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THE NUMERICAL FUNCTION OF A $*$–REGULARLY VARYING SEQUENCE

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Abstract. In this paper, we impose some conditions under which there is a close relation between the asymptotic behaviour of a $*$–regularly varying sequence and the asymptotic behaviour of its numerical function $\delta_c(x)$, $x > 0$.

1. Introduction and results

A sequence of positive numbers (c_n) is called *O–regularly varying* [2], if we have

$$\overline{k}_c(\lambda) = \overline{\lim}_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} < +\infty, \quad \lambda > 0. \tag{1}$$

The class of all O–regularly varying sequences is denoted ORV.

An O–regularly varying sequence (c_n) is called $*$–regularly varying [6], if it is nondecreasing, and if

$$\lim_{\lambda\to 1+} \overline{k}_c(\lambda) = 1. \tag{2}$$

The class of all $*$–regularly varying sequences is denoted $*$RV.

The above two classes of sequences represent the important objects in the sequential theory of regular variability in the Karamata sense [1], and in particular in the theory of statements of Tauberian type [4], as well as in some other parts of qualitative analysis of divergent processes [7].

The class K_c^* [5], consists of all $*$–regularly varying sequences which satisfy the condition

$$\underline{k}_c(\lambda) = \underline{\lim}_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} > 1, \quad \lambda > 1. \tag{3}$$

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Notice that, in particular, the class K_c^* contains all nondecreasing regularly varying sequences in the Karamata sense [1] whose index $\rho > 0$, and also all sequences whose general term is the n-th ($n \in N$) partial sum of a $*$–regularly varying sequence, but does not contain slowly varying sequences in the Karamata sense [1].

If next, (c_n) is an increasing sequence of positive numbers, then its numerical function $\delta_c(x)$, $x > 0$, is defined by $\delta_c(x) = \sum\limits_{c_n \le x} 1$, $x > 0$.

We shall prove several statements about the mentioned classes.

By $\asymp$ we shall denote the weak, while by $\sim$ the strong asymptotic equivalence.

Theorem 1. *Let (c_n) be an increasing sequence from the class K_c^* and assume that $g\colon [1, +\infty) \mapsto (0, +\infty)$ is a continuous and increasing function. Then we have*

$$c_n \sim g(n), \quad n \to \infty, \tag{4}$$

if and only if

$$\delta_c(x) \sim g^{-1}(x), \quad x \to +\infty. \tag{5}$$

Notice that if (c_n) is an arbitrary increasing sequence of positive number which is not in the class K_c^*, it is easy to construct a contionuous and increasing function $g\colon [1, +\infty) \mapsto (0, +\infty)$, so that (4) is true but not (5) or, (5) is true but not (4).

Corollary 1. *Let (c_n) be an increasing sequence from the class K_c^*, and (d_n) be an increasing sequence of positive numbers. Then we have*

$$c_n \sim d_n, \quad n \to \infty \tag{4'}$$

if and only if

$$\delta_c(x) \sim \delta_d(x), \quad x \to \infty. \tag{5'}$$

Corollary 1 follows easily from the theorem above.

Corollary 2. *Let (c_n) be an increasing sequence from the class K_c^* and let $g\colon [1, +\infty) \mapsto (0, +\infty)$ be a continuous and increasing function. If (4) holds, then we have*

$$\sum_{c_n \le x} c_n \asymp x\, g^{-1}(x), \quad x \to +\infty. \tag{6}$$

Corollary 3. *Let (c_n) be an increasing sequence from the class K_c^* and (d_n) be an increasing sequence of positive numbers. If (4') holds, then we have*

$$\sum_{c_n \le x} c_n \asymp \sum_{d_n \le x} d_n, \quad x \to +\infty. \tag{6'}$$

2. Proofs of statements

Proof of the theorem. Consider the function $f(x)$, $x \geq 1$, for which we have $c_n = f(n)$. It is obviously linear on intervals $[n, n+1]$, $n \in N$.

For any $\delta > 0$, there is some $n_0 = n_0(\delta) \in N$, so that for all $n \geq n_0$ we have $1 \leq 1+\dfrac{1}{n} \leq \delta+1$, so that we find $1 \leq \overline{\lim}_{n\to+\infty}\dfrac{c_{n+1}}{c_n} \leq \overline{k}_c(1+\delta)$. Since by assumption $(c_n) \in K_c^*$, it is ∗–regularly varying, so that $\lim\limits_{n\to\infty}\dfrac{c_{n+1}}{c_n} = 1$. If (4) holds true, then we have $f(x) \sim g(x)$, $x \to +\infty$, because for all $n \leq x < n+1$, $n \in N$, we have that

$$\frac{c_n}{c_{n+1}} \cdot \frac{c_{n+1}}{g(n+1)} \leq \frac{f(x)}{g(x)} \leq \frac{c_n}{g(n)} \cdot \frac{c_{n+1}}{c_n}.$$

Next, let for any $\lambda > 0$, $\overline{k}_f(\lambda) = \overline{\lim}_{x\to+\infty}\dfrac{f(\lambda x)}{f(x)}$. Then for every $\delta > 0$ we have

$$\begin{aligned}\overline{k}_c(\lambda) \leq \overline{k}_f(\lambda) &\leq \overline{\lim}_{x\to+\infty}\frac{f([\lambda x]+1)}{f([x])} \leq \\ &\leq \overline{\lim}_{x\to+\infty}\frac{c_{[\lambda[x]]}}{c_{[x]}} \cdot \overline{\lim}_{x\to+\infty}\frac{c_{[\lambda x]+1}}{c_{[\lambda[x]]}} \leq \\ &\leq \overline{k}_c(\lambda)\cdot\overline{k}_c(1+\delta),\end{aligned}$$

because

$$\lim_{x\to+\infty}\frac{[\lambda x]+1}{[\lambda[x]]} = 1+.$$

This means that for every $\lambda > 0$ we have $\overline{k}_c(\lambda) = \overline{k}_f(\lambda)$.

If we next redefine $f(x)$ by $f(0) = 0$, and on the interval $[0, 1]$ as a linear function, then we have that $f \in K_c^*$ (see [5]). If we in a similar way redefine $g(x)$ for $0 \leq x < 1$, and we suppose (4), then by [3] we have

$$f^{-1}(x) \sim g^{-1}(x), \quad x \to +\infty. \tag{7}$$

Since $\delta_c(x) = [f^{-1}(x)]$, $x > 0$, we obtain (5).

Conversely, supposing that (5) holds true, then with the so redefined functions f and g we have (7). Since $f \in K_c^*$, we get $f(x) \sim g(x)$, $x \to +\infty$, so that we obtain (4). □

Remark. If (c_n) is an increasing and unbounded ∗–regularly varying sequence, out the class K_c^*, then (5) implies (4) for every function g described in the Theorem. But it is not difficult to see that there is a function g which has properties from the Theorem, such that (4) does not implies (5).

If a sequence (c_n) is increasing and unbounded, and it is not ∗–regularly varying, it is not clear if, in the general case, (4) and (5) are equivalent to each other for an arbitrary function g described in the Theorem.

Proof of Corollary 2. By assumptions, we have that

$$\sum_{c_n \le x} c_n = \int_0^x t\, d\,\delta_c(t) \le x\,\delta_c(x), \quad x > 0. \tag{8}$$

On the other side, we have

$$\sum_{c_n \le x} c_n \ge \int_{x/2}^x t\, d\,\delta_c(t) \ge \frac{x}{2}(\delta_c(x) - \delta_c(\frac{x}{2})), \quad x > 0. \tag{9}$$

Since $(c_n) \in K_c^*$ we have that $\underline{k}_c(\lambda) > 1$, $\lambda > 1$, so that $\underline{k}_{\delta_c}(2) > 1$. In other words, $\overline{k}_{\delta_c}\left(\frac{1}{2}\right) < 1$. Next, define $p = 1 - \overline{k}_{\delta_c}\left(\frac{1}{2}\right)$. Then for all $x \ge x_0$ we have that

$$\frac{p}{4} \le \frac{\sum_{c_n \le x} c_n}{x\,\delta_c(x)} \le 1,$$

so that $\sum\limits_{c_n \le x} c_n \asymp x\,\delta_c(x)$, $x \to +\infty$. By assumptions of the colollary, and the Theorem, we have that then $\delta_c(x) \sim g^{-1}(x)$, $x \to +\infty$, so that (6) holds true. □

Finaly, Corollary 3 is a direct consequence of the Theorem and the Corollary 2.

3. References

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