Mathematica Moravica
Vol. **6** (2002), 93–96

A GENERAL FIXED POINT THEOREM FOR MAPPINGS IN PSEUDOCOMPACT TICHONOFF SPACES

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Abstract. In [2] Jain and Dixit have obtained some interesting results on fixed points in pseudocompact Tichonoff spaces. In this paper we present a general fixed point theorem in pseudocompact Tichonoff spaces for mappings satisfying an implicit relation which generalize Theorem 1 and 2 from [2] and some results from [3] and [4].

1. Introduction

A topological space P is said to be pseudocompact iff every real valued continuous function on P is bounded. Every compact space is pseudocompact but the converse is not true (Ex.5,pp.150 [1]).

Remark 1. In a metric space the notions "compact" and "pseudo-compact" coincide. By Tichonoff space we mean a completely regular Hausdorff space. It is observed that the product of two Tichonoff space is again a Tichonoff space, whereas the product of two paracompact spaces need be pseudocompact.

In [2] the following theorems are proved:

Theorem 1. *Let P be a pseudocompact Tichonoff space and f be a non-negative real valued continuous function over $P \times P$ satisfying*

$$\begin{cases} f(x,x) = 0 \text{ for all } x \in P \text{ and} \\ f(x,y) \leq f(x,z) + f(z,y) \text{ for all } x,y,z \in P \end{cases} \tag{1.1}$$

If $T : P \to P$ is a continuous mapping satisfying

$$\big[f(Tx,Ty)\big]^2 < f(x,Tx)f(y,Ty) + af(x,Ty)f(y,Tx) \tag{1.2}$$

for all distinct points x, $y \in P$, where $a \geq 0$, then T has a fixed point in P, which is unique whenever $a \leq 1$.

AMS (MOS) Subject Classification 1991. Primary: 54H25.
Key words and phrases: fixed point, pseudocompact spaces, Tichonoff space, implicit relation.

Theorem 2. *Let P and f be the same as defined in Theorem 1. If $T : P \to P$ is a continuous mapping satisfyiug*

$$\begin{aligned}&\big[f(Tx,Ty)\big]^2 < a\big[f(x,Tx)f(y,Ty)+f(x,Ty)f(y,Tx)\big]+\\ &+b\big[f(x,Tx)f(y,Tx)+f(x,Ty)f(y,Ty)\big]\end{aligned} \tag{1.3}$$

for all distinct points $x, y \in P$, where $a, b \geq 0$ and $0 < a + 2b < 1$.

Then T has a unique fixed point.

Remark 2. In the proofs of Theorems l and 2 the authors suppose that

$$f(x,y) = f(y,x) \quad \text{for all } x, y \in P.$$

The purpose of this paper is to prove a general fixed point theorem in pseudo-compact Tichonoff space for mappings satistying an imiplicit relation which generalize Theorems l and 2 and some results from [3] and [4].

2. Implicit relations

Let $\mathcal{F}_5$ be the set of all real coutinuous functions $F(t_1, \dots t_5)$: $R_+^5 \to R$ satisfying the following conditions:

(F$_1$): F is non-increasing in variable t_5;

(F$_2$) If $F(u, v, u, 0, u + v) < 0$ for $u, v > 0$, then $u < v$.

The function F satisfies the condition (F_U) if $F(u, 0, 0, u, u) \geq 0$, $\forall u \geq 0$.

Ex.l. $F(t_1, \dots t_5) = t_1^2 - t_2 t_3 - a t_4 t_5$ where $a \geq 0$.

(F_1): Obvious,

(F_2): $F(u, v, u, 0, u + v) < 0$ implies $u^2 - uv < 0$ which implies $u < v$ for $u, v > 0$.

(F_U): Let $0 < a \leq 1$ then $F(u, 0, 0, u, u) = u^2(1 - a) \geq 0$, $\forall u \geq 0$.

Ex.2. $F(t_1, \dots, t_5) = t_1^2 - a(t_2 t_3 + t_4 t_5) - b(t_2 t_4 + t_3 t_5)$ where $0 < a + 2b < 1$, $a \geq 0$, $b \geq 0$.

(F_1): Obvious.

(F_2): Let $F(u, v, u, 0, u + v) = u^2 - auv - buv - bu(u + v) < 0$. If $u \geq v$, then $u^2(1 - a - 2b) < 0$, a contradiction. Thus $u < v$.

Ex.3. $F(t_1, \dots, t_5) = t_1^2 - t_1(a t_2 + b t_3) - c t_4 t_5$ where $a, b, c \geq 0$ and $0 < a + b < 1$.

(F_1): Obvious.

(F_2): Let $F(u, v, u, 0, u + v) < 0$ then $u < \dfrac{a}{1-b} v < v$.

(F_U): If $0 < c \leq 1$, then $F(u, 0, 0, u, u) = u(1 - c) \geq 0$, $\forall u \geq 0$.

Ex.4. $F(t_1, \dots, t_5) = t_1^3 + t_1^2 t_3 + t_1 t_2^2 - c\dfrac{t_2^2 t_3^2 + t_4^2 t_5^2}{t_1 + t_2 + t_3 + 1}$ where $0 < c < 1$.

(F_1): Obvious.

(F_2): If $F(u,v,u,0,u+v) < 0$ then $u^3 + u^2v + uv^2 - c\dfrac{u^2v^2}{2u+v+1} < 0$ which implies $u^3 - \dfrac{cu^2v^2}{2u+v+1} < 0$ and thus $u < \dfrac{cv^2}{2u+v+1} < cv \leq v$.

(F_3): $F(u,0,0,u,u) = u^3 \cdot \dfrac{(1-c)u+1}{u+1} \geq 0$, $\forall u \geq 0$.

3. Main result

Theorem 3. *Let P and f be the same as defined in Theorem 1. If $T : P \to P$ is a continuous mapping satisying*

$$F\big(f(Tx,Ty), f(x,Tx), f(y,Ty), f(y,Tx), f(x,Ty)\big) < 0 \tag{1.4}$$

for all distinct points $x, y \in P$, where $F \in \mathcal{F}_5$, then T has a fixed point.

Furthermore if $f(x,y) = f(y,x)$ for all $x, y \in P$ and F satisfies property (F_U), the fixed point is unique.

Proof. We define $\varphi : P \to R$ by $\varphi(p) = f(p,Tp)$ for all $p \in P$. Clearly is continuous being the composite of two coutinuous function f and T. Since P is pseudocompact Tichonov space, every real valued function over P is bounded and attains its bouuds. Thus there exists a point $v \in P$ such that $\varphi(v) = \inf\{\varphi(p) : p \in P\}$. We now affirm that v is a fixed point for T. If not, let us suppose that $Tv \neq v$. Then using (4) we have successively

$$F\big(f(Tv,T^2u), f(v,Tv), f(Tv,T^2v), f(Tv,Tv), f(v,T^2v)\big) < 0$$
$$F\big(f(Tv,T^2v), f(v,Tv), f(Tv,T^2v), 0, f(v,Tv) + f(Tv,T^2v)\big) < 0$$

which implies by (F_2) that $f(Tv,T^2v) < f(v,Tv)$ i.e. $\varphi(Tv) < \varphi(v)$, a contradiction. Thus $v \in P$ is a fixed point for T. If $f(x,y) = f(y,x)$ and F satisfies property (F_U) then v is the unique fixed point.

Indeed, let $w \in P$ be another fixed point of T, i.e. $Tw = w$ and $v \neq w$. Then using (4) we have succesively,

$$F\big(f(Tv,Tw), f(v,Tv), f(w,Tw), f(w,Tv), f(v,Tw)\big) < 0$$
$$F\big(f(v,w), f(v,v), f(w,w), f(w,v), f(v,w)\big) < 0$$
$$F\big(f(v,w), 0, 0, f(v,w), f(v,w)\big) < 0$$

a contradiction of (F_U) which proves that $v \in P$ is the unique fixed point of T.

Corollary 1. *Theorem 1.*

Proof. It follows from Theorem 3 and Ex.l.

Corollary 2. *Theorem 2.*

Proof. It follows from Theorem 3 and Ex.2.

Theorem 4. *Let T be a continuous self mapping of a compact metric space (X,d) satisfying*

$$F\big(d(Tx,Ty),d(x,Tx),d(y,Ty),d(y,Tx),d(x,Ty)\big) < 0 \tag{1.5}$$

for all distinct points $x, y \in X$, where $F \in \mathcal{F}_5$. Then T has a fixed point. Furthermore, if T satisfies and properties (F_U), the fixed point is unique.

Proof. It follows from Theorem 3 and Remark 1.

Corollary 3. (Fisher [3]). *Let T be a continuous self mapping of a compact metric space (X,d) satisfying*

$$\big[d(Tx,Ty)\big]^2 \leq d(x,Tx)d(y,Ty) + ad(x,Ty)d(y,Tx) \tag{1.6}$$

for all distinct points $x, y \in X$, where $a \geq 0$. Then T has a fixed point. If $a \leq 1$, then the fixed point is unique.

Proof. It follows from Theorem 4 and Ex.l.

Corollary 4. (Pachpatte [4]). *If T is a continuous self mapping of a compact metric space (X,d) satisfying*

$$\begin{aligned}\big[d(Tx,Ty)\big]^2 < a\big(d(x,Tx)d(y,Ty)) + d(x,Ty)d(y,Tx) + \\ +b\big(d(x,Tx)d(y,Tx) + d(x,Ty)d(y,Ty)\big)\end{aligned} \tag{1.7}$$

for all distincts points $x, y \in X$, where $a, b \geq 0$, and $a + 2b < 1$, then T has a unique fixed point.

Proof. It follows from Theorem 4 and Ex.2.

4. References

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