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ROUGHLY d-CONVEX FUNCTIONS ON UNDIRECTED TREE NETWORKS

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Abstract. In this paper we establish some properties of roughly d-convex functions on undirected tree networks. It is pointed out that these roughly d-convex functions have the following properties concerning the property of minimum: each local minimum of a midpoint δ-d-convex or lightly γ-d-convex function is a global minimum, where a local minimizer has to yield the minimal function value in its neighborhood with radius equal to the roughness degree. Since every ρ-d-convex or δ-d-convex function is midpoint δ-d-convex and every γ-d-convex function is lightly γ-d-convex, this conclusion holds for them, too. We also state weaker but sufficient conditions for roughly d-convex functions. We adopt the definition of network as metric space introduced by Dearing P.M. and Francis R.L. in 1974.

1. Introduction

We recall first the definitions of undirected networks as metric space introduced in [1] by Dearing and Francis.

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, ..., w_m\}$ we associate a point v_i from an euclidian space X. This yields a finite subset $V = \{v_1, ..., v_m\}$ of X, called the **vertex set** of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called **edge** of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length l_{ij} and denote by U the set of all edges. We define the **network** $N = (V, U)$ by

$$N = \{x \in X \mid \exists\, (w_i, w_j) \in A \text{ such that } x \in [v_i, v_j]\}.$$

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It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in X. Suppose that for each $[v_i, v_j] \in U$ there is a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0,1]$ with $\theta_{ij}(v_i) = 0, \theta_{ij}(v_j) = 1$, and $\theta_{ij}([v_i, v_j]) = [0,1]$. We denote by T_{ij} the inverse function of θ_{ij}.

Any connected and closed subset of an edge bounded by two points x and y of $[v_i, v_j]$ is called a **closed subedge** and is denoted by $[x, y]$. If one or both of x, y are missing we say than the subedge is open in x, or in y or is open and we denote this by $(x, y]$, $[x, y)$ or (x, y), respectively. Using θ_{ij}, it is possible to compute the length of $[x, y]$ as

$$l([x,y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}.$$

Particularly we have

$$l([v_i, v_j]) = l_{ij}, \qquad l([v_i, x]) = \theta_{ij}(x)\, l_{ij}$$

and

$$l([x, v_j]) = (1 - \theta_{ij}(x))\, l_{ij}.$$

A path $L(x, y)$ linking two points x and y in N is a sequence of edges and at most two subedges at extremities, starting at x and ending at y. If $x = y$ then the path is called **cycle**. The **length of a path (cycle)** is the sum of the lengths of all its component edges and subedges and will be denoted by $l(L(x,y))$.

A network is connected if for any points $x, y \in N$ there is a path $L(x,y) \subset N$.

A connected network without cycles is called **tree**.

Let $L^*(x,y)$ be a shortest path between the points $x, y \in N$. This path is also called **geodesic**.

Definition 1. [1]*For any $x, y \in N$, the distance from x to y, $d(x,y)$ in the network N is the length of a shortest path from x to y:*

$$d(x,y) = l(L^*(x,y)).$$

It is obvious that (N, d) is a metric space.

For $x, y \in N$, we denote

$$\langle x, y\rangle = \{z \in N \mid d(x,z) + d(z,y) = d(x,y)\}, \tag{1}$$

and $\langle x, y\rangle$ is called the **metric segment** between x and y.

Definition 2. [1] *A set $D \subset N$ is called d-convex if $\langle x, y\rangle \subset D$ for all $x, y \in D$.*

Roughly d-convex functions are a generalization of roughly convex functions and respective of d-convex functions proposed by V. P. Soltan and P. S. Soltan in [14]. We recall that there is several kinds of roughly convex functions: ρ-convex functions, proposed by Klötzler and investigated by Hartwig and Söllner in [2], [13], δ-convex and midpoint δ-convex functions established by Hu, Klee, Larman in [3] and γ-convex, strictly γ-convex, lightly γ-convex, midpoint γ-convex, strictly r-convexlike functions, proposed and investigated by Phu in [6], [7], [8], [9], [10], [11] etc.

In the following lines we consider a network $N = (V, U)$ endowed with the metric defined in Definition 1. We denote $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$.

Definition 3. [14] *The function $f : N \to \overline{\mathbf{R}}$ is called d-convex on N if for any pair of points $x, y \in N, x \neq y$, and for every $z \in \langle x, y \rangle$ is satisfied the inequality*

$$f(z) \leq \frac{d(z,y)}{d(x,y)} f(x) + \frac{d(x,z)}{d(x,y)} f(y). \tag{2}$$

Extending Phu's observation at this function, we remark in [5] that the inequality (2) can be satisfied just for the points $x, y \in N$ with $d(x,y) \geq r$, r being a fixed positive real number convenient selected.

We consider the positive real numbers $r_\rho, r_\delta, r_\gamma, r$ and a d-convex set $D \subset N$.

Definition 4. [5] *The function $f : D \to R$ is called:*

1. **ρ-d-convex** on D with the roughness degree r_ρ if for any pair of points $x, y \in D$ with $d(x,y) \geq r_\rho$, is satisfied the inequality (2) for all $z \in \langle x, y \rangle$;
2. **δ-d-convex** on D with the roughness degree r_δ if for any pair of points $x, y \in D$ with $d(x,y) \geq r_\delta$, is satisfied the inequality (2) for all $z \in \langle x, y \rangle$ with $d(x,z) \geq r_\delta/2$, and $d(z,y) \geq r_\delta/2$;
3. **midpoint δ-d-convex** on D with the roughness degree r_δ if for any pair of points $x, y \in D$ with $d(x,y) \geq r_\delta$, is satisfied the inequality (2) for all $z \in \langle x, y \rangle$ with $d(x,z) = d(z,y) = d(x,y)/2$;
4. **γ-d-convex** on D with the roughness degree r_γ if for any pair of points $x, y \in D$ with $d(x,y) \geq r_\gamma$, is satisfied the inequality

   $$f\left(x'\right) + f\left(y'\right) \leq f(x) + f(y) \tag{3}$$

   for all pair of points $x', y' \in \langle x, y \rangle$ with $d\left(x, x'\right) = d\left(y, y'\right) = r_\gamma$;
5. **lightly γ-d-convex** on D with the roughness degree r_γ if for any pair of points $x, y \in D$ with $d(x,y) \geq r_\gamma$, is satisfied the inequality (2) for all $z \in \langle x, y \rangle$ with $d(x,z) = r_\gamma$ or for all $z \in \langle x, y \rangle$ with $d(z,y) = r_\gamma$;

6. **midpoint γ-d-convex** on D with the roughness degree r_γ if for any pair of points $x, y \in D$ with $d(x,y) = 2r_\gamma$, is satisfied the inequality (2) for all $z \in \langle x,y \rangle$ with $d(x,z) = d(z,y) = r_\gamma$;
7. **strictly γ-d-convex** on D with the roughness degree r_γ if for any pair of points $x, y \in D$ with $d(x,y) > r_\gamma$, is satisfied the inequality

$$f\left(x'\right) + f\left(y'\right) < f(x) + f(y), \tag{4}$$

for all pair of points $x', y' \in \langle x,y \rangle$ with $d\left(x, x'\right) = d\left(y, y'\right) = r_\gamma$;

8. **strictly r-d-convexlike** (or **strictly roughly d-convexlike**) on D with the roughness degree r if for any pair of points $x, y \in D$ with $d(x,y) > r$ there is $z \in \langle x,y \rangle, z \neq x, z \neq y$ such that is satisfied the inequality:

$$f(z) < \frac{d(z,y)}{d(x,y)} f(x) + \frac{d(x,z)}{d(x,y)} f(y). \tag{5}$$

The functions who satisfy one of the conditions (1)-(8) are called roughly d-convex.

We compared this kinds of roughly convex functions and we got the following scheme for the relation between them:

Theorem 1. *[5]Between some different kinds of roughly d-convex functions there is the following relations:*

$$\begin{array}{ccccc}
f\ d\text{-convex} \overset{\forall r_\rho > 0}{\Longrightarrow} f\ \rho\text{-}d\text{-convex} \overset{r_\rho \le r_\delta}{\Longrightarrow} f\ \delta\text{-}d\text{-convex} & \Longrightarrow & & & \begin{array}{c} f \text{ midpoint} \\ \delta\text{-}d\text{-convex} \end{array} \\
\Downarrow r_\rho \le r_\gamma & & & & \Downarrow r_\delta = 2r_\gamma \\
f\ \gamma\text{-}d\text{-convex} & \Longrightarrow & \begin{array}{c} f \text{ lightly} \\ \gamma\text{-}d\text{-convex} \end{array} & \Longrightarrow & \begin{array}{c} f \text{ midpoint} \\ \gamma\text{-}d\text{-convex} \end{array}
\end{array}$$

2. Some properties of roughly d-convex functions on tree networks

We consider now a tree network $N = (V, U)$ and a d-convex set $D \subset N$. We recall that in a tree network the metric segment $\langle x,y \rangle$ contain an unique path between x and y, for every $x, y \in N$.

Definition 5. *We say that the function $f : D \to R$ attains a r-local minimum at a point $x^* \in D$ if*

$$f(x) \ge f(x^*) \text{ for all } x \in D \text{ satisfying } d(x,y) < r.$$

Theorem 2. [5] *If $f : D \to R$ is a midpoint δ-d-convex function with the roughness degree $r_\delta > 0$, $x^* \in D$ and*

$$f(x) \ge f(x^*) \tag{6}$$

for all $x \in U_{r_\delta}(x^*) := \{z \in D \mid d(x^*, z) < r_\delta\}$, *then* $f(x) \geq f(x^*)$ *for all* $x \in D$ *(f attains its global minimum in D at* x^**).*

Remark. Since ρ-d-convexity and δ-d-convexity imply midpoint δ-d-convexity, ρ-d-convex functions and δ-d-convex functions have this property, too.

Theorem 3. *If* $f : D \to R$ *is a lightly* γ-d*-convex function with the roughness degree* $r_\gamma > 0$, $x^* \in D$ *and*

$$f(x) \geq f(x^*)$$

for all $x \in \overline{U_{r_\gamma}(x^*)} := \{z \in D \mid d(x^*, z) \leq r_\delta\}$, *then* $f(x) \geq f(x^*)$ *for all* $x \in D$ *(f attains its global minimum in D at* x^**).*

Proof. Assume the contrary that f does not attains its global minimum at x^*, then there is $x_0 \in D \backslash \overline{U_{r_\gamma}(x^*)}$ such that $f(x^*) > f(x_0)$. We consider now the points $s, x_1 \in \langle x_0, x^* \rangle$ such that

$$d(x^*, s) = r_\gamma \text{ and } d(x_1, x_0) = r_\gamma.$$

Since $f(x_0) < f(x^*) \leq f(s)$, the definition of lightly γ-d-convexity imply

$$f(x_1) \leq \frac{d(x_0, x_1)}{d(x_0, x^*)} f(x^*) + \frac{d(x_1, x^*)}{d(x_0, x^*)} f(x_0) < f(x^*).$$

We repeat this construction, and we get $x_i, i \in I \subset N$, with $f(x^*) > f(x_i)$ for all $i \in I$. Since $d(x_i, x^*) = d(x_{i-1}, x^*) - r_\gamma$, there is $i^* \in I$ such that $d(x_{i^*}, x^*) < r_\delta$ and hence for x_{i^*} we have $f(x_{i^*}) \geq f(x^*)$, which contradicts the relation $f(x^*) > f(x_i)$ for all $i \in I$. This contradiction completes our proof.

Remark. Since every γ-d-convex function is lightly γ-d-convex, this conclusion holds for γ-d-convex functions, too.

In the following line we will establish weaker but sufficient conditions for roughly d-convex functions $f : D \to R$, where D is a d-convex subset of a tree network $N = (V, U)$.

We consider a tree network $N = (V, U)$ and a d-convex set $D \subset N$.

Theorem 4. *[5]The function* $f : D \to R$ *is* γ-d*-convex on D with the roughness degree* $r_\gamma > 0$ *if and only if there is a* $\sigma > 0$ *such that*

$$f\left(x'\right) + f\left(y'\right) \leq f(x) + f(y) \tag{7}$$

is satisfied for any pair of points $x, y \in D$ *with*

$$r_\gamma \leq d(x, y) < r_\gamma + \sigma$$

and for $x', y' \in \langle x, y \rangle$ *with* $d\left(x, x'\right) = d\left(y, y'\right) = r_\gamma$.

Theorem 5. *The function $f : D \to R$ is midpoint δ-d-convex on D with the roughness degree $r_\gamma > 0$ if and only if the inequality (2) is satisfied for $z \in \langle x, y\rangle$ with $d(x,z) = d(z,y) = d(x,y)/2$, for any pair of points $x, y \in D$ satisfying*

$$r_\delta \le d(x,y) < 2r_\delta.$$

Proof. It is clear that we only need to prove the sufficiency. This is done by induction. We are going to show that (2) holds for any pair of points $x, y \in D$ satisfying

$$r_\delta \le d(x,y) < 2^i r_\delta, i = 1, 2, \dots$$

and for $z \in \langle x, y\rangle$ with $d(x,z) = d(z,y) = d(x,y)/2$.

By assumption, it holds for $i = 1$. We assume now that the assertion is true for some $n \in \mathbf{N}$. Let x, y be a pair of points in D with

$$2^n r_\delta \le d(x,y) < 2^{n+1} r_\delta.$$

We denote by z_1, z_2, z_3 the points from $\langle x, y\rangle$ such that

$$d(x,z_1) = d(z_1,z_2) = d(z_2,z_3) = d(z_3,y) = d(x,y)/4.$$

Then

$$r_\delta \le d(z_2,x) = d(z_3,z_1) = d(y,z_2) < 2^n r_\delta$$

implies

$$\begin{aligned} f(z_1) &\le (1/2)f(x) + (1/2)f(z_2) \\ 2f(z_2) &\le f(z_1) + f(z_3) \\ f(z_2) &\le (1/2)f(z_2) + (1/2)f(y). \end{aligned}$$

By addition of these three inequalities we get

$$f(z_2) \le (1/2)f(x) + (1/2)f(y).$$

Hence the assertion also holds for this pair of points $x, y \in D$.

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