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WEAK ASYMPTOTIC EQUIVALENCE RELATION AND INVERSE FUNCTIONS IN THE CLASS OR

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Abstract. If $f(x)$ is a continuous, strictly increasing and unbounded function defined on an interval $[a,+\infty)$ $(a>0)$, in this paper we shall prove that $f^{-1}(x)$ $(x \geq a)$ belongs to the Karamata class OR of all $\mathcal{O}$–regularly varying functions, if and only if for every function $g(x)$ $(x \geq a)$ which satisfies $f(x) \asymp g(x)$ as $x \to +\infty$, we have $f^{-1}(x) \asymp g^{-1}(x)$ as $x \to +\infty$. Here, $\asymp$ is the weak asymptotic equivalence relation. We shall also prove some variants of the previous theorem, in which, except the weak, we also deal with the strong asymptotic equivalence relation.

1. Introduction and results

A measurable function $f:[a,+\infty) \mapsto (0,+\infty)$ $(a>0)$ is called $\mathcal{O}$–regularly varying in the Karamata sense [1], if it satisfies

$$\overline{\lim}_{x\to+\infty}\frac{f(\lambda x)}{f(x)} = k_f(\lambda) < +\infty, \quad \lambda > 0. \tag{1}$$

The class of all such functions is denoted OR, and as is well known, this class is one of the essential objects in the qualitative analyse of divergent asymptotic processes [1].

An $\mathcal{O}$– regularly varying function $f:[a,+\infty) \mapsto (0,+\infty)$ $(a>0)$, is called *slowly varying* in the Karamata sense [1], if it satisfies

$$k_f(\lambda) = 1, \quad \lambda > 0. \tag{2}$$

The class of all such functions is denoted SV, and it is the most important object in the Karamata theory of regular variability [3].

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Two positive functions $f(x)$, $g(x)$ $(x \geq a)$ $(a > 0)$, are called *weakly asimptoticaly equivalent*, and denoted $f(x) \asymp g(x)$ as $x \to +\infty$, if there is some $\varepsilon > 1$ such that

$$\frac{1}{\varepsilon} \leq \frac{f(x)}{g(x)} \leq \varepsilon, \quad x \geq x_0(\varepsilon). \tag{3}$$

Next, they are called *strongly asymptotically equivalent*, and denoted $f(x) \sim g(x)$ as $x \to +\infty$, if (3) is satisfied for every $\varepsilon > 1$.

Next, let $\mathcal{A}$ be the class of all positive functions, defined for $x \geq a$, for a fixed $a > 0$, which are continuous, increasing and unbounded on the interval $[a, +\infty)$.

Assume that f and g are two functions from the class $\mathcal{A}$. We shall discuss some conditions under which we have that we have (4) implies (5), where (4) and (5) are the next relations:

$$f(x)\ \rho_1\ g(x), \quad x \to +\infty, \tag{4}$$

$$f^{-1}(x)\ \rho_2\ g^{-1}(x), \quad x \to +\infty, \tag{5}$$

and ρ_1 and ρ_2 are some relations from the set $\{\asymp, \sim\}$.

We notice that the case $\rho_1 = \rho_2 = \sim$ is considered in the paper [2].

Theorem 1. (a) *Suppose that f and g are two functions from the class $\mathcal{A}$, next at least one of the functions f^{-1}, g^{-1} belongs to the class OR, and relation (4) is satisfied for $\rho_1 = \asymp$. Then the relation (5) is also true with $\rho_2 = \asymp$.*

(b) *If $f \in \mathcal{A}$, and every function $g \in \mathcal{A}$ which satisfies (4) with $\rho_1 = \asymp$ also satisfies (5) with $\rho_2 = \asymp$, then $f^{-1} \in OR$, and $g^{-1} \in OR$.*

Theorem 2. (a) *Suppose that $f, g \in \mathcal{A}$, next at least on of the functions f^{-1}, g^{-1} belongs to the class SV, and relation (4) is true for $\rho_1 = \asymp$. Then the relation (5) is also true with $\rho_2 = \sim$.*

(b) *If $f \in \mathcal{A}$ and for every function $g \in \mathcal{A}$ which satisfies (4) for $\rho_1 = \asymp$, (5) is also true with $\rho_2 = \sim$, then $f^{-1} \in SV$, and $g^{-1} \in SV$.*

Theorem 3. (a) *Suppose that $f, g \in \mathcal{A}$, next at least one of the functions $f^{-1}, g^{-1} \in OR$, and relation (4) is true for $\rho_1 = \sim$. Then the relation (5) is also true for $\rho_2 = \asymp$.*

(b) *If $f \in \mathcal{A}$, and for every function $g \in \mathcal{A}$ which satisfies (4) with $\rho_1 = \sim$, (5) is also true with $\rho_2 = \asymp$, then $f^{-1} \in OR$ and also $g^{-1} \in OR$.*

We notice that previous theorems are in fact some characterizations of the Karamata classes $OR \cap \mathcal{A}$ and $SV \cap \mathcal{A}$.

2. Proofs of theorems

Proof of Theorem 1. (a) Without loos of generality, we can assume that the function $g^{-1} \in OR$. By relation $f(x) \asymp g(x)$ $(x \to +\infty)$, we have that $\underline{\lim}_{x\to+\infty}\dfrac{f(x)}{g(x)} = m > 0$. Therefore, there is a $\lambda_1 > \dfrac{1}{m}$ such that $f(x) \geq \dfrac{1}{\lambda_1}g(x)$ for $x \geq x_0(\lambda_1)$. Thus, for all enough large x we have

$$\frac{f^{-1}(x)}{g^{-1}(x)} \leq \frac{g^{-1}(\lambda_1 x)}{g^{-1}(x)}.$$

Hence we get

$$\overline{\lim}_{x\to+\infty}\frac{f^{-1}(x)}{g^{-1}(x)} \leq k_{g^{-1}}(\lambda_1) < +\infty.$$

Besides, we have that $\overline{\lim}_{x\to+\infty}\dfrac{f(x)}{g(x)} = M < +\infty$. Therefore, there is a positive number $\lambda_2 < \dfrac{1}{M}$ such that $f(x) \leq \dfrac{1}{\lambda_2}g(x)$, for $x \geq x_0(\lambda_2)$. This means that for all enough large x we have

$$\frac{f^{-1}(x)}{g^{-1}(x)} \geq \frac{g^{-1}(\lambda_2 x)}{g^{-1}(x)}.$$

Hence, we find that

$$\underline{\lim}_{x\to+\infty}\frac{f^{-1}(x)}{g^{-1}(x)} \geq \frac{1}{k_{g^{-1}}(1/\lambda_2)} > 0.$$

Therefore, we have that $f^{-1}(x) \asymp g^{-1}(x)$ as $x \to +\infty$.

(b) Suppose that $f \in \mathcal{A}$. Let $\lambda > 0$ and $g(x) = \lambda \cdot f(x)$ $(x \geq a, a > 0)$. Then we have that $g \in \mathcal{A}$ and $f(x) \asymp g(x)$, $x \to +\infty$. Therefore, $f^{-1}(x) \asymp g^{-1}(x)$, $x \to +\infty$, so that

$$\begin{aligned}
+\infty > A(\lambda) \geq \overline{\lim}_{x\to+\infty}\frac{f^{-1}(x)}{g^{-1}(x)} &= \\
&= \overline{\lim}_{t\to+\infty}\frac{f^{-1}(g(t))}{g^{-1}(g(t))} = \\
&= \overline{\lim}_{t\to+\infty}\frac{f^{-1}(\lambda f(t))}{t} = \\
&= \overline{\lim}_{t\to+\infty}\frac{f^{-1}(\lambda f(t))}{f^{-1}(f(t))} = \\
&= \overline{\lim}_{p\to+\infty}\frac{f^{-1}(\lambda p)}{f^{-1}(p)} = \\
&= k_{f^{-1}}(\lambda), \quad \lambda > 0.
\end{aligned}$$

Hence, the function $f^{-1} \in OR$. Besides, for every function $g \in \mathcal{A}$ for which $f(x) \asymp g(x)$, $x \to +\infty$, we have that $g^{-1}(x) = h(x) \cdot f^{-1}(x)$ for $0 < \frac{1}{A(g)} \le h(x) \le A(g) < +\infty$ if $x \ge x_0(g)$. Therefore,

$$k_{g^{-1}}(\lambda) \le k_{f^{-1}}(\lambda) \cdot A^2(g) < +\infty, \quad \lambda > 0.$$

Hence, $g \in OR$. □

Theorem 2 can be proved analogously as the Theorem 1, and the Theorem 3 (a) is a direct consequence of the Theorem 1 (a). So, we shall only prove Theorem 3 (b).

Proof of Theorem 3. (b) Suppose that $f \in \mathcal{A}$. Then $k_{f^{-1}}(\lambda) \le 1$ for $0 < \lambda \le 1$. Next notice that if $g \in \mathcal{A}$ and $f(x) \sim g(x)$, $x \to +\infty$, then $f^{-1}(x) \asymp g^{-1}(x)$, $x \to +\infty$, so that

$$\begin{aligned} +\infty > A(g) \ge \overline{\lim}_{x\to+\infty} \frac{f^{-1}(x)}{g^{-1}(x)} &= \\ = \overline{\lim}_{t\to+\infty} \frac{f^{-1}(g(t))}{g^{-1}(g(t))} &= \\ = \overline{\lim}_{t\to+\infty} \frac{f^{-1}(g(t))}{f^{-1}(f(t))}&. \end{aligned}$$

Next, let $\alpha(t)$ ($t \ge a$; $a > 0$) be an arbitrary positive continuous function such that $\alpha(t) \ge 1$ and $\alpha(t) \to 1+$ for $t \to +\infty$. We shall discuss the function $\beta(t) = \alpha(f(t))$ for $t \ge a$. If the function $h(t) = \beta(t)\, f(t)$, $t \ge a$, is increasing, then $h \in \mathcal{A}$ and we have $f(t) \sim h(t)$ as $t \to +\infty$. Hence we get

$$\begin{aligned} +\infty > A(h) \ge \overline{\lim}_{t\to+\infty} \frac{f^{-1}(\beta(t)f(t))}{f^{-1}(f(t))} &= \\ = \overline{\lim}_{t\to+\infty} \frac{f^{-1}(\alpha(f(t))\, f(t))}{t} &= \\ = \overline{\lim}_{p\to+\infty} \frac{f^{-1}(\alpha(p) \cdot p)}{f^{-1}(p)}&. \end{aligned}$$

If $h(t)$, $t \ge a$, is not increasing, then we can consider the function $r(t) = \max\limits_{a\le x\le t} h(x)$, $t \ge a$. This function is continuous, nondecreasing and satisfies $r(t) \to +\infty$, $t \to +\infty$, and $r(t) \ge \beta(t) \cdot f(t)$, $t \ge a$. Let $\varepsilon > 0$. Then there is a $t_0 \ge a$ such that

$$1 \le h(t)/f(t) < 1 + \varepsilon, \quad t \ge t_0,$$

and next there is a $t_1 > t_0$ such that

$$h(t) \ge \max_{a\le u\le t_0} h(u),$$

for all $t \geq t_1$. Then for every $t \geq t_1$ and a function $v(t) \in [t_0, t_1]$ we have that

$$\begin{aligned} 1 \leq \frac{r(t)}{f(t)} &= \frac{1}{f(t)} \max_{a \leq u \leq t} h(u) = \\ &= \frac{1}{f(t)} \max_{t_0 \leq u \leq t} h(u) = \frac{h(v(t))}{f(t)} \leq \\ &\leq \frac{h(v(t))}{f(v(t))} < 1 + \varepsilon. \end{aligned}$$

Hence we get $r(t) \sim f(t)$, $t \to +\infty$. Define next the function $r_1(t)$, $t \geq a$, with $r_1(t) = r(t) + u(t)$, where $u(t)$, $t \geq a$ is an increasing, continuous function such that $u(t) \to 1-$, $t \to +\infty$. Then $r_1 \in \mathcal{A}$ and we have that $r_1(t) \sim r(t) \sim f(t)$, $t \to +\infty$. Therefore we find that

$$\overline{\lim}_{t \to +\infty} \frac{f^{-1}(\beta(t) f(t))}{f^{-1}(f(t))} \leq \overline{\lim}_{t \to +\infty} \frac{f^{-1}(r_1(t))}{f^{-1}(f(t))} \leq A(r_1) < +\infty.$$

Hence,

$$\begin{aligned} \overline{\lim}_{t \to +\infty} \frac{f^{-1}(\beta(t) f(t))}{f^{-1}(f(t))} &= \overline{\lim}_{t \to +\infty} \frac{f^{-1}(\alpha(f(t)) f(t))}{f^{-1}(f(t))} = \\ &= \overline{\lim}_{p \to +\infty} \frac{f^{-1}(\alpha(p)\, p)}{f^{-1}(p)} \leq A(r_1) < +\infty. \end{aligned} \tag{6}$$

Now we shall prove that $\overline{\lim}_{\substack{\lambda \to 1+ \\ x \to +\infty}} \dfrac{f^{-1}(\lambda x)}{f^{-1}(x)} = A < +\infty$, where A is a finite real number.

On the contrary, suppose that there are some sequences (λ_n), (x_n) such that $\lambda_n \to 1+$ and $x_n \to +\infty$ as $n \to \infty$, such that

$$\frac{f^{-1}(\lambda_n x_n)}{f^{-1}(x_n)} \to +\infty, \quad n \to \infty.$$

Without loss of generality, we can assume that $x_n \geq a$ $(n \in N)$, next that (x_n) is an increasing sequence, and that $\lambda_n \geq 1$ for every $n \in N$. Define a function $\alpha(x)$, $x \geq a$, with $\alpha(x_n) = \lambda_n$ $(n \in N)$, and $\alpha(x) = \lambda_1$ for $x \in [a, x_1)$, and on the interval (x_k, x_{k+1}) $(k \geq 1)$ take the usual linear and continuous extension. The so obtained function $\alpha : [a, +\infty) \mapsto [1, +\infty)$ is continuous, and we have $\alpha(x) \to 1+$ as $x \to +\infty$. Consequently, we get

$$\overline{\lim}_{n \to +\infty} \frac{f^{-1}(\alpha(x_n)\, x_n)}{f^{-1}(x_n)} = \overline{\lim}_{n \to +\infty} \frac{f^{-1}(\lambda_n\, x_n)}{f^{-1}(x_n)} = +\infty,$$

what is a contradiction to (6).

Hence, for every $\varepsilon > 0$ there is an $x_0 \geq a$ and a $\delta > 0$, so that for all $x \geq x_0$ and all $\lambda \in [1, 1+\delta]$, we have

$$1 \leq \frac{f^{-1}(\lambda x)}{f^{-1}(x)} \leq A + \varepsilon.$$

Thus, if $\lambda \in (0, 1+\delta]$ we have that $k_{f^{-1}}(\lambda) \leq A + \varepsilon < +\infty$. Since f^{-1} is increasing, we have that $k_{f^{-1}}(\lambda) < +\infty$ for all $\lambda > 0$ (see e.g. [3]), so we find that $f^{-1} \in OR$.

The remaining part of the proof coincides with the corresponding part of the proof of Theorem 1 (b). □

Corollary. *Assume that both* $f, g \in \mathcal{A}$.

(a) *If at least one of the functions* $f^{-1}, g^{-1} \in OR$*, and* (4) *holds for* $\rho_1 = \asymp$ *or* $\rho_1 = \sim$*, then both functions* $f^{-1}, g^{-1} \in OR$.

(b) *If at least one of the functions* $f^{-1}, g^{-1} \in SV$*, and* (4) *is true for* $\rho_1 = \asymp$ *or* $\rho_1 = \sim$*, then both functions* $f^{-1}, g^{-1} \in SV$.

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