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A COMMENT ON NEAR-P-POLYAGROUPS (POLYAGROUPS)

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Abstract. In this article one proposition on $\{1,n\}-$neutral operation in near-P-polyagroups (polyagroups) is proved.

1. Introduction

1.1. Definition [1]: *Let $n \geq 2$ and let (Q,A) be an $n-$groupoid. We say that $(Q;A)$ is a Dörnete $n-$***group** *[briefly: $n-$group] iff is an $n-$semigroup and an $n-$quasigroup as well. (See, also [9].)*

1.2. Definition (Cf. [2],[3]): *Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q;A)$ be an $n-$groupoid. Then: we say that $(Q;A)$ is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

1° *For all $i,j \in \{1,\dots,n\}$ $i \equiv j$ (mod s), then $< i,j > -$associative law holds in $(Q;A)$; and*

2° *(Q,A) is an $n-$quasigroup.*

1.3. Definition [8]: *Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q;A)$ be an $n-$groupoid. Then: we say that $(Q;A)$ is a* **near-P-polyagroup** *[briefly: $NP-$polyagroup]* **of the type** $(s, n-1)$ *iff the following statements hold:*

$^\circ 1$ *For all $i,j \in \{1,\dots,n\}$ $(i < j)$ if $i,j \in \{t \cdot s + 1 | t \in \{0,1,\dots,k\}\}$, then the $< i,j > -$associative law holds in $(Q;A)$; and*

$^\circ 2$ *For all $i \in \{t \cdot s + 1 | t \in \{0,1,\dots,k\}\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality*

$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$

holds.[1)]

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[1)] For $s = 1$ $(Q;A)$ is a $(k+1)-$group, where $k+1 \geq 3;\ k > 1$.

1.4. Proposition: *Every polyagroup of the type* $(s, n-1)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$. [*By Def. 1.2 and by Def. 1.3.*]

2. Auxiliary propositions

2.1. Proposition [7]: *Let* $n \geq 2$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then, the following statements are equivalent:* (i) $(Q; A)$ *is an* $n-$*group;* (ii) *there are mapping* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the following laws hold in the algebra* $(Q; A,^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) *there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A,^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

2.3. Remark: $\mathbf{e}$ *is an* $\{1, n\}-$*neutral operation of* $n-$*groupoid* $(Q; A)$ *iff algebra* $(Q; A, \mathbf{e})$ *[of the type* $< n, n-2 >$*] satisfies the laws* (b) *and* $(\overline{b})$ *from 2.1 [4]. Operation* $^{-1}$ *from 2.1. [*$(c), (\overline{c})$*] is a generalization of the inverse operation in a group [5]. Cf. [9].*

2.4. Proposition[6]: *Let* $(Q; A)$ *be an* $n-$*group,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* $n \geq 3$. *Then, for every* $a_1^{n-2}, b_1^{n-2}, x \in Q$ *and for all* $i \in \{1, \ldots, n-1\}$ *the following equalities hold*

$A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x)$ *and*

$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) = A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))$.

2.4. Proposition [8] *Let* $k > 1$, $s \geq 1$ $n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then, the following statements are equivalent:* (i) $(Q; A)$ *is an* $NP-$**polyagroup of the type** $(s, n-1)$; (ii) *there are mapping* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A,^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,

$A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) *there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A,^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]*

$A(x_1^{(k-1)\cdot s} A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1}))$,
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$.
(Cf. Prop. 2.1 and Remark 2.2.)

3. Results

3.1. Theorem: *Let* $k > 1,\ s > 1,\ n = k\cdot s+1,\ (Q; A)$ *be an* $NP-$*polyagroup of the type* $(s, n-1)$ *and let* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation.*[2] *(Cf. Prop. 2.4.)*
Then, for all $i \in \{1, \ldots, k\}$, *for every* $x, b_1^{k-1} \in Q$ *and for every* $\overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$ *the following equalities hold:*

(1) $A(x, \overline{\overset{(j)}{y}{}_1^{s-1}, b_{i-1+j}}\Big|_{j=1}^{k-i}, \overset{(k-i+1)}{y}{}_1^{s-1}, \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, b_j}\Big|_{j=1}^{k-i}, \overset{(k)}{y}{}_1^{s-1}),$
$\overline{\overset{(j)}{y}{}_1^{s-1}, b_{j-(k-i+1)}}\Big|_{j=k-i+2}^{k}) = x$ *and*

(2) $A(\overline{b_{i-1+j}, \overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k-i}, \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, b_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), \overset{(k-i+1)}{y}{}_1^{s-1},$
$\overline{b_{j-(k-i+1)}, \overset{(j)}{y}{}_1^{s-1}}\Big|_{j=k-i+2}^{k}, x) = x.$[3]

Proof. Let $\overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1}$ [briefly: $\overline{\overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}$] be an arbitrary sequence over Q and let

$$Y \overset{def}{=} \overline{\overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}.$$

Also let

(3) $B_Y(x_1^{k+1}) \overset{def}{=} A(x_1, \overset{(1)}{y}{}_1^{s-1}, x_2, \overset{(2)}{y}{}_1^{s-1}, \ldots, x_k, \overset{(k)}{y}{}_1^{s-1}, x_{k+1})$
[briefly: $B_Y(x_1^{k+1}) \overset{def}{=} A(\overline{x_j, \overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}, x_{k+1})$, or $= A(x_1, \overline{\overset{(j)}{y}{}_1^{s-1}, x_{j+1}}\Big|_{j=1}^{k})$]
and

(4) $\mathbf{e}_Y(b_1^{k-1}) \overset{def}{=} \mathbf{e}(\overline{\overset{(j)}{y}{}_1^{s-1}, b_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1})$

for every $x_1^{k+1}, b_1^{k-1} \in Q$. Then, the following statements hold:

1° $(Q; B_Y)$ is a $(k+1)-$group; and

2° $\mathbf{e}_Y$ is a $\{1, k+1\}-$neutral operation of the $(k+1)-$group $(Q; B_Y)$.

The proof of 1° : By Def. 1.1 and by Def. 1.3.

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2) For $s = 1$ $(Q; A)$ is an $n-$group; $n = k+1 \geq 3$.

3) $\overline{\overset{(j)}{y}{}_1^{s-1}, b_{i-1+j}}\Big|_{j=t+1}^{t} \overset{def}{=} \emptyset,\ t \in N \cup \{0\}$.

The proof of 2° :
Let E be an $\{1, k+1\}-$neutral operation of the $(k+1)-$group $(Q; B_Y)$. (By 1° and by Prop. 2.1.) Whence, we have
(5_1) $B_Y(x, b_1^{k-1}, \mathsf{E}(b_1^{k-1})) = x$
for all $x, b_1^{k-1} \in Q$. Further on, we have

$$A(x, \overline{\overset{(j)}{y}\,{}_1^{s-1}, b_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}\,{}_1^{s-1}, \mathbf{e}(\overline{\overset{(j)}{y}\,{}_1^{s-1}, b_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}\,{}_1^{s-1})) = x$$

for all $x, b_1^{k-1} \in Q$. Whence, by (3), we obtain
(5_2) $B_Y(x, b_1^{k-1}, \mathbf{e}(\overline{\overset{(j)}{y}\,{}_1^{s-1}, b_j}\Big|_{j=1}^{k-1}, \overset{(k)}{y}\,{}_1^{s-1}))x$
for all $x, b_1^{k-1} \in Q$. Finally, by $(4), (5_1), (5_2), 1°$ and by Def. 1.1, we obtain $\mathsf{E} = \mathbf{e}_Y$.
In addition, by $1°, 2°$ and by Prop. 2.3, we have

$$B_Y(x, b_i^{k-1}, \mathbf{e}_Y(b_1^{k-1}), b_1^{i-1}) = x \text{ and}$$
$$B_Y(b_i^{k-1}, \mathbf{e}_Y(b_1^{k-1}), b_1^{i-1}, x) = x$$

for all $x, b_1^{k-1} \in Q$. Whence, since Y is an arbitrary sequence over Q, by (3) and by (4), we have (1) and (2) for all $x, b_1^{k-1} \in Q$ and for every sequence $\overset{(1)}{y}\,{}_1^{s-1}, \ldots, \overset{(k)}{y}\,{}_1^{s-1}$ over Q. □

By Th. 3.1 and by Prop. 1.4, we obtain:

3.2. Theorem: *Let $k > 1, s > 1, n = k \cdot s + 1, (Q; A)$ be an polyagroup of the type $(s, n-1)$ and let $\mathbf{e}$ its $\{1, n\}-$neutral operation. (Cf. Prop. 1.4 and Prop. 2.4.) Then, for all $i \in \{1, \ldots, k\}$, for every $x, b_1^{k-1} \in Q$ and for every $\overset{(1)}{y}\,{}_1^{s-1}, \ldots, \overset{(k)}{y}\,{}_1^{s-1} \in Q$ the equalities (1) and (2) hold.* □

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