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INFINITELY DISTRIBUTIVE ELEMENTS IN POSETS

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Abstract. Infinitely distributive and codistributive elements in posets are studied. It is proved that an element a in a poset P has these properties if and only if the image of a has the corresponding properties in the Dedekind MacNeille completion of P. An application of the order theoretical results to a poset of weak congruences is presented.

1. Preliminaries

1.1 Special elements in lattices

An element a of a lattice L is **infinitely distributive** iff for every family $\{x_i | i \in I\} \subseteq L$:

$$a \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (a \vee x_i).$$

An element a which satisfies the dual law is called **infinitely codistributive**.

Elements satisfying the corresponding laws with finite families I usually are called distributive (or codistributive).

In paper [20] the relationship between infinite and finite distributive (codistributive) elements has been investigated.

Element $a \in L$ is infinitely codistributive if and only if the mapping $m_a : L \longrightarrow a \downarrow$ defined by $m_a(x) = a \wedge x$ is a complete lattice homomorphism (homomorphism which is compatible with all suprema and infima). This homomorphism induces a complete congruence on L [18]. Moreover, if lattice L is algebraic, classes of the induced congruence always have the top elements.

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The dual complete homomorphism connected with the completely distributive elements will be denoted by n_a.

For a poset $(P, \leq)$ and $X \subseteq P$ we introduce the following notions and notations.

Let $L_P(X)$ be the set of all lower bounds of X in P, and $U_P(X)$ the set of all upper bounds:

$$L_P(X) = \{y \in P \mid y \leq x \quad \text{for all} \quad x \in X\},$$
$$U_P(X) = \{y \in P \mid x \leq y \quad \text{for all} \quad x \in X\}.$$

In no confusion can occur, subscripts will be omitted and we shall write L(X) and U(X).

If $X = \{x_1, \dots x_n\}$ is finite then instead of $L(X)$ and $U(X)$, we also use the notation $L(x_1, \dots, x_n)$ and $U(x_1, \dots, x_n)$. $L(X \cup Y)$ will be denoted by $L(X, Y)$, and similarly $U(X \cup Y)$, by $U(X, Y)$.

Throughout the paper $DM(P)$ denotes the Dedekind-McNeille completion of P. In this context, $G(P)$ is the sublattice of $DM(P)$ generated by the set $\{L(x) \mid x \in P\}$. This lattice is called, according to [6], characteristic lattice of poset P.

Let e_P be a natural mapping from a poset P to its Dedekind-McNeille completion of P.

$$e_P : P \to DM(P) \quad \text{defined by} \quad e_P(x) := L(x).$$

A mapping $f : P \to Q$ is **ω-stable** [6] if there is a lattice homomorphism $f^* : G(P) \to G(Q)$ such that $e_Q \circ f = f^* \circ e_P$.

An equivalence on P is defined to be a **congruence on** P if it is a kernel of a ω-stable mapping on P [6].

Theorem 1. [6] *A relation on a poset P is a congruence on P if and only if it corresponds to a restriction of a congruence on lattice $G(P)$.* □

We call a relation on a poset P **complete congruence** if it is a restriction of a complete congruence relation on the lattice $DM(P)$ (complete congruences are equivalence relations compatible with arbitrary suprema and infima).

Throughout the paper, infima and suprema in P are denoted by inf_P and sup_P, and infima and suprema in lattices $DM(P)$ and $G(P)$ are denoted by $\wedge$ and $\vee$. The ordering relation in the poset and the related Dedekind MacNeille completion is denoted by the same symbol $\leq$.

The following lemma is a consequence of the fact that mapping e_P is an order embedding.

Lemma 1. *Let P be a poset and $DM(P)$ its Dedekind MacNeille completion. Then for $a \in P$, $X \subseteq P$,*

$$a \in U_P(X) \quad \text{if and only if} \quad e_P(a) \geq \bigvee_{x \in X} e_P(x),$$

where $\bigvee$ is the supremum in the $DM(P)$. □

The dual lemma is also valid.

1.2 Identities in posets

Identities on posets are introduced and studied in several papers (see e.g. [8], [12], [13]).

A partially ordered set P is **distributive** [8] if for all $x, y, z \in P$,

$$L(x, U(y, z)) = L(U(L(x, y), L(x, z))).$$

It is proved by Lamerová and Rachunek ([8]) that this condition is equivalent with its dual:

$$U(x, L(y, z)) = U(L(U(x, y), U(x, z))).$$

It turned out that the distributivity of a poset is connected with the distributivity of the corresponding characteristic lattice.

Theorem 2. (**Niederle** [12]): *Poset P is distributive if and only if it is a doubly dense subset of a distributive lattice.* □

Theorem 3. (**Niederle** [13]): *Poset P is distributive if and only if the lattice $G(P)$ is distributive.* □

2. Special elements in posets

A large class of special elements in posets has been introduced and studied in [23] and [24].

In this section we introduce notions of infinitely distributive and codistributive elements in posets. We characterize these elements by infinite (co) distributivity in the lattice $DM(P)$.

Element a in a poset P is **infinitely distributive** if and only if for every family $\{x_i \mid i \in I\}$ of elements from P, $U(a, L(\{x_i \mid i \in I\})) = U(L(\bigcup_{i \in I} U(a, x_i)))$.

Element a in a poset P is **infinitely codistributive** if for every family $\{x_i \mid i \in I\}$ of elements from P, $L(a, U(\{x_i \mid i \in I\})) = L(U(\bigcup_{i \in I} L(a, x_i)))$.

Theorem 4. *Element $a \in P$ is infinitely distributive in P if and only if $e_P(a)$ is an infinitely distributive element in the lattice $DM(P)$.*

Proof. Let $a \in P$ and let $e_P(a)$ be an infinitely distributive element in $DM(P)$. Recall that P is a double dense subset of $DM(P)$. Let

$$t \in U(a, L(\{x_i \mid i \in I\})).$$

Then, $t \geq a$ and for any $x \in L\{x_i \mid i \in I\}$, $t \geq x$. Therefore, $e_P(t) \geq e_P(a)$ and $e_P(t) \geq e_P(x)$ for all $x \in L\{x_i \mid i \in I\}$. In lattice $DM(P)$, $e_P(x) \leq e_P(x_i)$, for all $i \in I$, and therefore $e_P(x) \leq \bigwedge_{i\in I} e_P(x_i)$. Hence, $e_P(t) \geq e_P(x)$ for all $x \leq \bigwedge_{i\in I} x_i$. Since $\bigwedge_{i\in I} e(x_i) = \bigvee\{e(x) \mid e(x) \leq \bigwedge_{i\in I} e(x_i)\}$, we have that $e_P(t) \geq \bigwedge_{i\in I} e(x_i)$. Further on, $e_P(t) \geq e_P(a) \vee \bigwedge_{i\in I} e_P(x_i) = \bigwedge_{i\in I}(e_P(a) \vee e_P(x_i))$ by the infinite distributivity of $e_P(a)$ in $DM(P)$.

Let $p \in L(\bigcup_{i\in I} U(a, x_i))$. Then, $p \leq q$ for every $q \in \bigcup_{i\in I} U(a, x_i)$, and by the similar arguments as above, $e_P(p) \leq e_P(a) \vee e_p(x_i)$. Therefore, $e_P(p) \leq \bigwedge_{i\in I}(e_P(a) \vee e_P(x_i))$. Hence, $e_P(p) \leq e_P(t)$, and $p \leq t$. Hence, $t \in U(L(\bigcup_{i\in I} U(a, x_i)))$.

Therefore, we proved $U(a, L\{x_i \mid i \in I\}) \subseteq U(L(\bigcup_{i\in I} U(a, x_i)))$. The other inclusion is always fulfilled.

Now, we suppose that a is an infinitely distributive element of P. P is a double dense subset in $DM(P)$.

Firstly, we prove that $e_P(a) \vee \bigwedge_{i\in I} e_P(x_i) = \bigwedge_{i\in I}(e_P(a) \vee e_P(x_i))$ is satisfied for all $x_i \in P$.

Let $y \in P$. Then,
$e_P(y) \geq \bigwedge_{i\in I}(e_P(a) \vee e_P(x_i))$ if and only if
$e_P(y) \in U(\bigwedge_{i\in I}(e_P(a) \vee e_P(x_i)))$ if and only if
$y \in U(L(\bigcup_{i\in I} U(a, x_i)))$ if and only if
$y \in U(a, L\{x_i \mid i \in I\})$ if and only if
$e_P(y) \in U(e_P(a), \bigwedge_{i\in I} e_P(x_i))$ if and only if
$e_P(y) \geq e_P(a)$ and $e_P(y) \geq \wedge_{i\in I} e_P(x_i)$ if and only if
$e_P(y) \geq e_P(a) \vee \bigwedge_{i\in I} e_P(x_i)$.

If $\{x_i \mid i \in I\}$ is a family of elements from $DM(P)$, then every x_i is an infimum of a family of images of elements from P, $x_i = \wedge_{j\in J_i} e_P(z_j)$, for $z_j \in P$.

Therefore,

$$e_P(a) \vee \bigwedge_{i\in I} x_i = e_P(a) \vee \bigwedge_{i\in I} \bigwedge_{j\in J_i} e_P(z_j) = \bigwedge_{i\in I} \bigwedge_{j\in I_j} (e_P(a) \vee e_P(z_j)) \geq$$

$$\geq \bigwedge_{i\in I}(e_P(a) \vee \bigwedge_{j\in I_j} e_P(z_j)) = \bigwedge_{i\in I}(e_P(a) \vee x_j).$$

The other inequality is always satisfied. □

The dual theorem is also satisfied.

Theorem 5. *Element $a \in P$ is infinitely codistributive if and only if $e_P(a)$ is an infinitely codistributive element in the lattice $DM(P)$.* □

By the Lemma 1 in [18] we obtain following consequences:

Corollary 1. *Element $a \in P$ is infinitely distributive if and only if relation θ_a on P, defined by*

$$x\theta_a y \text{ if and only if } e_P(x) \vee e_P(a) = e_P(y) \vee e_P(a)$$

is a complete congruence on poset P. □

Corollary 2. *Element $a \in P$ is infinitely codistributive if and only if relation θ_a on P, defined by*

$$x\theta_a y \text{ if and only if } e_P(x) \wedge e_P(a) = e_P(y) \wedge e_P(a)$$

is a complete congruence on poset P. □

2.1 Weak congruence lattice

In this section we recall the notion of weak congruences which will serve as a justification of introduction of the terms of distributive and codistributive elements in posets.

Let $\mathcal{A} = (A, F)$ be an algebra. Let $Cw\mathcal{A}$ be a set of all weak congruences (symmetric and transitive and compatible relations) on $\mathcal{A}$. $(Cw\mathcal{A}, \subseteq)$ is an algebraic lattice. It is a lattice of all congruences on all subalgebras together with the empty set in case when there is no nullary operation in the similarity type.

The diagonal relation Δ is always an infinitely codistributive element in $Cw\mathcal{A}$. The filter $\Delta\uparrow$ is the congruence lattice $Con\mathcal{A}$, and the ideal $\Delta\downarrow$ is isomorphic with the subuniverse lattice $\mathrm{Sub}\,\mathcal{A}$.

The top elements of the congruence classes determined by the homomorphism $m_\Delta : x \mapsto x \wedge \Delta$ are squares of subuniverses.

An algebra $\mathcal{A}$ has the **congruence intersection property (CIP)** iff Δ is a distributive element in the lattice $Cw\mathcal{A}$.

An algebra $\mathcal{A}$ has the infinite congruence intersection property (*CIP) if and only if for an arbitrary family of weak congruences $\{\rho_i | i \in I\}$,

$$\Delta \vee (\bigwedge_{i\in I} \rho_i) = \bigwedge_{i\in I} (\Delta \vee \rho_i).$$

2.2 Weak congruences under different order

Let $Cw\mathcal{A}$ be a set of all weak congruences of an algebra $\mathcal{A}$, and Δ the diagonal relation.

Let ρ, θ be two weak congruences, and $\rho \in Con\mathcal{C}$, $\theta \in Con\mathcal{B}$. We introduce a new operation on $Cw\mathcal{A}$:

$$\rho * \theta = (B^2 \wedge \rho) \vee (C^2 \wedge \theta),$$

and $\emptyset * \theta = \emptyset$.

Use of such an operation, which is also a graphical composition was proposed by M. Ploščica in [17].

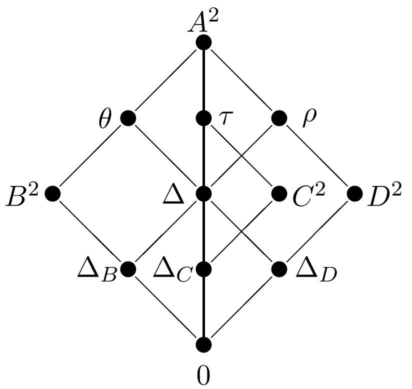
In the sequel, $(Cw\mathcal{A}, \wedge, \vee)$ or $(Cw\mathcal{A}, \subseteq)$ is a weak congruence lattice of an algebra $\mathcal{A}$, and $(Cw\mathcal{A}, \leq_*)$ is the poset of weak congruences, where the relation $\leq_*$ is defined by the operation $*$:

$$\rho \leq_* \theta \quad \text{if and only if} \quad \rho * \theta = \theta.$$

Theorem 6. *[9] Let $Cw\mathcal{A}$ be a weak congruence lattice, and $*$ and $\leq_*$ be defined as above. Then:*

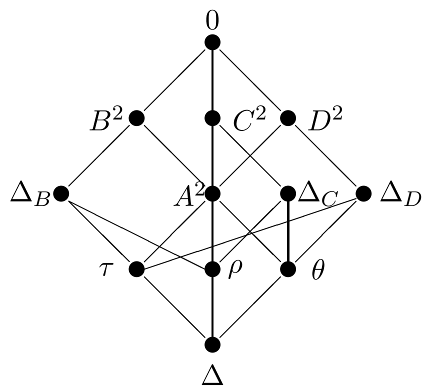
(**i**) *$\Delta \leq_* \rho$, for all $\rho \in Cw\mathcal{A}$.*
(**ii**) *If $\rho, \theta \in [\Delta_B, B^2]$, then $\rho * \theta = \rho \vee \theta$, for $B \in \mathrm{Sub}\,\mathcal{A}$.*
(**iii**) *$\rho \subseteq \theta$ if and only if $\rho \leq_* \theta$, for $\rho, \theta \in [\Delta_B, B^2]$.*
(**iv**) *The interval $[\Delta_B, B^2]_*$ is a lattice $Con\mathcal{B}$, for $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$.*
(**v**) *$B^2 * C^2 = B^2 \wedge C^2$.*
(**vi**) *$B^2 \leq_* C^2$ if and only if $C^2 \subseteq B^2$.*
(**vii**) *$A^2 \leq_* \rho$ if and only if $\rho = B^2$, for $\rho \in Con\mathcal{B}$.*
(**viii**) *$Cw\mathcal{A}$ is equal to the union of intervals $[\Delta_B, B^2]_*$, for all $B \in \mathrm{Sub}\,\mathcal{A}$.*
(**ix**) *The filter $A^2{\uparrow}_*$ is anti-isomorphic with $\mathrm{Sub}\,\mathcal{A}$.*
(**x**) *If $\rho \in Con\mathcal{B}$, then $\rho * A^2 = B^2$.* □

Example 1. A lattice of weak congruences $(Cw\mathcal{G}, \subseteq)$ for a four-element Klein's group $\mathcal{G}$ is given in Figure 2 a). A poset of weak congruences $(Cw\mathcal{G}, \leq_*)$ for the same group is given in Fig. 2 b).



$(Cw\mathcal{G}, \subseteq)$

Figure 1 a)



$(Cw\mathcal{G}, \leq_*)$

Figure 1 b)

3. Special elements in poset of weak congruences

Let A be an algebra, and $(Cw\mathcal{A}, \leq_*)$ the poset of weak congruences.

Being the bottom element, the diagonal relation Δ is always an infinitely distributive and infinitely codistributive element in this poset.

Lemma 2. *In the poset $(Cw\mathcal{A}, \leq_*)$, for every $\rho \in Con\mathcal{B}$, $B \in \operatorname{Sub}\mathcal{A}$*

$$\sup\{A^2, \rho\} = B^2.$$

Proof. Since $A^2 * \rho = B^2$, B^2 is an upper bound for elements A^2 and ρ. Let $\theta \in Cw\mathcal{A}$ be another upper bound, i.e., let $A^2 \leq_* \theta$ and $\rho \leq_* \theta$. Let $\theta \in Con\mathcal{C}$, for $C \in \operatorname{Sub}\mathcal{A}$. Hence,

$$\theta = A^2 * \theta = (A^2 \wedge \theta) \vee (A^2 \wedge C^2) = \theta \vee C^2 = C^2,$$

and

$$C^2 = \rho * C^2 = (B^2 \wedge C^2) \vee (\rho \wedge C^2) = B^2 \wedge C^2.$$

Thus, $C^2 \subseteq B^2$ and $C \leq B$. By the Theorem 6. (vi), $B^2 \leq_* C^2$ and B^2 is the required supremum. □

Lemma 3. *In the poset $(Cw\mathcal{A}, \leq_*)$, for every family $\{B_i \in \operatorname{Sub}\mathcal{A} \,|\, i \in I\}$*

$$\inf_{i\in I} {B_i}^2 = \bigvee_{i\in I} {B_i}^2,$$

where the operation $\vee$ at the right is the supremum in the weak congruence lattice.

Proof. By Theorem 6., $\bigvee_{i\in I} {B_i}^2$ is a maximim lower bound for elements $\{B_i | i \in I\}$. Suppose that $\rho \in Con\mathcal{D}$ is another lower bound for elements B_i for $i \in I$: $\rho \leq_* B_i$, for all $i \in I$. Hence, $B_i \subseteq D$, for all $i \in I$ and thus $\bigvee_{i\in I} {B_i}^2 \subseteq D$. Therefore, $\rho*(\bigvee_{i\in I} {B_i}^2) = (\rho\wedge(\bigvee_{i\in I} {B_i}^2))\vee(D^2\wedge(\bigvee_{i\in I} {B_i}^2)) = (\bigvee_{i\in I} {B_i}^2)$. Hence, $\bigvee_{i\in I} {B_i}^2$ is the required infimum. □

Theorem 7. *A^2 is an infinitely distributive element in poset $(Cw\mathcal{A}, \leq_*)$.*

Proof. Let $\{\rho_i \mid i \in I\}$ be a family of weak congruences, where $\rho_i \in Con\mathcal{B}_i$ for $\mathcal{B}_i \in \operatorname{Sub}\mathcal{A}$.

By Lemma 2., $\sup\{A^2, \rho_i\} = B_i^2$ for each $i \in I$, where suprema are considered under the ordering $\leq_*$.

Hence, $U(A^2, \rho_i) = B_i^2\uparrow$, for every $i \in I$. By Lemma 3

$$U\left(L\left(\bigcup_{i\in I} U(A^2,\rho_i)\right)\right) = \left(\bigvee_{i\in I} B_i\right)\uparrow .$$

On the other hand, let ρ belong to $U(A^2, L(\{\rho_i \mid i \in I\}))$. Since $\rho \geq A^2$, and by Theorem 6. (vii), $\rho = D^2$, for some subalgebra $\mathcal{D}$. $D^2 \geq \theta$ for any $\theta \in L(\{\rho_i \mid i \in I\}))$. We prove that $\Delta_S \leq \rho_i$ for any $i \in I$, where $S = \bigvee_{i\in I} B_i$. Indeed,

$$\Delta_S * \rho_i = (S^2 \wedge \rho_i) \vee (\Delta_S \wedge B_i^2) = \rho_i \vee \Delta_{B_i} = \rho_i,$$

and thus $\Delta_S \leq_* \rho_i$. Hence, $\Delta_S \in L(\{\rho_i \mid i \in I\}))$ and $D^2 \geq_* \Delta_S$, therefore $(D^2 \wedge \Delta_S) \vee (D^2 \wedge S^2) = D^2$ and $D^2 \leq S^2$, and $D \leq S$ in subalgebra lattice. By , $\rho = D^2 \in (\bigvee_{i\in I} B_i)\uparrow$ and

$$U\left(A^2, L(\{\rho_i \mid i \in I\})\right) \subseteq U\left(L\left(\bigcup_{i\in I} U(A^2,\rho_i)\right)\right).$$

The other inclusion is always satisfied. □

Since A^2 is an infinitely distributive element, we obtain the natural decomposition to congruence classes of the poset $(Cw\mathcal{A}, \leq_*)$.

Corollary 3. *Each block of the congruence on poset $(Cw\mathcal{A}, \leq_*)$ induced by $\rho \mapsto \sup\{\rho, A^2\}$ is a congruence lattice of a subalgebra $\mathcal{B}$, where ρ belongs to $Con\mathcal{B}$. Therefore, poset $(Cw\mathcal{A}, \leq_*)$ is the union of intervals $Con\mathcal{B}$ which are the congruences lattices on all the subalgebras of $\mathcal{A}$.* □

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