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SOME REMARKS NEAR-P-POLYAGROUPS AND POLYAGROUPS

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Abstract. In this paper the Hosszú–Gluskin Theorem for near-P-polyagroups (polyagroups) is proved.

1. Introduction

1.1. Definition [1]: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. We say that $(Q; A)$ is a Dörnte $n-$***group** *[briefly: $n-$group] iff is an $n-$semigroup and an $n-$guasigroup as well. (See, also [12].)*

1.2. Definition *(cf. [9],[10]): Let $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then: we say that $(Q; A)$ is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

1° *For all $i, j \in \{1, \dots, n\}$ $(i < j)$ if $i \equiv j (mod\ s)$, then $< i, j > -$associative law holds in $(Q; A)$; and*

2° *$(Q; A)$ is an $n-$quasigroup.*

1.3. Definition[11]: *Let $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then: we say that $(Q; A)$ is a* **near-P-polyagroup** *[briefly: $NP-$polyagroup]***of the type** $(s, n-1)$ *iff the following statements hold:*

$^\circ 1$ *For all $i, j \in \{1, \dots, n\}$ $(i < j)$ if $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \dots, k\}\}$, then the $< i, j > -$associative law holds in $(Q; A)$; and*

$^\circ 2$ *For all $i \in \{t \cdot s + 1 | t \in \{0, 1, \dots, k\}\}$, and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality*

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds.[1)]

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[1)] For $s = 1$ $(Q; A)$ is a $(k+1)-$group, where $k + 1 \geq 3;\ k > 1$.

1.4. Proposition: *Every polyagroup of the type* $(s, n-1)$ *is an* NP*-polyagroup of the type* $(s, n-1)$. *[By Def.1.2 and by Def. 1.3.]*

2. Auxiliary propositions

2.1. Proposition [8]: *Let* $n \geq 2$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then, the following statements are equivalent:* (i) $(Q; A)$ *is an* $n-$*group;* (ii) *there are mappings* $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A, ^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) *there are mappings* $^{-1}$ *and* $\mathbf{e}$*, respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A, ^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

2.2. Remark: $\mathbf{e}$ *is an* $\{1, n\}-$*neutral operation of* $n-$*groupoid* $(Q; A)$ *iff algebra* $(Q; A, \mathbf{e})$ *[of the type* $< n, n-2 >$*] satisfies the laws* (b) *and* $(\overline{b})$ *from 2.1 [5]. Operation* $^{-1}$ *from 2.1. [*$(c), (\overline{c})$*] is a generalization of the inverse operation in a group [6]. Cf. [12].*

2.3. Definition[7]: *We say that an algebra* $(Q; B, \varphi, b)$ *[of the type* $< 2, 1, 0 >$ *is a* **Hosszú-Gluskin algebra of order** n $(n \geq 3)$ *[briefly:* $nHG-$*algebra] iff the following statements hold:*

$(Q; B)$ *is a group;*

$\varphi \in Aut\ (Q; B)$;

$\varphi(b) = b$; *and*

For every $x \in Q, B(\varphi^{n-1}(x), b) = B(b, x)$.

2.4. Proposition *(Hosszú-Gluskin Theorem [2,3])[7]: Let* $(Q; A)$ *be an* n*-group,* $\mathbf{e}$ *its* $\{1, n\}$*-neutral operation (cf. 2.2) and* $n \geq 3$*. Let also,* c_1^{n-2} *be an arbitrary (fixed) sequence over a set* Q*, and let*

$$B(x, y) \overset{def}{=} A(x, c_1^{n-2}, y),$$

$$\varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \text{ and}$$

$$b \overset{def}{=} A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|)$$

for all $x, y \in Q$*. Then, the following statements hold:*

(1) $(Q; B, \varphi, b)$ *is an* $nHG-$*algebra; and*

(2) For every $x_1^n \in Q$ *the equality*

$$A(x_1^n) = \overset{n}{B}(x_1, \varphi(x_2), \ldots, \varphi^{n-1}(x_n), b)^{2)}$$

holds.[3)]

2.5. Proposition [11]: *Every* $NP-$*polyagroup has a* $\{1,n\}-$*neutral operation.*

3. Results

3.1. Theorem: *Let* $k > 1,\ s \geq 1,\ n = k \cdot s + 1$, $(Q; A)$ *be an* $NP-$*polyagroup of the type* $(s, n-1)$, $\mathbf{e}$ *its* $\{1,n\}-$*neutral operation and* Y *stands for sequence* $\overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$ *[=* $\overline{\overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}$*] over* Q. *Also let* c_1^{k-1} *be an arbitrary (fixed) sequence over a set* Q. *Further on, let*

(1) $\mathbf{B}(Y, x, y) \overset{def}{=} A(x, \overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}, y)$,

(2) $\boldsymbol{\varphi}(Y, x) \overset{def}{=} A(\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), x, \overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1})$ *and*

(3) $\mathbf{b}(Y) \overset{def}{=} A(\overline{\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}}\Big|_{i=1}^{k}, \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}))$

for all $x, y, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$. *Finally, let for all* $x, y \in Q$

$\widehat{1}$ $B_Y(x, y) \overset{def}{=} \mathbf{B}(Y, x, y)$,

$\widehat{2}$ $\varphi_Y(x) \overset{def}{=} \boldsymbol{\varphi}(Y, x)$ *and*

$\widehat{3}$ $b_Y \overset{def}{=} \mathbf{b}(Y)$,

where Y *is an arbitrary (fixed) sequence over* Q. *Then, the following statements hold:*

(i) *For all sequence* Y *over* Q $(Q; B_Y, \varphi_Y, b_Y)$ *is an* $(k+1)HG-$*algebra; and*

(ii) *For all* $x_1^{k+1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$ *the following equality holds*

$$A(\overline{x_j, \overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}, x_{k+1}) =$$

$$\overset{k+1}{\mathbf{B}}(Y, x_1, Y, \varphi(Y, x_2), \ldots, Y, \varphi^k(Y, x_{k+1}), Y, \mathbf{b}(Y)),$$

where $\boldsymbol{\varphi}^1 \overset{def}{=} \boldsymbol{\varphi}$ *and for all* $x \in Q$, *for all sequence* Y *over* Q *and for every* $i \in N$ $\boldsymbol{\varphi}^{i+1}(Y, x) \overset{def}{=} \boldsymbol{\varphi}(Y, \boldsymbol{\varphi}^i(Y, x))$.

[2)] $\overset{1}{B} \overset{def}{=} B$ and $\overset{t+1}{B}(x_1^{(t+1)(n-1)+1}) \overset{def}{=} B(\overset{t}{B}(x_1^{t(n-1)+1}), x_{t(n-1)+2}^{(t+1)(n-1)+1}), t \in N$; cf. [12], VI-6.

[3)] The formulation and the proof of the theorem follow the idea of E.I. Sokolov from [4]. See, also [12]; Chapter IV and Appendix 2.

Proof. Firstly, we observe that under the assumptions the following statements hold

1° Let Y be an arbitrary (fixed) sequence over a set Q. Also let

(4) $\mathbf{A}(x_1^{k+1}) \overset{def}{=} A(\overline{x_j, \overset{(j)}{y}{}_1^{s-1}}\,\Big|_{j=1}^{k}, x_{k+1})$

for all $x_1^{k+1} \in Q$. Further on, let c_1^{n-2} be an arbitrary (fixed) sequence over a set Q. Then $(Q;\mathbf{A})$ is an $(k+1)-$group;

2° Let $(Q;A)$ $(k+1)-$group from 1°. Also let

(5) $\mathsf{E}(a_1^{n-2}) \overset{def}{=} \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, a_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1})$

for all $a_1^{n-2} \in Q$. Then E is an $\{1, k+1\}-$neutral operation of the $(k+1)-$group $(Q;\mathbf{A})$; and

3° Let $(Q;\mathbf{A})$ $(k+1)-$group from 1°. Then:

3_a° $(Q; B_Y, \varphi_Y, b_Y)$ is an $(k+1)HG-$algebra; and

3_b° $\mathbf{A}(x_1^{k+1}) = \overset{k+1}{B_Y}(x_1, \varphi_Y(x_2), \ldots, \varphi_Y^k(x_{k+1}), b_Y)$ for all $x_1^{k+1} \in Q$.

The proof of 1° : Def. 1.1 and by Def. 1.3.

Sketch of the proof of 2° :

$$\mathbf{A}(\mathsf{E}(a_1^{n-2}), a_1^{n-2}, x) \overset{(5)}{=} \mathbf{A}(\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, a_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), a_1^{n-2}, x)$$
$$\overset{(4)}{=} A(\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, a_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), \overline{\overset{(j)}{y}{}_1^{s-1}, a_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}, x)$$
$$\overset{2.5}{=} x.$$

Whence, by Def. 1.1, Prop. 2.1 and by Rem. 2.2, we obtain 2°.

Sketch of the proof of 3° :

$$B_Y(x,y) \overset{\hat{(1)},(1)}{=} A(x, \overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}, y)$$
$$\overset{(4)}{=} \mathbf{A}(x, c_1^{k-1}, y),$$

$$\varphi_Y(x) \overset{\hat{(2)},(2)}{=} A(\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), \overset{(1)}{y}{}_1^{s-1}, x, \overline{\overset{(j)}{y}{}_1^{s-1}, c_{j-1}}\,\Big|_{j=2}^{k})$$

$$\overset{(5)}{=} A(\mathsf{E}(c_1^{k-1}), \overset{(1)}{y}{}_1^{s-1}, x, \overline{\overset{(j)}{y}{}_1^{s-1}, c_{j-1}}\,\Big|_{j=2}^{k})$$

$$\overset{(4)}{=} \mathbf{A}(\mathsf{E}(c_1^{k-1}), x, c_1^{k-1}) \text{ and}$$

$$b_Y \overset{\hat{(3)},(3)}{=} A(\overline{\mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}}\,\Big|_{i=1}^{k-1}, \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, c_j}\,\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}))$$

$$\overset{(4),(5)}{=} \mathbf{A}(\overset{k+1}{\overline{\mathsf{E}(c_1^{k-1})}}\,|).$$

Whence, by Prop. 2.4, we obtain 3°.

In addition, by 3°, since Y is an arbitrary sequence over Q, we conclude that the statement (i) holds.

Finally, by 3° *[*3°_b*]*,$(\widehat{1})-(\widehat{3})$ and $(1)-(3)$, since Y is an arbitrary sequence over Q, we obtain also (ii). □

By Th.3.1 and by Prop.1.4, we have:

3.2. Theorem: *Let* $k > 1, s > 1, n = k \cdot s + 1, (Q; A)$ *be an polyagroup of the type* $(s, n-1)$, $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* Y *stands for sequence* $\overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} [= \overline{\overset{(j)}{y}{}_1^{s-1}} \Big|_{j=1}^{k}]$ *over* Q*. Also let* c_1^{k-1} *be an arbitrary (fixed) sequence over* Q*. Further on, let* $(1)-(3)$ *from Th.3.1 for all* $x, y, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$*. Finally, let for all* $x, y \in Q$ $(\widehat{1}) - (\widehat{3})$ *form Th.3.1, where* Y *is an arbitrary (fixed) sequence over* Q*. Then, the statements* (i) *and* (ii) *from Th. 3.1 hold.*

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