$ such that

$$\text{(As)} \qquad \lim_{n\to\infty} \varphi^n(t) = +\infty \quad \text{for every } t > c$$

and such that

$$M_{Tu,Tv}(x) \le$$
$$\le \max\Big\{M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{u,Tv}(\varphi(x)), M_{v,Tu}(\varphi(x))\Big\}$$

for all $u, v \in X$ and for every $x > c$.

Theorem 2.1. *Let (X, ρ) be a complete transversal upper intervally T-space, where the upper transverse $\rho[u, v] = M_{u,v}(x)$ and the upper bisection function $g : [a, b] \times [a, b] \to [a, b]$ is nondecreasing such that $g(t, t) \le t$ for all $t \in [a, b]$. If T is any intervally upper contraction mapping of X into itself, then there is a unique point $p \in X$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in X$.*

Proof. For this proof the following inequalities are essential. Namely, from the conditions for the function $g : [a, b] \times [a, b] \to [a, b]$ we obtain the following inequalities

$$\text{(1)} \qquad g(s, t) \le g\Big(\max\{s, t\}, \max\{s, t\}\Big) \le \max\{s, t\}$$

for all $s, t \in [a, b]$. On the other hand, since X is a transversal upper intervally space, for some $c \in \mathbb{R}$ and for every $x \geq c$ we have from (1) the following inequalities

$$M_{u,v}(x) \leq \max\Big\{M_{u,r}(x), M_{r,v}(x), g(M_{u,r}(x), M_{r,v}(x))\Big\} \leq$$

$$\leq \cdots \leq \max\Big\{M_{u,r}(x), M_{r,v}(x)\Big\}. \tag{2}$$

To prove the existence of the fixed point, consider an arbitrary $u \in X$, and define $u_n = T^n(u)$, for $n \in \mathbb{N} \cup \{0\}$. We show that the sequence $\{u_n\}_{n \in \mathbb{N} \cup \{0\}}$ is fundamental in X. Then for $t > c$ and $m > n$ $(m, n \in \mathbb{N})$ from (2) is

$$M_{u_n,u_m}(t) \leq \max\Big\{M_{u_n,u_{n+1}}(t), \ldots , M_{u_{m-1},u_m}(t)\Big\}. \tag{3}$$

On the other hand, since T is an intervally upper contraction mapping, from (2) we obtain for every $t > c$ the following inequalities

$$M_{u_n,u_{n+1}}(t) = M_{Tu_{n-1},Tu_n}(t) \leq$$

$$\leq \max\Big\{M_{u_{n-1},u_n}(\varphi(t)), M_{u_n,Tu_n}(\varphi(t)), M_{u_n,Tu_{n-1}}(\varphi(t)), M_{u_{n-1},Tu_n}(\varphi(t))\Big\} =$$

$$= \max\Big\{M_{u_{n-1},u_n}(\varphi(t)), M_{u_n,u_{n+1}}(\varphi(t)), M_{u_n,u_n}(\varphi(t)), M_{u_{n-1},u_{n+1}}(\varphi(t))\Big\} \leq$$

$$\leq \max\Big\{M_{u_{n-1},u_n}(\varphi(t)), M_{u_n,u_{n+1}}(\varphi(t)), M_{u_{n-1},u_{n+1}}(\varphi(t))\Big\} \leq \tag{4}$$

$$\leq \max\Big\{M_{u_{n-1},u_n}(\varphi(t)), M_{u_n,u_{n+1}}(\varphi(t))\Big\}$$

and thus

$$M_{u_n,u_{n+1}}(\varphi(t)) \leq \max\Big\{M_{u_{n-1},u_n}(\varphi^2(t)), M_{u_n,u_{n+1}}(\varphi^2(t))\Big\}. \tag{5}$$

From (4) and (5) it follows by induction that for every integer $k \in \mathbb{N}$ the following inequality holds

$$M_{u_n,u_{n+1}}(t) \leq \max\Big\{M_{u_{n-1},u_n}(\varphi(t)), M_{u_n,u_{n+1}}(\varphi^k(t))\Big\},$$

that is, when $k \to +\infty$, we obtain $M_{u_n,u_{n+1}}(t) \leq M_{u_{n-1},u_n}(\varphi(t))$ for every $n \in \mathbb{N}$, i. e.,

$$M_{u_n,u_{n+1}}(t) \leq M_{u_0,u_1}(\varphi^n(t))$$

for every $n \in \mathbb{N}$. Hence, from the former inequality (3), we obtain

$$M_{u_n,u_m}(t) \leq \max\Big\{M_{u_0,u_1}(\varphi^n(t)), \ldots , M_{u_0,u_1}(\varphi^{m-1}(t))\Big\},$$

that is, $M_{u_n,u_m}(t) \leq M_{u_0,u_1}(\varphi^n(t))$. Hence, $\{u_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a fundemental sequence in X. Since X is a complete space, there is an $p \in X$ such that $u_n \to p$, that is $T^n(u) \to p$.

In further, we observe that by the condition (Eq) fact $u \neq v$ implies

(6) $$M_{u,v}(\varphi(x)) < M_{u,v}(x) \text{ for some } x > c.$$

On the other hand, since T is an intervally upper contraction, from the former facts, we have

(7) $$M_{T(u_n),T(p)}(t) \leq$$
$$\leq \max\left\{M_{u_n,p}(\varphi(t)), M_{p,Tp}(\varphi(t)), M_{u_n,T(p)}(\varphi(t)), M_{p,T(u_n)}(\varphi(t))\right\}$$

for every $t > c$ and for every $n \in \mathbb{N}$. Thus, since $T(u_n) = u_{n+1}$ for $n \to \infty$ from (7) we obtain

$$M_{p,Tp}(t) \leq M_{p,Tp}(\varphi(t))$$

for every $t > c$. This means, from (6), that is $Tp = p$.

We further prove the uniqueness. Suppose $p \neq q$ and $Tp = p$, $Tq = q$. Then, there exists an $x > c$ and an $a < r \leq b$, such that $M_{p,q}(x) = r$. However, since T is an intervally upper contraction mapping, for each $n \in \mathbb{N}$ we have

$$M_{p,q}(x) = M_{Tp,Tq}(x) \leq$$
$$\leq \max\left\{M_{p,q}(\varphi(x)), M_{p,p}(\varphi(x)), M_{q,q}(\varphi(x)), M_{p,q}(\varphi(x)), M_{p,q}(\varphi(x))\right\}$$

and thus $M_{p,q}(x) \leq M_{p,q}(\varphi(x))$, i.e., by induction,

$$r = M_{p,q}(x) \leq M_{p,q}(\varphi(x)) \leq \cdots \leq M_{p,q}(\varphi^n(x));$$

and hence, since $M_{p,q}(\varphi^n(x)) \to a$ as $n \to \infty$, it follows that $r = a$, i.e., $p = q$. This contradicts the choice of $a < r \leq b$, and therefore, the fixed point is unique. The proof is complete.

3. Further consequences

In connection with the preceding statement, from our the Principle of Symmetry (see: *Math. Japonica*, **35** (1990), p. 661), we obtain as an immediate consequence of Theorem 1 the following result.

Theorem 3.2. *Let (X, ρ) be a complete transversal upper intervally T-space, where the upper transverse $\rho[u, v] = M_{u,v}(x)$ and the upper bisection function $g : [a, b] \times [a, b] \to [a, b]$ is nondecreasing such that $g(t, t) \leq t$ for all $t \in [a, b]$. If there exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some $c \in \mathbb{R}$ such that (As) and if for each $u \in X$ there is a positive integer $n = n(u)$ such that*

$$M_{T^n(u),T^n(v)}(x) \leq$$
$$\leq \max\left\{M_{u,v}(\varphi(x)), M_{u,T^n u}(\varphi(x)), M_{v,T^n v}(\varphi(x)), M_{u,T^n v}(\varphi(x)), M_{v,T^n u}(\varphi(x))\right\}$$

for every $v \in X$ and for every $x > c$, then T has exactly one fixed point $p \in X$ and $T^n q \to p$ for every $q \in X$.

On the other hand, from the preceding facts, since every upper intervally space is a transversal upper intervally T-space, hence Theorems 1 and 2 hold and for upper intervally spaces; similar, and for upper statistical spaces.

Also, as immediate consequences of the preceding Theorem 1 we obtain directly the following interesting cases of upper intervally contractive mappings:

(M) There exists a constant $0 < k < 1$ such that for each $p, q \in X$, for some $c \in \mathbb{R}$ and for every $x > c$ the following inequality holds

$$M_{Tp,Tq}(x) \leq M_{p,q}\left(\frac{x}{k}\right).$$

(N) There exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some $c \in \mathbb{R}$ with the property (As) such that for each $u, v \in X$ and for every $x > c$ the following inequality holds

$$M_{Tu,Tv}(x) \leq M_{u,v}(\varphi(x)).$$

(R) There exists a constant $0 < k < 1$ such that for each $u, v \in X$, for some $c \in \mathbb{R}$ and for every $x > c$ the following inequality holds

$$M_{Tu,Tv}(kx) \leq \max\Big\{M_{u,v}(x), M_{u,Tu}(x), M_{v,Tv}(x), M_{u,Tv}(x), M_{v,Tu}(x)\Big\}.$$

4. Transversal lower intervally spaces

In connection with the preceding, the function $\rho : X \times X \to [a, b] \subset \mathbb{R}_+^0$ for $a < b$ is called a **lower (intervally) transverse** on X (or *lower intervally transversal*) iff: $\rho[x, y] = \rho[y, x]$ and if there is a **lower (intervally) bisection function** $d : [a, b] \times [a, b] \to [a, b]$ such that

$$\text{(Am)} \qquad \rho[x, y] \geq \min\Big\{\rho[x, z], \rho[z, y], d\Big(\rho[x, z], \rho[z, y]\Big)\Big\}$$

for all $x, y, z \in X$. A **transversal lower intervally space** is a set X together with a given lower intervally transverse $\rho : X \times X \to [a, b] \subset \mathbb{R}_+^0$ for $a < b$ on X.

Otherwise, a **transversal intervally space** is an upper and a lower transversal intervally space simultaneous.

As an important example of transversal lower intervally spaces we have a Menger's (probabilistic) space. Karl Menger introduced in 1942 the notion of probabilistic metric space.

In this sense, a mapping $N : \mathbb{R} \to [a, b] \subset \mathbb{R}_+^0$ for $a < b$ is called a **lower (distribution) function** if it is nondecreasing, left-continuous with $\inf N = a$ and $\sup N = b$. We will denote by $\mathfrak{L}$ the set of all lower (distribution) functions.

A **lower statistical space** is a pair $(X, \mathcal{F})$, where X is an abstract set and $\mathcal{F}$ is a mapping of $X \times X$ into the set of all lower (distribution) functions $\mathcal{L}$. We shall denote the lower (distribution) function $\mathcal{F}(p,q)$ by $N_{p,q}(x)$ or $N_{p,q}$, whence the symbol $N_{p,q}(x)$ will denote the value of $N_{p,q}$ at $x \in \mathbb{R}$. The functions $N_{p,q}$ are assumed to satisfy the following conditions: $N_{p,q} = N_{q,p}, N_{p,q}(c) = a$ for some $c \in \mathbb{R}$, and

(Em) $$N_{p,q}(x) = b \text{ for } x > c \text{ if and only if } p = q,$$

and if $N_{p,q}(x) = b$ and $N_{q,r}(y) = b$ implies $N_{p,r}(x+y) = b$ for all $p, q, r \in X$ and for all $x, y \in \mathbb{R}$.

In view of the condition $N_{p,q}(c) = a$ for some $c \in \mathbb{R}$, which evidently, implies that $N_{p,q}(x) = a$ for all $x \leq c$. Thus the condition (Em) is equivalent to the statement: $p = q$ if and only if $N_{p,q}(x) = H(x)$, where $H(x) = a$ if $x \leq c$ and $H(x) = b$ if $x > c$.

Every metric space may be regarded as a statistical lower space of a special kind. One has only to set $N_{p,q}(x) = H(x - \rho(p,q))$ for every pair of points (p,q) in the metric space (X, ρ).

An example of transversal lower intervally space is a **lower intervally space** which is a nonempty set X together with the functions $N_{p,q}(x)$ with the following properties: $N_{p,q} = N_{q,p}, N_{p,q}(c) = a$ for some $c \in \mathbb{R}$, (Em), and if there is a nondecreasing function $\tau : [a,b] \times [a,b] \to [a,b]$ with the property $\tau(t,t) \geq t$ for all $t \in [a,b]$ such that

(Nm) $$N_{p,q}(x+y) \geq \tau(N_{p,r}(x), N_{r,q}(y))$$

for all $p, q, r \in X$ and for all $x, y \geq c$.

We notice, if we chosen a lower (intervally) bisection function $d : [a,b] \times [a,b] \to [a,b]$ such that $d = \tau$ (from (Nm)), then we immediate obtain that every lower intervally space, for $\rho[p,q] = N_{p,q}$, is a transversal lower intervally space; because in this case from (Nm) the following inequalities hold:

$$\rho[p,q] = N_{p,q}(x) \geq \tau\Big(N_{p,r}(x-y), N_{r,q}(y)\Big) :=$$

(8) $$:= d\Big(\rho[p,r], \rho[r,q]\Big) \geq \min\Big\{\rho[p,r], \rho[r,q], d(\rho[p,r], \rho[r,q])\Big\}.$$

In connection with the preceding, a **transversal lower intervally T-space** is a pair (X, ρ), where X is a transversal lower intervally space and where the lower (intervally) transverse $\rho[u,v] = N_{u,v}(x)$ satisfying: $N_{u,v} = N_{v,u}$, $N_{u,v}(c) = a$ for some $c \in \mathbb{R}$ and (Em).

This space is a very characteristic example of transversal lower intervally spaces for further work.

A **Menger space (or lower probabilistic MT-space)** is a triplet $(X, \mathcal{F}, \triangle)$, where $(X, \mathcal{F})$ is a lower statistical space where $\rho[u,v] = F_{u,v}(x) : X \times X \to [0,1]$ and $\tau \in \mathcal{B}$ satisfies the preceding triangle inequality (Nm).

If we chosen a lower bisection function $d : [0,1]\times[0,1] \to [0,1]$ such that $d = \tau$ (for $\tau \in \mathcal{B}$) then from (8) we immediate obtain that every Menger's space, for $\rho[p,q] = F_{p,q}$, is a transversal lower intervally space. Every Menger's space is and a lower intervally space, also.

The concept of a neighborhood in a lower transversal intervally space X for the lower intervally transverse $\rho[p,q] = N_{p,q}(x)$ in $[a,b] \subset \mathbb{R}^0_+$ for $a < b$ is the following. If $p \in X$, $\mu > c$ for some $c \in \mathbb{R}$, and σ a positive real, then an (μ,σ)-**neighborhood** of p denoted by $\mathcal{O}_p(\mu,\sigma)$, is defined by

$$\mathcal{O}_p(\mu,\sigma) = \Big\{q \in X : \rho[p,q] = N_{p,q}(\mu) > b - \sigma\Big\}.$$

Lemma 5. *Let (X,ρ) be a transversal lower intervally T-space, where the upper transverse $\rho[p,q] = N_{p,q}(x)$. If $\varepsilon_1 \le \varepsilon_2$ and $r_1 \le r_2$, then $\mathcal{O}_p(\varepsilon_1,r_1) \subset \mathcal{O}_p(\varepsilon_2,r_2)$.*

Proof. Supposse $q \in \mathcal{O}_p(\varepsilon_1,r_1)$ so that $N_{p,q}(\varepsilon_1) > b - r_1$. Then $N_{p,q}(\varepsilon_2) \ge N_{p,q}(\varepsilon_1) > b - r_1 \ge b - r_2$, whence, by definition, $q \in \mathcal{O}_p(\varepsilon_2,r_2)$. The proof is complete.

Lemma 6. *Let (X,ρ) be a transversal lower intervally T-space, where the lower transverse $\rho[u,v] = N_{u,v}(x)$ and the lower bisection function $d : [a,b]\times[a,b] \to [a,b]$ is nondecreasing such that $d(t,t) \ge t$ for all $t \in [a,b]$. Then (X,ρ) is a Hausdorff space in topology induced by the familly*

$$\Big\{\mathcal{O}_p(\mu,r) :\ p \in X \text{ and } \mu > c \text{ for some } c \in \mathbb{R} \text{ and } r > 0\Big\}$$

of neighborhoods.

The proof of this statement is a totally analogous with the coresponding proof of Lemma 2; and thus we omit it.

From the preceding facts, the above topology satisfies the first axiom of countability. In this topology a sequence $\{p_n\}_{n\in\mathbb{N}}$ in X **converges to a point** $p \in X$ (in notation $p_n \to p$) for some $c \in \mathbb{R}$ and for every $\mu > c$ and every $\sigma > 0$, there exists an integer $\mathcal{M}(\mu,\sigma)$ such that $p_n \in \mathcal{O}_p(\mu,\sigma)$, i.e., $\rho[p,p_n] = N_{p,p_n}(\mu) > b - \sigma$ whenever $n \ge \mathcal{M}(\mu,\sigma)$.

Lemma 7. *Let (X,ρ) be a transversal lower intervally T-space, where the lower transverse $\rho[u,v] = N_{u,v}(x)$. If $p_n \to p$, then $N_{p,p_n} \to N_{p,p} = H$, i.e., for some $c \in \mathbb{R}$ and every $x > c$, $N_{p,p_n}(x) \to N_{p,p}(x) = H(x)$, and conversely.*

Proof. If $x > c$, then for every $\varepsilon > 0$ there exists an integer $\mathcal{M}(x,\varepsilon)$ such that $N_{p,p_n}(x) > b - \varepsilon$ whenever $n \ge \mathcal{M}(x,\varepsilon)$. But this means that $\lim_{n\to\infty} N_{p,p_n}(x) = b = N_{p,p}(x)$. If $x = c$, then for every n, $N_{p,p_n}(c) = a$ and hence $\lim_{n\to\infty} N_{p,p_n}(c) = a = N_{p,p}(c)$. The converse is immediate. The proof is complete.

Lemma 8. *Let (X,ρ) be a transversal lower intervally T-space, where the lower transverse $\rho[u,v]=N_{u,v}(x)$ and the lower bisection function $g:[a,b]\times[a,b]\to[a,b]$ is nondecreasing such that $d(t,t)\geq t$ for all $t\in[a,b]$. Then the lower (distribution) function $\mathfrak{F}$ is a lower semicontinuous function of points, i.e., for every fixed x, if $p_n\to p$ and $q_n\to q$, then*

$$\liminf_{n\to\infty} N_{p_n,q_n}(x)=N_{p,q}(x).$$

The proof of this statement is a totally analogous with the proof of Lemma 4.

In connection with the preceding, the sequence $\{p_n\}_{n\in\mathbb{N}}$ will be called **fundamental** in X if for some $c\in\mathbb{R}$ and for each $\mu>c$, $\sigma>0$ there is an integer $\mathcal{M}(\mu,\sigma)$ such that $\rho[p_n,p_m]=N_{p_n,p_m}(\mu)>b-\sigma$ whenever $n,m\geq\mathcal{M}(\mu,\sigma)$. In analogy with the completion concept of metric space, a transversal lower intervally space X will be called **complete** if each fundamental sequence in X converges to an element in X.

In [34] we introduced a notion of a lower probabilistic contraction on a transversal lower probabilistic space and proved fixed point theorems which are extensions some former results.

In this paper, a mapping T of a transversal lower intervally space (X,ρ) into iteself, for $\rho[u,v]=N_{u,v}(x)$, will be called a **lower intervally contraction** iff there exists a nondecreasing function $\varphi:[c,+\infty)\to[c,+\infty)$ for some $c\in\mathbb{R}$ such that (As) and

$$N_{Tu,Tv}(x)\geq$$
$$\geq\min\Big\{N_{u,v}(\varphi(x)),N_{u,Tu}(\varphi(x)),N_{v,Tv}(\varphi(x)),N_{u,Tv}(\varphi(x)),N_{v,Tu}(\varphi(x))\Big\}$$

for all $u,v\in X$ and for all $x>c$.

Theorem 4.3. *Let (X,ρ) be a complete transversal lower intervally T-space, where the lower transverse $\rho[u,v]=N_{u,v}(x)$ and the lower bisection function $d:[a,b]\times[a,b]\to[a,b]$ is nondecreasing such that $d(t,t)\geq t$ for all $t\in[a,b]$. If T is any lower intervally contraction mapping of X into itself, then there is a unique point $p\in X$ such that $Tp=p$. Moreover, $T^nq\to p$ for each $q\in X$.*

The proof of this statement is a totally analogous with the proof of Theorem 1.

A brief proof of this statement based on the preceding facts, is special case for transversal lower probabilistic T-spaces, may be found in Tasković [34].

As immediate consequences of the preceding Theorem 3 we obtain directly the following interesting cases of lower intervally contractive mappings:

(M') There exists a constant $0 < k < 1$ such that for each $p, q \in X$, for some $c \in \mathbb{R}$ and for every $x > c$ the following inequality holds

$$N_{Tp,Tq}(x) \geq N_{p,q}\left(\frac{x}{k}\right).$$

(N') There exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some $c \in \mathbb{R}$ with the property (As) such that for each $u, v \in X$ and for every $x > c$ the following inequality holds

$$N_{Tu,Tv}(x) \geq N_{u,v}(\varphi(x)).$$

(R') There exists a constant $0 < k < 1$ such that for each $u, v \in X$, for some $c \in \mathbb{R}$ and for every $x > c$ the following inequality holds

$$N_{Tu,Tv}(kx) \geq \min\Big\{N_{u,v}(x), N_{u,Tu}(x), N_{v,Tv}(x), N_{u,Tv}(x), N_{v,Tu}(x)\Big\}.$$

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Matematički fakultet
11000 Beograd, P.O. Box 550
Serbia & Montenegro

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