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ON SOME FIXED POINT THEOREMS FOR MAPPINGS SATISFYING A NEW TYPE OF IMPLICIT RELATION

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Abstract. In this paper we introduce a new class of functions $F : R_+^6 \to R$ such that the fulfilment of the inequality of type (3) for x, y in X, ensures the existence and the uniqueness of a fixed point.

1. Introduction

The notion of contractive mapping has been introduced by Banach in [1].

In the last thirty years different types of generalizations of this concept appeared. The connection between them have been studied in different papers, for example [2], [3], [5]-[9].

Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ a mapping in essence, T is a generalized contraction if an inequality of type

$$(1)\qquad d(Tx, Ty) \leq f\Big(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big)$$

holds for $x, y \in X$, where $f : R_+^5 \to R$ satisfies some properties or has a special form.

In [4], the present author established a class of mappings $F : R_+^6 \to R$ such as the fulfilment of the inequality of type

$$(2)\qquad F\Big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big) \leq 0$$

for $x, y \in X$, ensures the existence and the uniqueness of a fixed point for T.

The purpose of this paper is to introduce a new class of mappings $F : R_+^6 \to R$ such that the fulfilment of the inequality of type

$$(3)\qquad F\Big(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Ty), d(y, T^2x), d(y, Tx)\Big) \leq 0$$

for $x, y \in X$, ensures the existence and the uniqueness of a fixed point for T.

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2. Implicit relations

Let Φ be the set of all real continuous functions $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:

$\emptyset_1$:F is nonincreasing in variable t_5,

$\emptyset_h$: there exists $h \in [0,1)$ such that for every $u, v \geq 0$

$$F(u, v, v, u, u, 0) \leq 0 \text{ implies } u \leq hv;$$

$\emptyset_u : F(u, u, 0, 0, u, u) > 0, \forall u > 0.$

Ex.1. $F(t_1, ..., t_6) = t_1^2 - at_5t_6 - bmax\{t_2^2, t_3^2, t_4^2\}$, where $a > 0, b \geq 0$ and $a + b < 1$.

($\emptyset_1$): Obviously.

($\emptyset_h$): Let $u > 0, v \geq 0$ and $F(u, v, v, u, u, 0) = u^2 - bmax\{u^2, v^2\} \leq 0$. If $u \geq v$ then $u^2(1 - b) < 0$, a contradiction. Thus $u < v$ and $u \leq hv$, where $h = \sqrt{b} < 1$.

If $u = 0$ and $v \geq 0$ then $u \leq hv$.

$(\emptyset_u) F(u, u, 0, 0, u, u) = u^2(1 - a - b) > 0, \forall u > 0.$

Ex.2. $F(t_1, ..., t_6) = t_1^2 - at_5t_6 - t_1(bt_2 + ct_3 + dt_4)$, where $a, b, c, d \geq 0$ and $a + b + c + d < 1$.

($\emptyset_1$): Obviously.

($\emptyset_h$): Let $u > 0, v \geq 0$ and $F(u, v, v, u, u, 0) = u^2 - u(bv + cv + du) \leq 0$. Then $u \leq hv$, where $h = c + b/1 - d < 1$. If $u = 0, v \geq 0$ then $u \leq hv$.

$(\emptyset_u) : F(u, u, 0, 0, u, u) = u^2(1 - a - b) > 0, \forall u > 0.$

Ex.3. $F(t_1, ..., t_6) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2t_3t_4 - dt_5^2t_6$, where $a > 0, b, c, d \geq 0$ and $a + b + c + d < 1$.

($\emptyset_1$): Obviously.

($\emptyset_2$): Let $u > 0, v \geq 0$ and $F(u, v, v, u, u, 0) = u^3 - au^2v - buv^2 - cuv^2 \leq 0$, which implies $u^2 - auv - (b + c)v^2 \leq 0$. If $b = c = 0$, then $u \leq hv$, where $0 < h = a < 1$.

If $b + c > 0$ then $f(t) = (b + c)t^2 + at - 1 \geq 0$, where $t = v/u > 0$. Since $f(1) = (a + b + c) - 1 < 0$, let $r > 1$ be the root of equation $f(t) = 0$. Then $f(t) > 0$ for $t > r$ which implies $u \leq hv$, where $h = 1/r < 1$. If $u = 0$ then $u \leq hv$.

$(\emptyset_u) : F(u, u, 0, 0, u, u) = u^3(1 - a - b - d) > 0, \forall u > 0.$

3. Fixed points in complete metric spaces

Theorem 1. *Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a mapping satisfying the inequality (3) for every $x, y \in X$, where F satisfies condition $(\emptyset_u)$. Then T has at most one fixed point.*

Proof. Suppose that T has two fixed points u and v with $u \neq v$. Then by (3) we have successively

$$F\Big(d(Tu,Tv),d(u,v),d(u,Tu),d(v,Tv),d(v,T^2u),d(v,Tu)\Big) \leq 0$$

$$F\Big(d(u,v),d(u,v),0,0,d(u,v),d(u,v)\Big) \leq 0,$$

a contradiction of $(\emptyset_u)$.

Theorem 2. *Let (X,d) be a metric space and $T:(X,d)\to(X,d)$ be a mapping such that there exists $h\in[0,1)$ with $d(T^2x,Tx)\leq hd(x,Tx)$ for every $x\in X$. Then for every $x\in X$ the sequence $\{T^nx\}$ is a Cauchy sequence.*

Proof. Let $x\in X$ and the sequence $\{T^nx\}$. Since $d(T^2x,Tx)\leq hd(x,Tx)$ by induction we have $d(T^{n+1}x,T^nx)\leq h^nd(x,Tx)$. By a routine calculation it follows that $\{T^nx\}$ is a Cauchy sequence.

Theorem 3. *Let (X,d) be a complete metric space and $T:(X,d)\to(X,d)$ a mapping satisfying the inequality (3) for every $x,y\in X$ where $F\in\Phi$. Then T has a unique fixed point.*

Proof. Let x be arbitrary in X. We shall show that the sequence defined by $x_{n+1}=Tx_n$ is a Cauchy sequence. From (3) for $y=Tx$ we have

$$F\Big(d(Tx,T^2x),d(x,Tx),d(x,Tx),d(Tx,T^2x),d(Tx,T^2x),0\Big) \leq 0.$$

By $(\emptyset_h)$ we have $d(T^2x,Tx)\leq h\cdot d(x,Tx)$. By Theorem 2, $x_{n+1}=T^nx$ is a Cauchy sequence. Since (X,d) is complete, there exists $u\in X$ such that $\lim x_n=u$.

By (3) we have successively

$$F\Big(d(Tx_n,Tu),d(x_n,u),d(x_n,Tx_n),d(u,Tu),d(u,T^2x_n),d(u,Tx_n)\Big) \leq 0.$$

$$F\Big(d(x_{n+1},Tu),d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(u,x_{n+2}),d(u,x_{n+1})\Big) \leq 0.$$

Letting n tend to infinity we have successively

$$F\Big(d(u,Tu),0,0,d(u,Tu),0,0\Big) \leq 0,$$

$$F\Big(d(u,Tu),0,0,d(u,Tu),d(u,Tu),0\Big) \leq 0,$$

which implies by $(\emptyset_h)$ that $u=Tu$. By Theorem 1 u is the unique fixed point of T.

Corollary 1. *Let (X,d) be a complete metric space and $T:(X,d)\to(X,d)$ satisfying one of the following inequalities:*

(1.1) $d^2(Tx,Ty)\leq ad(y,T^2x)d(y,Tx)+b\max\{d^2(x,y),d^2(x,Tx),d^2(y,Ty)\}$, *where* $a > 0, b \geq 0$ *and* $a + b < 1$, *or*

$$d^2(Tx, Ty) \leq \tag{1.2}$$
$$ad(y, T^2x)d(y, Tx) + d(Tx, Ty)(bd(x, y) + cd(x, Tx) + dd(y, Ty),$$

where $a, b, c, d \geq 0$ *and* $a + b + c + d < 1$, *or*

$$d^3(Tx, Ty) - ad^2(Tx, Ty)d(x, y) - bd(Tx, Ty)d^2(x, y) - \tag{1.3}$$
$$cd(x, y)d(x, Tx)d(y, Ty) - d \cdot d^2(y, T^2x) \cdot d(y, Tx) \leq 0,$$

where $a > 0, b, c, d \geq 0$ *and* $a + b + c + d < 1$ *for all* x, y *in* X, *then* T *has unique fixed point.*

Remark 1. Let Ψ be the set of all real continuous functions $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:

(Ψ_1): F is nonincreasing in variable t_5,

(Ψ_h): there exists $h \in [0, 1)$ such that for every $u, v \geq 0, F(u, v, u, v, u, 0) \leq 0$ implies $u \leq h \cdot v$,

$(\Psi_u) : F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

Theorem 4. *If the inequality*

$$F\Big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, T^2y), d(x, Ty)\Big) \leq 0 \tag{4}$$

holds for all x, y *in* X, *where* $F \in \Psi$, *then* F *has a unique fixed point.*

Proof. The proof is similar to the proof of Theorem 3.

4. Fixed points in compact metric spaces

Let $\overline{\Phi}$ be the set of all real continuous functions $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:

($\overline{\Phi_h}$): For every $u \geq 0, v > 0, F(u, v, v, u, u, 0) < 0$ implies $u < v$,

($\overline{\Phi_u}$): $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

Remark 2. The functions F from Ex. 1-3 satisfies conditions ($\overline{\Phi_h}$) and ($\overline{\Phi_u}$).

Remark 3. There exists functions $F \in \overline{\Phi}$ which is increasing in variable t_5.

Ex.4. $F(t_1, ..., t_6) = t_1^3 - c\dfrac{t_2t_3t_4}{1 + t_5 + t_6}$, where $0 < c < 1$.

($\overline{\Phi_h}$): Let $u, v > 0$ and $F(u, v, v, u, u, 0) = u^3 - c\dfrac{v^2u}{1 + u} < 0$, then $u^2 < \dfrac{c}{1 + v}v^2$ which implies $u < v$. If $u = 0$, then $u < v$.

$(\overline{\Phi_u}) : F(u, u, 0, 0, u, u) = u^3 > 0, \forall u > 0$.

Theorem 5. *Let T be a continuous mapping of the compact metric space (X, d) into itself such that*

$$F\Big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(y, T^2x), d(y, Tx)\Big) < 0 \tag{5}$$

for every $x \neq y$ in X, where $F \in \overline{\Phi}$. Then T has a unique fixed point.

Proof. Let $f(x) = d(x, Tx)$ for all $x \in X$. Since T is continuous, there exists a point $z \in X$ such that $f(z) = inf\{f(x) : x \in X\}$. Suppose that $z \neq Tz$.

By (5) for $x = z$ and $y = Tz$ we obtain

$$F\Big(d(Tz, T^2z), d(z, Tz), d(z, Tz), d(Tz, T^2z), d(Tz, T^2z), 0\Big) < 0$$

which implies $d(Tz, T^2z) < d(z, Tz) = inf\{d(x, Tx) : x \in X\}$. A contradiction.

Hence, $z = Tz$. From Theorem 1 z is the unique fixed point of T.

Corollary 2. *Let T be a continuous mapping of the compact metric space (X, d) into itself such that*

$$d^3(Tx, Ty) < c\frac{d(x, y)d(x, Tx)d(y, Ty)}{1 + d(y, T^2y) + d(y, Tx)}$$

for all $x \neq y$ in X and $0 < c < 1$. Then T has a unique fixed point.

Proof. The proof follows from Theorem 5 and Ex.4.

Remark 4. A Corollary analogous to Corollary 1 is obtained by Ex.1-3.

Remark 5. A Theorem similar to Theorem 4 is obtained for compact metric space.

5. References

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