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A NOTE ON NEARLY PARACOMPACTNESS

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Abstract. The purpose of the present paper is to study some properties of nearly paracompactness, α-Hausdorff subset, almost closed mappings and closed graphs.

1. Preliminaries

Our notation is standard. No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is regular open iff $IntClA = A$. A subset A of a space X is regular closed iff $ClIntA = A$, [13].

A set P is said to be δ-closed if for each point $x \notin P$, there exists an open set G containing x such that $\alpha(G) \cap P = \emptyset (\alpha(G) = \bar{G}^0)$. A set G is δ-open iff its complement is δ-closed, [15].

For every topological space (X, τ), the collection of all δ-open sets forms a topology for X, which is weaker than τ. This topology has a base consisting of all regular open subsets in (X, τ). We shall denote this topology by τ^*. An open cover $\mathcal{U}$ is *even* if there exists a neighborhood V of the diagonal in $X \times X$ such that $V[x] \subset U$, $(V[x] = \{y : (x, y) \in V\})$ for some $U \in \mathcal{U}$ [2].

A subset A of a space X is α-paracompact (α-nearly paracompact) with respect to a subset B iff for every open (regular open) cover $\mathcal{U} = \{U_i : i \in I\}$ of A there is an open family $\mathcal{V} = \{V_j : j \in J\}$ such that:

1. $\mathcal{V}$ refines $\mathcal{U}$,
2. $A \subset \cup\{V_j : j \in J\}$,
3. $\mathcal{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are mutually α-paracompact (α-nearly paracompact) iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and B is α-paracompact (α-nearly paracompact) with respect to the subset A [8].

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A subset A of a space X is α-Hausdorff iff for any two points a, b of a space X, where $a \in A$ and $b \in X \backslash A$, there are disjoint open sets U and V containing a and b respectively.

A subset A of a space X is α-regular (α-almost regular) iff for any point $a \in A$ and any open (regular open) set U containing a, there is an open set V such that $a \in V \subset ClV \subset U$, [11].

A mapping $f : X \to Y$ is almost closed iff for every regular closed set F in X, the set $f(F)$ is closed, [13].

A mapping $f : X \to Y$ has a closed graph $G(f)$ iff $G(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$, [1].

Let X be any topological space. Let (X, ρ) be a graph, where ρ is an equivalence relation in X. Let $\rho(X)$ denote ρ-class of x. Let X/ρ denote the set of all ρ-classes of equivalence relation ρ in X.

A subset A of a space X is admissible if it is the union of members of X/ρ, [2].

The equivalence relation ρ is upper semicontinuous (almost upper semicontinuous) if for each $D \in X/\rho$ and any open (regular open) neighborhood U of D in X there is an open admissible set V such that

$$D \subset V \subset U. \quad [2]$$

2. Results

Definicija 2.1. *An open cover $\mathcal{U}$ is δ-even if there exists a τ_p^*-open neighborhood V of the diagonal in $(X \times X, \tau_p)$, (where τ_p is the product topology), such that $V[x] \subset U$ for some $U \in \mathcal{U}$.*

Theorem 1. *An almost-regular space (X, τ) is nearly paracompact iff every regular open cover is δ-even.*

Proof. Necessity: Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open covering of X.

Since X is almost-regular and nearly paracompact, there exists by Theorem 1.6 in [15], a regular open star refinement $\mathcal{V} = \{V_\beta : \beta \in J\}$. Let $V = U\{V_\beta \times V_\beta : \beta \in J\}$. Then V is τ_ρ^*-open neighborhood of the diagonal in $X \times X$. We shall prove that for each $x \in X$, $V[x] \subset U_\alpha$ for some $\alpha \in I$.

If $y \in V[x]$, then $(x, y) \in V$ and hence there exists $\beta \in J$ such that $(x, y) \in V_\beta \times V_\beta$. Then $x \in V_\beta$ and $y \in V_\beta$. Hence, $y \in St(x, \mathcal{V})$. Then, $V[x] \subset St(x, \mathcal{V})$. Since $\mathcal{V}$ is star refinement of $\mathcal{U}$, therefore there exists $\alpha \in I$ such that $V[x] \subset St(x, \mathcal{V}) \subset U_\alpha$. Hence the result.

Sufficiency: We shall prove that (X, τ^*) is paracompact.

Let $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ be any τ^*-open covering of X. Then, there exists a τ-regular open refinement $\mathcal{U} = \{U_\beta : \beta \in J\}$ of $\mathcal{G}$.

Since, every regular open cover is δ-even, therefore there exists a τ_p^* open neighborhood V of diagonal in $X \times X$ such that for each $x \in X$, $V[x] \subset U_\beta$ for some $\beta \in J$. Then, (X, τ^*) is the space, which is regular and every τ^*-open cover is even, therefore (X, τ^*) is paracompact by Theorem 5.28 in [2].

Hence (X, τ) is nearly paracompact.

It would be interesting to see that ρ is closed in X^2, where ρ is an equivalence relation in X. Some results will be given in that sense.

In [2] the following result is proved.

Theorem A. *If X/ρ is a Hausdorff space, then ρ is closed in X^2. If the projection $P : X \cdot X/\rho$ is an open mapping and ρ is a closed subset of X^2, then X/ρ is Hausdorff.*

This result will be used in the proves of some theorems.

Theorem 2. *Let (X, ρ) be a graph, where ρ is an upper semicontinuous equivalence relation in X. For each $x \in X$, let the equivalence class $\rho(x)(X \in \rho(x))$ be an α-Hausdorff α-paracompact subset of X. Then ρ is closed in X^2.*

Proof. It follows that the quotient space X/ρ is Hausdorff (Theorem 2.1 in [11]. Thus, by Theorem A, ρ is closed in X^2.

Theorem 3. *Let X be any space. Let A be a compact subset of X such that every two points of A can be strongly separated by open subsets of X. Let $X \backslash A$ be an α-Hausdorff α-nearly paracompact subset of X. Let ρ be an equivalence relation in X defined as follows*

$$\rho = \{(x, y) : x, y \in X \backslash A \quad or \quad x = y\}.$$

Then the quotient space X/ρ is Hausdorff and compact. ρ is closed in X^2.

Proof. The equivalence relation ρ is upper semicontinuous such that $\rho(x)$ is α-Hausdorff α-nearly paracompact, for each $x \in X$. Thus, X/ρ is Hausdorff. Hence ρ is closed in X^2.

Now, let

$$\mathcal{U} = \{U_i : i \in I\}$$

be any open covering of X/ρ. Then

$$\{P^{-1}(U_i) : i \in I\}$$

is an open covering of X. Since A is compact, there is a finite subset I_0 of I such that

$$A \subset \cup\{P^{-1}(U_i) : i \in I_0\}.$$

Let U^* be such element of $\mathcal{U}$ such that

$$P(X\backslash A) \subset U^*.$$

($X\backslash A$ is a point of the quotient space X/ρ). Since

$$X = \cup\{P^{-1}(U_i) : i \in I_0\} \cup P^{-1}(U^*)$$

it follows that the family

$$\{U_i : i \in I_0, U^*\}$$

is a finite open covering of X/ρ, hence X/ρ is compact.

Theorem 4. *Let X be any regular space. Let X be an open subset of X such that every two points of A can be strongly separated by open subsets of X. Let $X\backslash A$ be α-Hausdorff. Let ρ be an equivalence relation in X defined by*

$$\rho = \{(x, y) : x, y \in X\backslash A \ or \ x = y\}.$$

Then ρ is closed in X^2.

Proof. It would be proved that X/ρ is a-Hausdorff space. If m and n are two points of A, then there are open sets U and V of X such that

$$m \in U,\ n \in V,\ U \cap V = \emptyset.$$

Let

$$U_1 = U \cap A,\ V_1 = V \cap A.$$

Then $P(U_1)$ and $P(V_1)$ are disjoint open sets of X/ρ containing $P(m)$ and $P(n)$ respectively.

Now, let

$$m \in A.$$

Since $X\backslash A$ is α-regular, it follows that there are two open subsets U and V of X such that

$$x \in U,\ X\backslash V,\ U \cap V = \emptyset.$$

Then

$$P(U) \text{ and } P(V)$$

are two disjoint open subsets of X/ρ containing $P(m)$ and $P(X\backslash A)$ respectively. Hence X/ρ is Hausdorff. By Theorem A it follows that ρ is closed in X^2.

In [2] the following result is proved.

Theorem B. *If f_1 and f_2 are two continuous mappings of a space X into a Hausdorff space Y, then*

$$F = \{x : f_1(x) = f_2(x)\}$$

is closed in X. If F is a dense subset of $X(C1F = X)$ then $f_1 = f_2$.

Similar results is not always true for two equivalence relation in X, as it is shown by the following example

Example 1. Let R be the real line (the set of all real numbers endowed with the usual topology in P). Define an equivalence relation ρ_1 in R as

$$\rho_1 = \{(x, y) : x, y \in [0, 1] \text{ or } x = y\}.$$

Define an equivalence relation ρ_2 in R as follows

$$\rho_2 = \{(x, y) : x, y \in [0, 2] \text{ or } x = y\}.$$

The equivalence relation ρ_1 is upper semicontinuous such that, for each $x \in R$, $\rho_1(x)$ is an α-Hausdorff compact subset of R, thus R/ρ_1 is Hausdorff. Hence ρ_1 is closed in R^2.

The equivalence relation ρ_2 is upper semicontinuous such that, for each $x \in R$, $\rho_2(x)$ is an α-Hausdorff compact subset of R, thus R/ρ_2 is Hausdorff. Hence ρ_2 is closed in R^2.

Now, we have

$$\rho_1(x) = \rho_2(x) \text{ for each } x \in A = R\backslash[0, 2]$$

but the subset A is not closed in R.

Theorem 5. *Let ρ_1 and ρ_2 be any two equivalence relation in X and let*

$$\mathcal{U} = \{\rho_1(x) : \rho_1(x) = \rho_2(x)\}.$$

If each member of $\mathcal{U}$ is an α-Hausdorff α-paracompact subset and $\mathcal{U}$ is locally finite, then

$$U = \cup(\rho_1(x) : \rho_1(x) \in \mathcal{U}\}$$

is closed in X. If U is dense, then $\rho_1 = \rho_2$, i.e. $X/\rho_1 = X/\rho_2$.

Proof. Since every α-Hausdorff α-paracompact subset is closed ([11]) it follows that $\rho_1(x)$ is closed for each $\rho_1(x) \in \mathcal{U}$. Since $\mathcal{U}$ is locally finite, it follows that

$$\begin{aligned} C1U &= C1 \cup \{\rho_1(x) : \rho_1(x) \in \mathcal{U}\} = \cup\{C1\rho_1(x) : \rho_1(x) \in \mathcal{U}\} = \\ &= \cup\{\rho_1(x) : \rho_1(x) \in \mathcal{U}\} = U. \end{aligned}$$

Since, any union of α-Hausdorff subsets is α-Hausdorff, thus U is α-Hausdorff. If U is dense in X, then $C1U = X$. Thus

$$\rho_1 = \rho_2 \text{ i.e. } X/\rho_1 = X/\rho_2.$$

3. Some examples of quotient spaces

Example 2. Let $\equiv_1$ be a congruence in the space R

$$(x \equiv_1 y) = ((\exists n \in Z)x = y + n).$$

Every equivalence class is a closed paracompact subset of R. The projection $P : R \to R/\equiv_1$ is an open (not a closed) mapping. The space $R/\equiv_1$ is Hausdorff. Thus, $\equiv_1$ is closed in R^2.

Example 3. Let

$$A = [0, 1] \subset R$$

have a subspace topology with respect usual topology. Let $\equiv_1$ be a congruence in the space R.

Every equivalence class is a closed compact subset of A. The projection $P : A \to A/\equiv_1$ is a closed (not an open) mapping. The space $A/\equiv_1$ is Hausdorff, hence $\equiv_1$ is closed in A^2 i.e. in R^2 (A^2 is closed in R^2). Since every continuous image of compact space is compact, it follows that $A/\equiv_1$ is compact.

Example 4. Let

$$A = [-1, 1] \subset R$$

have a subspace topology with respect usual topology. Define an equivalence relation ρ in A by taking

$$\rho = \{(x, y) : y = x \vee y = -x \text{ for } x \in (-1, 1)\} \cup \{(1, 1), (-1, 1)\}.$$

Every equivalence class is a closed compact subset of A. The projection $P : A \to A/\rho$ is an open mapping, hence A/ρ is not Hausdorff (ρ is not closed in A^2). A/ρ is T_1 (every point is closed).

Example 5. Define an equivalence relation ρ in R as follows

$$\rho = \{(x, y) : x, y \in Z \text{ or } x = y\}.$$

Every equivalence class is a closed α-paracompact subset of R.

It follows that the quotient space R/ρ is Hausdorff, thus ρ is closed in R^2. The space R/ρ is not locally compact (for point $P(Z)$ there is not an open subset V such that $C1V$ is compact in R/ρ).

Example 6. Let

$$A = [0, \infty] \subset R$$

have a subspace topology with respect usual topology. Define an equivalence relation ρ by taking

$$\rho = \{(x, y) : y = x \vee y = 1/x \text{ for } x \notin 0\} \cup \{(0, 0)\}.$$

Every equivalence class is a closed compact subset of A. ρ is closed in A^2. The projection P is an open (not a closed) mapping). Hence A/ρ is Hausdorff. The space A/ρ is compact.

4. References

[1] T.R. Hamlet and L.L. Herrington, *The closed graph and P-closed graph properties in general topology*, Amer. Math. Soc. Providence R.I. (1981), 1–68.

[2] J.L. Kelley, *General Topology*, Van Nostrand, Princeton, N.J., 1961.

[3] I. Kovačević, *A note on almost closed mappings and nearly paracompactness*, Univ. u N. Sadu, Zb. rad. Prir. mat. Fak. Mat. Ser. **13** (1983), 87–101.

[4] I. Kovačević, *A note on quotient spaces and paracompactness*, Univ. u N. Sadu, Zb. rad. Prir. mat. Fak. Mat. Ser. **11** (1981), 89–99.

[5] I. Kovačević, *On α-Hausdorff subsets, almost closed mappings and almost upper semicontinuous decomposition*, Indian J. Pure Appl. Math. **20(4)** (1989), 334–340.

[6] I. Kovačević, *On almost closed mappings, paracompactness and partial equivalence relations*, Indian J. Pure Appl. Math. **25(9)** (1994), 949–954.

[7] I. Kovačević, *On M_N subsets*, J. Math. Novi Sad **27(1)** (1997), 45–52.

[8] I. Kovačević, *On subsets, almost closed mappings and paracompactness*, Glasnik matematički **24(44)** (1989), 125–132.

[9] I. Kovačević, *Some properties of M_N subset and almost closed mappings*, Indian J. Pure Appl. Math. **27(9)** (1996), 875–881.

[10] I. Kovačević, *Some properties of nearly and almost compact spaces*, Math. Balkanica **7** (1977), 195–200.

[11] I. Kovačević, *Subsets and paracompactness*, Zb. rad. Prir. mat. Fak. Univ. u N. Sadu **14(2)** (1984), 79–87.

[12] T. Noiri, *Almost-continuity and some separation axioms*, Glasnik matematički **9(29)** (1974), 131–135.

[13] M.K. Singal and A.R. Singal, *Almost continuous mappings*, Yokohoma Math. J. **16**(1968), 63–73.

[14] M.K. Singal and Asha Mathur, *On nearly compact spaces*II, Boll. Un. Mat. Ital. (4) **2**(1969), 702–710.

[15] M.K. Singal and S.P. Arya, *On nearly paracompact spaces*, Matem. vesnik **6**(21) (1969), 3–16.

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