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ON nB-ALGEBRAS

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Abstract. In the present paper: 1) we define nB-algebra $(Q; B, \mathbf{e})$ of the type $< n, n-2 >$, so that (among others) for $n = 2$ $(Q; B, \mathbf{e})$ is a B-algebra; and 2) nB-algebra $(Q; B, \mathbf{e})$ is described as an n-group $(Q; A)$.

1. Introduction

1.1. Definition (Cf. [6]) *Let (Q, B) be a groupoid. Let also e be a (fixed) element of the set Q. $(Q; B, e)$ is said to be a B-***algebra** *iff the following laws hold:*

(1) $B(x, x) = e$,
(2) $B(B(x, y), z) = B(x, B(z, B(e, y)))$ *and*
(3) $B(x, e) = x$.

1.2. Proposition: *Let $(Q; B, e)$ be a B-algebra. Then the following laws hold:*
(4) $B(B(x, a), B(e, a)) = x$ $[B(x, a) = u \Leftrightarrow x = B(u, B(e, a))]$ *and*

(5) $B(a, B(B(e, b), B(e, a))) = b$ $[B(a, x) = b \Leftrightarrow x = B(B(e, b), B(e, a))]$.

(See, also [6].)

1.3. Definition *Let $n \geq 2$ and let $(Q; B, \mathbf{e})$ be an algebra of the type $< n, n-2 >$. Then, we shall say that $(Q; B, \mathbf{e})$ is a nB-***algebra** *iff the following laws hold:*

$(\breve{1})$ $B(x, a_1^{n-2}, x) = \mathbf{e}(a_1^{n-2})$,

$(\breve{2})$ $B(B(x, y, b_1^{n-2}), z, a_1^{n-2}) = B(x, B(z, a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)), b_1^{n-2})$,

$(\breve{3})$ $B(B(x, a_1^{n-2}, y), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)) = x$ *and*

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($\breve{4}$) $B(a, a_1^{n-2}, B(B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, b), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a))) = b.$

1.4. Proposition: *Let $(Q; B, \mathbf{e})$ be an nB-algebra. Then the following law holds:*

($\breve{5}$) $B(B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x.$

Proof. Put $b = a$ in ($\breve{4}$), we have

$B(a, a_1^{n-2}, B(B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a))) = a,$

whence, by ($\breve{1}$), we obtain ($\breve{5}$). □

1.5. Remark: *For $n = 2$ ($\breve{1}$) – ($\breve{5}$). reduces, respectively, to following laws:* (1), (2), (4), (5), (3).

2. Auxiliary propositions

2.1. Definition [1] *Let $n \geq 2$ and let (Q, A) be an n-groupoid. We say that $(Q; A)$ is a Dörnte n-group [briefly: n-group] iff is an n-semigroup and an n-quasigroup as well. (See, also [8].)*

2.2. Proposition[4]: *Let $n \geq 2$ and let $(Q; A)$ be an n-groupoid. Then, the following statements are equaivalent: (i) $(Q; A)$ is an n-group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q; A,^{-1}, \mathbf{e})$ [of the type $< n, n-1, n-2 >$]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1})),$

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A(a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q; A,^{-1}, \mathbf{e})$ [of the type $< n, n-1, n-2 >$]

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}),$

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A, (a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}).$

2.3. Remark: $\mathbf{e}$ *is an $\{1, n\}$-neutral operation of n-groupoid $(Q; A)$ iff algebra $(Q; A, \mathbf{e})$ [of the type $< n, n-2 >$] satisfies the laws (b) and $(\overline{b})$ from 2.2 [2]. Operation $^{-1}$ from 2.2. [$(c), (\overline{c})$] is a generalization of the inverse operation in a group [3]. Cf. [8].*

2.4. Proposition[5]: *Let $n \geq 2$ and let $(Q; B)$ be an n-groupoid. Let also the following laws*

(α) $B(B(x, z, b_1^{n-2}), B(y, a_1^{n-2}, z), a_1^{n-2}) = B(x, y, b_1^{n-2})$ *and*

(β) $B(a, a_1^{n-2}, B(B(B(u, a_1^{n-2}, u), a_1^{n-2}, b), a_1^{n-2}, B(B(v, a_1^{n-2}, v), a_1^{n-2}, a))) = b$

hold in the n-groupoid $(Q;B)$*. Then, there is an n-group* $(Q;A)$ *such that the following equality holds* ${}^{-1}A = B$*, where*
(o) ${}^{-1}A(x, a_1^{n-2}, y) = z \overset{def}{\Leftrightarrow} A(z, a_1^{n-2}, y) = x$
for all $x, y, z \in Q$ *and for every sequence* a_1^{n-2} *over* Q.
(See, also [8]/XIII.)

2.5. Proposition[5]: *Let* $n \geq 2$ *and let* $(Q;B)$ *be an n-groupoid. Furthermore, let* $B =^{-1} A$ *(cf. (o) from 2.4). Then the* (α) *and* (β) *[from 2.4] hold in the n-groupoid* $(Q;B)$*. Moreover, for all* $x, y \in Q$ *and for every sequence* a_1^{n-2} *over* Q *the following equality holds*

$$(\gamma)\ B(x, a_1^{n-2}, y) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{-1}),$$

where ${}^{-1}$ *is an inverse operation of the n-group* $(Q;A)$ *[cf. 2.3]. (See, also [8]/VIII,XIII.)*

3. Results

3.1. Theorem: *Let* $n \geq 2$ *and let* $(Q;B,\mathbf{e})$ *be an nB-algebra. Then, there is an n-group* $(Q;A,^{-1},\mathsf{E})$ *such that the following equalities hold* ${}^{-1}A = B$ *and* $\mathsf{E} = \mathbf{e}$.

Proof. 1) Firstly, we prove that the assumptions the following statements hold:
1° For all $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q the following implication holds

$$B(x, a_1^{n-2}, a) = B(y, a_1^{n-2}, a) \Rightarrow x = y;$$

2° For all $x, y, u \in Q$ and for every sequence a_1^{n-2} over Q the following equivalence holds

$$(\varepsilon)\quad \begin{cases} B(u, a_1^{n-2}, y) = z \Leftrightarrow \\ u = B(z, a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)); \end{cases}$$

3° Law (α) from 2.4 holds in the n-groupoid $(Q;B)$; and
4° Law (β) from 2.4 holds in the n-groupoid $(Q;B)$.
Sketch of the proof of 1° :
$B(x, a_1^{n-2}, a) = B(y, a_1^{n-2}, a) \Rightarrow$
$B(B(x, a_1^{n-2}, a), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a)) =$
$B(B(y, a_1^{n-2}, a), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a)) \overset{(\breve{3})}{\Longrightarrow}$
$x = y.$
Sketch of the proof of 2° :
$B(u, a_1^{n-2}, y) = z \overset{1^\circ}{\Longleftrightarrow}$
$B(B(u, a_1^{n-2}, y), a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)) =$

$B(z, a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)) \overset{(\breve{3})}{\Longrightarrow}$
$u = B(z, a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y))$.
The proof of 3° :

Put (ε) /from 2°/ in $(\breve{2})$, we obtain
$B(B(x, y, b_1^{n-2}), B(u, a_1^{n-2}, y), a_1^{n-2}) = B(x, u, b_1^{n-2})$,
i.e., (α) from 2.4.
The proof of 4° :

Put $(\breve{1})$ in $(\breve{4})$, we have (β) from 2.4.

2) Finally, the following statement also holds:

5° There is an n-group $(Q; A,^{-1}, \mathsf{E})$ such that the following equalities hold

a) ${}^{-1}A = B$; and *b*) $\mathsf{E} = \mathbf{e}$.
The proof of $5^\circ - a)$: By $3^\circ, 4^\circ$ and by Prop. 2.4.
The proof of $5^\circ - b)$:

By $5^\circ - a)$, by Prop. 2.2 – (b) and by $(\breve{1})$ from 1.3, we hawe $\mathsf{E} = \mathbf{e}$. □

3.2. Theorem: *Let $n \geq 2$ and let $(Q; A,^{-1}, \mathbf{e})$ be an n-group. Let also $B =^{-1} A$. Then $(Q; B, \mathbf{e})$ is an nB-algebra.*

Proof. 1) By Prop. 2.2 – (a) and by (o) from Prop. 2.4, we conclude that the $(\breve{1})$ from Def. 1.3 holds in n-groupoid $(Q; B)$.

2) Firstly, by (c) from Prop. 2.2 and by (o) from 2.4, we have
$(a_1^{n-2}, y)^{-1} = B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, y)$.
Whence, by (γ) from Prop. 2.5, we conclude that the following equivalence holds

$$(\overline{\varepsilon}) \quad \begin{cases} B(y, a_1^{n-2}, z) = u \Leftrightarrow \\ y = B(u, a_1^{n-2}, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, z)). \end{cases}$$

Addition, by Prop. 2.5, (α) holds in n-groupoid $(Q; B)$. Finally, put $(\overline{\varepsilon})$ in (α), we have $(\breve{2})$.

3) By $(\overline{\varepsilon})$, we obtain $(\breve{3})$

4) By Prop. 2.5, (β) holds in n-groupoid $(Q; B)$. Put $(\breve{1})$ in (β), we have $(\breve{4})$. □

3.3. Remark: *For $n = 2$ see, also [7].*

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