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FIXED POINTS AND ADMISSIBLE SETS

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Abstract. This paper is to present two fixed points theorems for nonexpansive mappings, defined on general convex topological spaces. This theorems, for mappings defined on metric spaces are given by Y. Kijma and W. Takahashi ([3]).

1. Definitions

Definicija 1.1. *Let X be a topological space and let $A : X \times X \to [0, +\infty)$ be a function. If there exists a mapping $G : X \times X \times [0, 1] \to X$, satisfying the following inequality*

$$A\Big(z, G(x, y, \lambda)\Big) \le \max\Big\{A(z, x), A(z, y)\Big\}, \tag{S}$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0, 1]$, then we say that X possesses general convex structure denoted by $G(x, y, \lambda)$. The topological space X with general convex structure is called a general convex topological space. We say that X is a bounded general convex topological space if A is a bounded function.

Definicija 1.2. *A subset K of X, is general convex if $G(x, y, \lambda) \in K$, for all $x, y \in K$ and arbitrary $\lambda \in [0, 1]$.*

This definiton of general convex topological spaces has been given by M. Tasković (see [5]). Condition (S) from this definition provide that sets $B(x, \rho) = \{y \in X : A(x, y) \le \rho\}$ (which we can call closed balls), are general convex sets. This definition is one natural generalisation of Takahashi's [4] definition of convex structure in convex metric spaces.

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Definicija 1.3. *For $S \subseteq X$, the diameter of S is defined by*

$$\delta(S) := \sup\{A(x,y) : x, y \in S\}.$$

Point $x \in S$ is a diametral point of S if $\delta(S) := \sup\{A(x,y) : y \in S\}$.

Definicija 1.4. *A subset $S \subseteq X$, is said to be admissible if it is an intersection of closed balls.*

2. Results

Theorem 1. *Let X be a bounden general convex topological space with general convex structure $G(x,y,\lambda)$ satisfying conditions:*

(1) *If a family of closed balls has finite intersection property, then the intersection of the family is nonempty.*
(2) *Each admissible subset which contains more than one point contains a nondiametral point.*

If f is a nonexpansive mapping of X into X than f has a fixed point.

Proof. Let Φ be a family of all nonempty, admissible subsets of X, each of which is mapped into itself by f. The family Φ is partially ordered, by usual set inclusion, and nonempty, because A is a bounded function, and follows that $X \in \Phi$. We shall provide that the family Φ has the minimal element. Let $\{S_i : i \in I\}$ be a totally ordered subfamily of Φ. Then, set $S = \cap_{i \in I} S_i$ is a lower bound of the family $\{S_i : i \in I\}$. Because S_i is a admissible set, we can write

$$S_i = \cap\{B(x_j, r_j) : j \in J_i\}.$$

From last follows that S can be interpreted in the following form

$$S = \cap\{B(x_j, r_j) : j \in J\},$$

where J is the union of index sets J_i. Let

$$B\Big(x_{j_1}, r_{j_1}\Big), \dots, B\Big(x_{j_n}, r_{j_n}\Big),$$

be an arbitrary finite subfamily of $\{B(x_j, r_j) : j \in J\}$. Each of balls $B(x_{j_k}, r_{j_k})$, where $k = 1...n$, contains some of sets S_{l_k}, from totally ordered subfamily $\{S_i : i \in I\}$, and follows that $\cap_{k=1}^{n} S_{l_k} \subseteq \cap_{k=1}^{n} B(x_{j_k}, r_{j_k})$. Since the subfamily $\{S_i : i \in I\}$ is a totally ordered, $\cap_{k=1}^{n} S_{l_k}$ is nonempty and from previous we can observe that $\cap_{k=1}^{n} B(x_{j_k}, r_{j_k})$ is a nonempty.

From (1) follows that family $\{B(x_j, r_j) : j \in J\}$ has nonempty intersection and so $S \neq \emptyset$. We conclude that S is the lower bound of family $\{S_i : i \in I\}$. By Zorn's lemma, we conclude that Φ has a minimal element, denoted by F. Like in the paper [3] we can prove that F has only one element which is the fixed point of mapping f.

Theorem 2. *Let X be a bounden general convex topological space with general convex structure $G(x, y, \lambda)$ satisfying conditions (1) and (2). Let $\mathcal{F} = \{T \mid T : X \to X\}$ be a finite commuting family of nonexpansive mappings of X into X than $\mathcal{F}$ has a common fixed point.*

Proof. Let Φ be a family of all nonempty, admissible subsets of X, each of which is mapped into itself by f. The family Φ is partially ordered, by usual set inclusion, and nonempty.

From Theorem 1 follows that Φ has the minimal element, denoted by F. Denote $\mathcal{F} = \{T_1, ..., T_n\}$. Let us note a set $W = \{x \in F : T_1 T_2 \cdots T_n(x) = x\}$.

From Theorem 1 follows that mapping $T_1 T_2 \cdots T_n : F \to F$ has a fixed point, and it follows that $W \neq \emptyset$. It is easy to prove that $T_i(W) = W$. If $x \in W$, than the element $T_1 T_2 \cdots T_{i-1} T_{i+1} \cdots T_n(x) \in W$ is mapped by T_i to x, and follows that $x \in T_i(W)$. Conversely, if $x \in T_i(W)$, than exists $y \in W$ so that $T_i(y) = x$.

From $x = T_i(y) = T_i T_1 T_2 \cdots T_n(y) = T_1 T_2 \cdots T_n T_i(y) = T_1 \cdots T_n(x)$, follows that $x \in W$. Let K be the least admissible set containing $W \subseteq F$. Since F is a admissible set containing W, follows that $K \subseteq F$. We shall show that set K has only one point. In this goal, let us suppose that K contains at least two elements. Then, from (2) it follows that point $x_0 \in K$ exists, such that

$$\delta_{x_0}(K) := r < \delta(K).$$

The set $C := F \cap [\cap\{B(z, r) : z \in K\}]$ is a nonempty, admissible subset of F. We can note that $C = C_1$ where $C_1 := F \cap [\cap\{B(z, r) : z \in W\}]$. Since $W \subseteq K$, it follows that $C \subseteq C_1$. Conversely, if $x \in C_1$, then $A(x, z) \leq r$, for all $z \in W$, and $W \subseteq B(x, r)$. Because K is the least admissible set containing W, then $K \subseteq B(x, r)$ and $A(x, z) \leq r$, for all $z \in K$, from which it follows that $x \in C$. Because T_i is nonexpansive mapping for all $c \in C$ and $w \in W$ follows that

$$A(T_i(c), T_i(w)) \leq A(c, w) \leq r.$$

From $T_i(W) = W$, we conclude that $T_i(C) \subseteq C$, and since C is nonempty, admissible and $C \subseteq F$, then, from minimality of F, follows that $C = F$. Finally, from

$$\delta(K) = \delta(F \cap K) = \delta(C \cap K) \leq r < \delta(K),$$

follows contradiction. So, set K contains only one point, and this point is the common fixed point for family $\mathcal{F}$.

3. References

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