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A COMMON FIXED POINT ON TRANSVERSAL PROBABILISTIC SPACES

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Abstract. In this paper we shall prove a common fixed point theorem for family of commuting mappings defined on transversal probabilistic spaces. This result extends some previous results.

1. Definitions and previous results

Next definitions are given by Tasković (see [3,4]).

Definition 1. *Let X be a nonempty set. The symmetric function $\rho : X \times X \to [0,1]$ is called a* **lower probabilistic transversal** *on X if there is a function $d : [0,1] \times [0,1] \to [0,1]$ such that*

$$\rho(x,y) \geq \min\{\rho(x,z), \rho(z,y), d(\rho(x,z), \rho(z,y))\}$$

for all $x, y, z \in X$. A **lower transversal probabilistic space** *is a set X together with a given lower probabilistic transversal on X. The function d is called* **lower (probabilistic) bisection function**.

Definition 2. *Let $\mathcal{F}$ denote the family of distribution functions denoted by $F_{u,v}$, for all $u, v \in X$ (a distribution function is nondecreasing and left continuous mapping of reals into $[0,1]$ with the properties $\inf\limits_{x\in R} F_{u,v}(x) = 0$ and $\sup_{x\in R} F_{u,v}(x) = 1$). The functions $F_{u,v}$ are assumed to satisfy the following conditions: $F_{u,v}(x) = 1$ for $x > 0$ iff $u = v$, $F_{u,v}(0) = 0$ and $F_{u,v}(x) = F_{v,u}(x)$ for all $x \in R$.*

In further, with $(X, \mathcal{F}, \rho)$, we shall denote lower transversal probabilistic space, together with family of distribution functions defined on it. The lower probabilistic transversal is defined with $\rho(u,v) = F_{u,v}(x)$ for all $x \in R$.

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Definition 3. *(a) A sequence $(p_n)_{n\in N}$ in $(X,\mathcal{F},\rho)$ convergates to a point $p\in X$ iff for any $\varepsilon>0$ and any $\lambda>0$, there exists an integer $M_{\varepsilon,\lambda}$, such that $F_{p,p_n}(\varepsilon)>1-\lambda$, whenever $n\geq M_{\varepsilon,\lambda}$.*

(b) The sequence $(p_n)_{n\in N}$ will be called fundamental in $(X,\mathcal{F},\rho)$ if for each $\varepsilon>0$ and each $\lambda>0$, exists an integer $M_{\varepsilon,\lambda}$, such that $F_{p_m,p_n}(\varepsilon)>1-\lambda$, whenever $m,n\geq M_{\varepsilon,\lambda}$. A lower transversal probabilistic space will be called complete if each fundamental sequence in X converges to an element in X.

Definition 4. *A mapping T of a lower transversal probabilistic space $(X,\mathcal{F},\rho)$ will be called a* **probabilistic contraction** *if there exists a nondecreasing function $\varphi:[0,+\infty)\to[0,+\infty)$ such that*

$$\text{(As)}\qquad \lim_{n\to\infty}\varphi^n(t)=+\infty,\quad \textit{for every } t>0,$$

satisfying the condition:

$$\text{(Pc)}\qquad F_{Tu,Tv}(x)\geq\min\big\{F_{u,v}(\varphi(x)),F_{u,Tu}(\varphi(x)),\\ F_{v,Tv}(\varphi(x)),F_{v,Tu}(\varphi(x)),F_{u,Tv}(\varphi(x))\big\}$$

for all $u,v\in X$ and for every $x\in(0,+\infty)$. M. Tasković has proven the next theorem (see [4]).

Theorem 1. *Let $(X,\mathcal{F},\rho)$ be a complete lower transversal probabilistic space, where the lower probabilistic transversal is defined with $\rho(u,v)=F_{u,v}$ and the lower bisection function $d:[0,1]\times[0,1]\to[0,1]$ is nondecreasing such that $d(t,t)\geq t$ for every $t>0$. If T is any probabilistic contraction mapping of X into itself, then there is a unique point $p\in X$ such that $Tp=p$.*

2. Main results

As an extension of previous theorem, in this section we shall formulate and prove a common fixed point theorem.

Theorem 2. *Let $(X,\mathcal{F},\rho)$ be a complete lower transversal probabilistic space where the lower probabilistic transversal is defined with $\rho(u,v)=F_{u,v}$ and the lower bisection function $d:[0,1]\times[0,1]\to[0,1]$ is nondecreasing such that $d(t,t)\geq t$ for every $t>0$. Let (T_n), for $n\in\mathbf{N}$ be a sequence of mappings from X into itself and $S:X\to X$ be a continuous bijective function commuting with each of T_n, satisfying condition $T_n(X)\subseteq S(X)$, for all $n\in\mathbf{N}$. Let exists a nondecreasing function $\varphi:[0,+\infty)\to[0,+\infty)$, such that condition (As) holds. If for all points $u,v\in X$ and all mappings T_i and T_j the inequality*

$$\text{(Pcd)}\qquad F^2_{T_iu,T_jv}(x)\geq\min\big\{F^2_{Su,Sv}(\varphi(x)),F^2_{Su,T_iu}(\varphi(x)),F^2_{Sv,T_jv}(\varphi(x)),\\ F_{Su,T_jv}(\varphi(x))F_{Sv,T_iu}(\varphi(x)),F_{Su,T_jv}(\varphi(x))F_{Su,T_iu}(\varphi(x))\big\},$$

holds for every $x > 0$ then there is a unique common fixed point $p \in X$ for S and all of mappings T_n.

Proof. Let u_0 be an arbitrary point from X. Let us define sequence (u_n), for $n \in \mathbf{N}$ as follows:

$$u_n = S^{-1}(T_n(u_{n-1})), \quad \text{for } n \in \mathbf{N} \tag{1}$$

We show that the sequence $v_n = S(u_n) = T_n(u_{n-1})$, for $n \in \mathbf{N}$ is fundamental in X.

From condition (Pcd) and for all $a > 0$ the next inequalities follow:

$$\begin{aligned} F^2_{Su_{n-1},Su_n}(a) = F^2_{T_{n-1}u_{n-2},T_nu_{n-1}} \geq \\ \min\big\{F^2_{Su_{n-2},T_{n-1}u_{n-2}}(\varphi(a)), F^2_{Su_{n-1},T_nu_{n-1}}(\varphi(a)), F^2_{Su_{n-2},Su_{n-1}}(\varphi(a)), \\ F_{Su_{n-2},T_nu_{n-1}}(\varphi(a))F_{Su_{n-1},T_{n-1}u_{n-2}}(\varphi(a)), \\ F_{Su_{n-2},T_nu_{n-1}}(\varphi(a))F_{Su_{n-2},T_{n-1}u_{n-2}}(\varphi(a))\big\} = \\ \min\big\{F^2_{Su_{n-2},Su_{n-1}}(\varphi(a)), F^2_{Su_{n-1},Su_n}(\varphi(a)), \\ F_{Su_{n-2},Su_n}(\varphi(a))F_{Su_{n-1},Su_{n-1}}(\varphi(a)), \\ F_{Su_{n-2},Su_n}(\varphi(a))F_{Su_{n-2},Su_{n-1}}(\varphi(a))\big\}. \end{aligned} \tag{2}$$

Since the space is lower probabilistic transversal then for every $x \geq 0$ the following inequalities hold:

$$\begin{aligned} F_{a,b}(x) \geq \min\big\{F_{a,c}(x), F_{c,b}(x), d(F_{a,c}(x), F_{c,b}(x)\big\} \geq \\ \min\big\{F_{a,c}(x), F_{c,b}(x)\big\}, \end{aligned} \tag{$*$}$$

because $d(a,b) \geq d(\min\{a,b\}, \min\{a,b\}) \geq \min\{a,b\}$. From previous follows that

$$F_{Su_{n-2},Su_n}(\varphi(a)) \geq \min\big\{F_{Su_{n-2},Su_{n-1}}(\varphi(a)), F_{Su_{n-1},Su_n}(\varphi(a))\big\}. \tag{3}$$

Then, from inequality (3) and the fact that values of distribution functions are in interval $[0,1]$ next inequalities follow:

$$\begin{aligned} F_{Su_{n-2},Su_n}(\varphi(a))F_{Su_{n-1},Su_{n-1}}(\varphi(a)) = F_{Su_{n-2},Su_n}(\varphi(a)) \geq \\ \min\big\{F_{Su_{n-2},Su_{n-1}}(\varphi(a)), F_{Su_{n-1},Su_n}(\varphi(a))\big\} \geq \\ \min\big\{F^2_{Su_{n-2},Su_{n-1}}(\varphi(a)), F^2_{Su_{n-1},Su_n}(\varphi(a))\big\}. \end{aligned} \tag{4}$$

$$\begin{aligned} F_{Su_{n-2},Su_n}(\varphi(a))F_{Su_{n-2},Su_{n-1}}(\varphi(a)) \geq \\ \min\big\{F^2_{Su_{n-2},Su_{n-1}}(\varphi(a)), F_{Su_{n-1},Su_n}(\varphi(a))F_{Su_{n-2},Su_{n-1}}(\varphi(a))\big\}. \end{aligned} \tag{5}$$

From the fact that $\min\{a^2, b^2, ab\} = \min\{a^2, b^2\}$, for all $a, b \in [0,1]$, inequalities (2), (4) and (5) imply:

$$F^2_{Su_{n-1},Su_n}(a) \geq \min\big\{F^2_{Su_{n-2},Su_{n-1}}(\varphi(a)), F^2_{Su_{n-1},Su_n}(\varphi(a))\big\}. \tag{6}$$

From last follows:

$$F_{Su_{n-1},Su_n}(a) \geq \min\left\{F_{Su_{n-2},Su_{n-1}}(\varphi(a)), F_{Su_{n-1},Su_n}(\varphi(a))\right\}. \tag{7}$$

Since φ is a nondecreasing function and $\varphi(a) > 0$, $\varphi(a) > a$ for every $a > 0$ it follows by induction that for every $k \in \mathbf{N}$ the following inequality holds:

$$F_{Su_{n-1},Su_n}(a) \geq \min\left\{F_{Su_{n-2},Su_{n-1}}(\varphi(a)), F_{Su_{n-1},Su_n}(\varphi^k(a))\right\}, \tag{8}$$

and when $k \to +\infty$ we get that for every $n \in \mathbf{N}$:

$$F_{Su_{n-1},Su_n}(a) \geq F_{Su_{n-2},Su_{n-1}}(\varphi(a)). \tag{9}$$

By induction we can prove the inequality (10) for the sequence $\{v_n\}$.

$$F_{v_{n-1},v_n}(a) \geq F_{v_0,v_1}(\varphi^{n-1}(a)). \tag{10}$$

From $(*)$, and last inequality, for $m > n$ and arbitrary $\varepsilon > 0$, follows:

$$F_{v_n,v_m}(\varepsilon) \geq \min\left\{F_{v_n,v_{n+1}}(\varepsilon), \dots, F_{v_{m-1},v_m}(\varepsilon)\right\} \geq$$
$$\min\left\{F_{v_0,v_1}(\varphi^n(\varepsilon)), ..., F_{v_0,v_1}(\varphi^{m-1}(\varepsilon))\right\} = F_{v_0,v_1}(\varphi^n(\varepsilon)).$$

From (As) we conclude that exists a natural $N \in \mathbf{N}$ such that $F_{v_0,v_1}(\varphi^N(\varepsilon)) > 1-\lambda$. We can take that $n, m \geq N$ and we conclude that v_n is a fundamental sequence in $(X, \mathcal{F}, \rho)$. Since the space is complete, then there is a point $p \in X$ such that $v_n \to p$.

We shall prove that p is a common fixed point for S and T_n. Since S commutates with each of T_n, then from (1) and the fact that $T_n Su_{n-1} = ST_n u_{n-1} = SSu_n$ follows:

$$F^2_{SSu_n,T_kp}(a) = F^2_{ST_nu_{n-1},T_kp}(a) = F^2_{T_nSu_{n-1},T_kp}(a) \geq$$
$$\min\{F^2_{SSu_{n-1},Sp}(\varphi(a)), F^2_{SSu_{n-1},T_nSu_{n-1}}(\varphi(a)), F^2_{Sp,T_kp}(\varphi(a)),$$
$$F_{SSu_{n-1},T_kp}(\varphi(a))F_{Sp,T_nSu_{n-1}}(\varphi(a)), F_{SSu_{n-1},T_kp}(\varphi(a))F_{SSu_{n-1},T_nSu_{n-1}}\} =$$
$$\min\{F^2_{SSu_{n-1},Sp}(\varphi(a)), F^2_{SSu_{n-1},SSu_n}(\varphi(a)), F^2_{Sp,T_kp}(\varphi(a)),$$
$$F_{SSu_{n-1},T_kp}(\varphi(a))F_{Sp,SSu_n}(\varphi(a)), F_{SSu_{n-1},T_kp}(\varphi(a))F_{SSu_{n-1},SSu_n}\}.$$

From continuity of S and because $Su_n \to p$ when $n \to +\infty$, we get that for every $k \in \mathbf{N}$ follows:

$$F^2_{Sp,T_kp}(a) \geq \min\left\{F^2_{Sp,Sp}(\varphi(a)), F^2_{Sp,Sp}(\varphi(a)), F^2_{Sp,T_kp}(\varphi(a)),\right. \tag{11}$$
$$\left.F_{Sp,T_kp}(\varphi(a))F_{Sp,Sp}(\varphi(a)), F_{Sp,T_kp}(\varphi(a))F_{Sp,Sp}(\varphi(a))\right\} = F^2_{Sp,T_kp}(\varphi(a)).$$

Because all of the functions in last inequality are nondecreasing we conclude that for each $m \in \mathbf{N}$ the inequality $F_{Sp,T_kp}(a) \geq F_{Sp,T_kp}(\varphi^m(a))$ holds. When $m \to +\infty$, for every $a > 0$, we obtain $F_{Sp,T_kp}(a) = 1$. From this, for every $k \in \mathbf{N}$ we obtain $(**)$ $S(p) = T_k(p)$. In following text we shall show that p is a common fixed point for all of mappings T_n.

From inequality:

$$F^2_{Su_n,T_kp}(a) = F^2_{T_nu_{n-1},T_kp}(a) \geq \min\big\{F^2_{Su_{n-1},Sp}(\varphi(a)), F^2_{Su_{n-1},Su_n}(\varphi(a)), F^2_{Sp,T_kp}(\varphi(a)), F_{Su_{n-1},T_kp}(\varphi(a))F_{Sp,Su_n}(\varphi(a)), F_{Su_{n-1},T_kp}(\varphi(a))F_{Su_{n-1},Su_n}(\varphi(a))\big\}, \tag{12}$$

when $n \to +\infty$, because $(**)$ holds, we conclude that:

$$F^2_{p,T_kp}(a) \geq \min\big\{F^2_{p,T_kp}(\varphi(a)), F^2_{p,p}(\varphi(a)), F^2_{T_kp,T_kp}(\varphi(a)), F_{p,T_kp}(\varphi(a))F_{T_kp,p}(\varphi(a)), F_{p,T_kp}(\varphi(a))F_{p,p}(\varphi(a))\big\}, \tag{13}$$

From last, we obtain that for each $a > 0$ holds the following:

$$F_{p,T_kp}(a) \geq F_{p,T_kp}(\varphi(a)). \tag{14}$$

Next, we obtain that for every $m \in \mathbf{N}$ follows $F_{p,T_kp}(a) \geq F_{p,T_kp}(\varphi^m(a))$, and when $m \to +\infty$, we conclude that for every $a > 0$ the fact $F_{p,T_kp}(a) = 1$ holds, and it implies that for each $k \in \mathbf{N}$ we get $p = T_kp = Sp$.

Let us prove uniqueness of common fixed point p. Suppose that there is another common fixed point $q \neq p$. From

$$F^2_{p,q}(a) = F^2_{T_ip,T_jq}(a) \geq \min\big\{F^2_{Sp,Sq}(\varphi(a)), F^2_{Sp,p}(\varphi(a)), F^2_{Sq,q}(\varphi(a)), F_{Sp,q}(\varphi(a))F_{Sq,p}(\varphi(a)), F_{Sp,q}(\varphi(a))F_{Sp,p}(\varphi(a))\big\} = F^2_{p,q}(\varphi(a)). \tag{15}$$

follows that for every $a > 0$ holds that $F_{p,q}(a) \geq F_{p,q}(\varphi(a))$, and so, for every $m \in \mathbf{N}$, we obtain that $F_{p,q}(a) \geq F_{p,q}(\varphi^m(a))$, and when $m \to +\infty$, we conclude that for every $a > 0$ holds the fact $F_{p,q}(a) = 1$. From conditions for distribution functions we get that $p = q$. This completes the proof.

3. Consequences and comments

The next theorem is consequence of Theorem 1, for $S = id$.

Theorem 3. *Let $(X, \mathcal{F}, \rho)$ be a complete lower transversal probabilistic space where the lower probabilistic transversal is defined with $\rho(u, v) = F_{u,v}$ and the lower bisection function $d : [0, 1] \times [0, 1] \to [0, 1]$ is nondecreasing such that $d(t, t) \geq t$ for every $t > 0$. Let (T_n), for $n \in \mathbf{N}$ be a sequence of mappings from X into itself. Let exists a nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$, such that condition (As) holds. If for all points $u, v \in X$ and all mappings T_i and T_j the inequality*

$$F^2_{T_iu,T_jv}(x) \geq \min\big\{F^2_{u,v}(\varphi(x)), F^2_{u,T_iu}(\varphi(x)), F^2_{v,T_jv}(\varphi(x)), F_{u,T_jv}(\varphi(x))F_{v,T_iu}(\varphi(x)), F_{u,T_jv}(\varphi(x))F_{u,T_iu}(\varphi(x))\big\}, \tag{Pcd1}$$

holds for every $x > 0$, then there is a unique common fixed point $p \in X$ for all of mappings T_n.

C. Bylka (see [1]) has proven the next theorem for mapping defined on Menger's space.

Theorem 4. *Let $(X, \mathcal{F}, t)$ be a complete probabilistic Menger space, where t is a continuous t-norm satisfying $t(x,x) \geq x$ for each $x \in [0,1]$, and T a mapping of X into itself. Let exists a nondecreasing function $\varphi : [0,+\infty) \to [0,+\infty)$, such that condition (As) holds. If for all points $u, v \in X$ and every $x > 0$ the condition:*

$$(***) \qquad F_{Tu,Tv}(x) \geq F_{u,v}(\varphi(x))$$

holds, then T has a unique fixed point $p \in X$.

Proof. It is easy to prove that every Menger space is a lower probabilistic transversal space (see [3]). Chosen us $T = T_i = T_j$ and $d = t$. From $(***)$ follows condition (Pcd1):

$$F^2_{Tu,Tv}(x) \geq F^2_{u,v}(\varphi(x)) \geq \min\big\{F^2_{u,v}(\varphi(x)), F^2_{u,Tu}(\varphi(x)), F^2_{v,Tv}(\varphi(x)), F_{u,Tv}(\varphi(x))F_{v,Tu}(\varphi(x)), F_{u,Tv}(\varphi(x))F_{u,Tu}(\varphi(x))\big\}.$$

Theorem 4. follows from Theorem 3.

Comment. It is easy to prove that mappings from Theorem 3 are probabilistic contractions. For $T = T_i = T_j$, because for all $a, b, c > 0$ the inequality $\min\{ab, ac\} \geq \min\{a^2, b^2, c^2\}$ holds, then from (Pcd1) follows:

$$F^2_{Tu,Tv}(x) \geq \min\big\{F^2_{u,v}(\varphi(x)), F^2_{u,Tu}(\varphi(x)), F^2_{v,Tv}(\varphi(x)), F_{u,Tv}(\varphi(x))F_{v,Tu}(\varphi(x)), F_{u,Tv}(\varphi(x))F_{u,Tu}(\varphi(x))\big\} \geq \min\big\{F^2_{u,v}(\varphi(x)), F^2_{u,Tu}(\varphi(x)), F^2_{v,Tv}(\varphi(x)) F^2_{u,Tv}(\varphi(x)), F^2_{v,Tu}(\varphi(x))\big\}.$$

From last inequality follows (Pc). Hence, Theorem 2 and Theorem 3, are common fixed point theorem for probabilistic contraction mappings.

4. References

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