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TAUBERIAN THEOREMS FOR SEQUENCES WITH MODERATELY OSCILLATORY CONTROL MODULO

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Abstract. We introduce a general control modulo of the oscillatory behavior or order m of $\{u_n\}$, which leads new Tauberian conditions and consequently new Tauberian theorems. Also the notion of moderately oscillatory and regularly generated sequences is presented and studied. In the first section we give basic definitions, notations and a brief survey of classical results. Next we establish Tauberian theorems by using the general control modulo. The proofs of these theorems are based on the classical and neoclassical Tauberian results, in a particular on the corollary to Karamata's Hauptsatz. Finally in the last section we consider the class of moderately oscillatory regularly generated sequences and prove some theorems similar to Tauberian theorems.

1. Introduction

1.1 Definitions and notations

In the classical and neoclassical Tauberian theory, the convergence restoration problem of $\{u_n\}$ out of the existence of the limit

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n \tag{1}$$

and some additional properties of $\{u_n\}$ reduces to proving that

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n. \tag{2}$$

These additional conditions on $\{u_n\}$ are so-called Tauberian conditions which control the behavior of $\{\triangle u_n\}$, i.e., they control the oscillatory behavior of $\{u_n\}$.

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The following denotations will be frequently used in what follows. For a real or complex sequence $u = \{u_n\}$, denote for some integer $m \geq 0$,

$$\sigma_n^{(m)}(u) = \begin{cases} \frac{1}{n+1}\sum_{k=0}^{n} \sigma_k^{(m-1)}(u) = u_o + \sum_{k=1}^{n} \frac{V_k^{(m-1)}(\Delta u)}{k} & \text{for } m \geq 1 \\ u_n & \text{for } m = 0 \end{cases}$$

where

$$V_n^{(m)}(\Delta u) = \begin{cases} \frac{1}{n+1}\sum_{k=0}^{n} V_k^{(m-1)}(\Delta u) & \text{for } m \geq 1 \\ \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k & \text{for } m = 0 \end{cases}$$

and

$$\Delta u_n = \begin{cases} u_n - u_{n-1} & \text{for } n \geq 1 \\ u_o & \text{for } n = 0 \end{cases}$$

and $\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(m)}(\Delta u)$.

The Kronecker identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(o)}(\Delta u), \tag{3}$$

will be also used in the various steps of the proofs. It can be rewritten as

$$u_n = V_n^{(o)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(o)}(\Delta u)}{k} + u_o. \tag{4}$$

The above form of $\{u_n\}$ is more suitable to set conditions on the generator sequence $\{V_n^{(o)}(\Delta u)\}$ of $\{u_n\}$ rather than the sequence itself.

The concept of slowly oscillating sequences such as $\{u_n\}$, $\{V_n^{(o)}(\Delta u)\}$ or others, plays an important role in obtaining (2) from the existence of the limit (1). The definition of slow oscillation given by Landau [1] and later by Schmidt [2] are rather cumbersome for proving our results. In our work we shall use a more suitable definition of slow oscillation given in [3]. However, as noticed in [4] we can define slow oscillation in normed linear spaces. Let be a normed linear space with norm $||.||$

Definition 1. *A sequence $\{u_n\}$ from B is slowly oscillating in norm, (or $||.||$-slowly oscillating) if*

$$\lim_{\lambda\to 1+} \overline{\lim_{n}} \max_{n+1\leq k\leq[\lambda n]} \left\|\sum_{j=n+1}^{k} (u_j - u_{j-1}\right\| = 0.$$

From (4) we may redefine slow oscillation of $\{u_n\}$ via its generator sequence $\{V_n^{(o)}(\Delta u)\}$. Namely we shall say that a sequence $\{u_n\}$ in B is slowly oscillating in norm if and only if

$$V_n^{(o)}(\Delta u) = O(1), \quad n \to \infty \tag{i}$$

and

(ii) $$\{V_n^{(o)}(\Delta u)\}$$

is slowly oscillating in norm.

Clearly, the above definition of slow oscillation in norm and (i) and (ii) hold for real or complex sequences $\{u_n\}$. In our work we shall study real or complex sequences. If (i) and (ii) hold, $\{u_n\}$ is slowly oscillating due to the representation (4). On the other hand, if $\{u_n\}$ is slowly oscillating, it is shown in [5] that $\{V_n^{(o)}(\Delta u)\}$ is bounded. Indeed, for $\lambda > 1$, define $w_n(u,\lambda) = \max_{n+1\le k\le[\lambda n]} \left|\sum_{j=n+1}^{k} \Delta u_j\right|$ and rewrite the finite sum $\sum_{k=1}^{n} k\Delta u_k$ as the series $\sum_{j=0}^{\infty} \sum_{\frac{n}{2^{j+1}}\le k<\frac{n}{2^j}} k\Delta u_k$. Hence

$$\left|\sum_{k=1}^{n} k\Delta u_k\right| \le \sum_{j=0}^{\infty}\left|\sum_{\frac{n}{2^{j+1}}\le k<\frac{n}{2^j}} k\Delta u_k\right| \le \left(\sum_{j=0}^{\infty}\frac{n}{2^j}\right) w_{\left[\frac{n}{2^{j+1}}\right]}(u,\lambda) \le$$
$$nC_1\sum_{j=0}^{\infty}\frac{1}{2^j} = 2nC_1 = nC,$$

where $C > 0$. Consequently we have $V_n^{(o)}(\Delta u) = \frac{1}{n}\sum_{k=1}^{n} k\Delta u_k = O(1)$, $n\to\infty$. Therefore $\{\sigma_n^{(1)}(u)\} = \left\{u_o + \sum_{k=1}^{n}\frac{V_k^{(o)}(\Delta u)}{k}\right\}$ is slowly oscillating. Thus $\{V_n^{(o)}(\Delta u)\}$ is also slowly oscillating.

Hardy and Littlewood [6] conjectured that

(5) $$V_n^{(o)}(|\Delta u|,p) = \frac{1}{n+1}\sum_{k=0}^{n} k^p|\Delta u_k|^p = O(1), \quad n\to\infty, \quad p>1$$

together with the existence of the limit (1) implies (2). This conjecture was proved later by Szasz [7]. The condition (5), the Hardy-Littlewood condition, is of considerable interest in this dissertation. Observe that if (5) holds, then the sequence $\{u_n\}$ is slowly oscillating. Indeed,

$$\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| \le \max_{n+1\le k\le[\lambda n]}\sum_{j=n+1}^{k}|\Delta u_j| \le \sum_{j=n+1}^{[\lambda n]} j\frac{|\Delta u_j|}{j}$$

$$\le \frac{1}{n+1}\sum_{j=n+1}^{[\lambda n]} j|\Delta u_j| = \frac{[\lambda n]-n}{n+1}\frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]} j|\Delta u_j|$$

$$\le \frac{[\lambda n]-n}{n+1}\frac{1}{([\lambda n]-n)^{\frac{1}{p}}}\left(\sum_{j=n+1}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

$$= \frac{([\lambda n]-n)^{1-\frac{1}{p}}}{n+1}\left(\sum_{j=n+1}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}} = \frac{([\lambda n]-n)^{\frac{1}{q}}}{n+1}\left(\sum_{j=n+1}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

$$\le \frac{([\lambda n]-n)^{\frac{1}{q}}}{(n+1)^{\frac{1}{q}}}\left(\frac{[\lambda n]+1}{n+1}\frac{1}{[\lambda n]+1}\sum_{j=0}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

$$= \frac{([\lambda n]-n)^{\frac{1}{q}}}{(n+1)^{\frac{1}{q}}}\frac{([\lambda n]+1)^{\frac{1}{p}}}{(n+1)^{\frac{1}{p}}}\left(\frac{1}{[\lambda n]+1}\sum_{j=0}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

Taking limsup of both sides, we get

$$\overline{\lim_n}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right|$$

$$\le \overline{\lim_n}\left(\frac{[\lambda n]-n}{n+1}\right)^{\frac{1}{q}}\left(\frac{[\lambda n]+1}{n+1}\right)^{\frac{1}{p}}\overline{\lim_n}\left(\frac{1}{[\lambda n]+1}\sum_{j=0}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

$$= \lim_n\left(\frac{[\lambda n]-n}{n+1}\right)^{\frac{1}{q}}\left(\frac{[\lambda n]+1}{n+1}\right)^{\frac{1}{p}}\overline{\lim_n}\left(\frac{1}{[\lambda n]+1}\sum_{j=0}^{[\lambda n]} j^p|\Delta u_j|^p\right)^{\frac{1}{p}}$$

$$\le (\lambda-1)^{\frac{1}{q}}\lambda^{\frac{1}{p}}C,$$

where C is the constant from (5). Finally we have

$$\lim_{\lambda\to 1+}\overline{\lim_{n}}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right|\le C\lim_{\lambda\to 1+}(\lambda-1)^{\frac{1}{q}}\lambda^{\frac{1}{p}}=0.$$

For $p=1$, (5) becomes

$$V_n^{(o)}(|\Delta u|)=V_n^{(o)}(|\Delta u|,1)=\frac{1}{n+1}\sum_{k=0}^{n}k|\Delta u_k|^p=O(1),\quad n\to\infty,$$

which is no longer a Tauberian condition [8]. In [8] this situation was illustrated by the following example. Consider the sequence defined by

$$\Delta u_n=\begin{cases}1 & \text{if}\quad n=2^m,\quad m=1,2\dots,\\ -1 & \text{if}\quad n=2^m+1,\quad m=1,2\dots,\\ 0 & \quad\text{otherwise}\end{cases}$$

which is the sequence of the first backward differences of the sequence $\{u_n\}$. Then the sequence $\{V_n^{(o)}(|\Delta u|)\}$ is bounded. This follows from the way that the sequence $\{\Delta u_n\}$ is constructed. Indeed, for $n=2^s$, we have $V_{2^s}^{(o)}(|\Delta u|)=\frac{1}{2^s}\sum_{j=1}^{s-1}(2.2^j+1)+1<4$. Clearly, $\{V_{2^s+1}^{(o)}(|\Delta u|)\}$ is also bounded. Next we have to show that the limit (1) exists. Consider the series $f(\Delta u,x)=\sum_{n=1}^{\infty}\Delta u_n x^n$, where $\{\Delta u_n\}$ is defined as above. We may rewrite this series as $f(\Delta u,x)=\sum_{n=1}^{\infty}(x^{2^n}-x^{2^n+1})$. Notice that if $0\le x<1$, then $f(\Delta u,x)\ge 0$. Hence it follows that $\underline{\lim}_{x\to 1-}f(\Delta u,x)\ge 0$. Also observe that from the rewritten form of $f(\Delta u,x)$ we have

$$\begin{aligned}f(\Delta u,x)&=(1-x)\sum_{n=1}^{\infty}x^{2^n}=(1-x)\left(x^2+x^4+x^8+\sum_{n=4}^{\infty}x^{2^n}\right)\\&\le(1-x)\left(x^2+x^4+x^8+\int_0^{\infty}x^{t^2}dt\right)\\&=(1-x)\left(x^2+x^4+x^8+\int_0^{\infty}e^{\left(-\ln\frac{1}{n}\right)t^2}dt\right)\\&=(1-x)\left(x^2+x^4+x^8+\frac{1}{\sqrt{\ln\frac{1}{x}}}\int_0^{\infty}e^{-u^2}dt\right)\\&=(1-x)\left(x^2+x^4+x^8+C\left(\sqrt{\ln\frac{1}{x}}\right)^{-1}\right).\end{aligned}$$

Thus we obtain $\overline{\lim}_{x\to 1-} f(\Delta u, x) \le 0$ because $\ln\left(\frac{1}{x}\right) \sim 1-x$ as $x \to 1-$. Finally $\lim_{x\to 1-} f(\Delta u, x) = \lim_{x\to 1-}(1-x)\sum_{n=1}^{\infty} u_n x^n = 0$. But it is clear from the construction of $\{\Delta u_n\}$ that the sequence $\{u_n\}$ diverges.

In [8] it is also shown that for $p=1$, $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$ provided that $\lim_n V_n^{(o)}(|\Delta u))$ exists. Since (5), for $p=1$, is not a Tauberian condition for recovering convergence of $\{u_n\}$ from the existence of the limit (1), this situation motivated a new way to study the control devices of oscillatory behavior of $\{u_n\}$, more general than the slow oscillation.

Definition 2. *A sequence $\{u_n\}$ is moderately oscillatory [3] if, for $\lambda > 1$,*

$$\overline{\lim_n} \max_{n+1\le k\le [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| < \infty.$$

For instance, in [9] it is shown that if

$$V_n^{(o)}(|\Delta u|) = V_n^{(o)}(|\Delta u|, 1) = \frac{1}{n+1}\sum_{k=0}^{n} k|\Delta u_k| = O(1), \quad n \to \infty,$$

then the sequence $\{u_n\}$ is moderately oscillatory. Assuming the existence of the limit $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ together with $V_n^{(o)}(|\Delta u|) = O(1)$, $n\to\infty$, it is shown in [10, 11] that $u_n = O(1)$, $n \to \infty$.

Now we shall introduce a new device for the control of the oscillatory behavior of the sequence $\{u_n\}$. Denote by $\omega_n^{(o)} = n\Delta u_n$, the classical control modulo of the oscillatory behavior. For each integer $m \ge 1$, and for all positive integers n, define recursively as in [12, 13] $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$, the general control modulo of the oscillatory behavior of order m, which generates new Tauberian conditions, and consequently new Tauberian theorems.

In [12, 14] the notion of regularly generated sequences is introduced as follows.

Definition 3. *Let L be any linear space and let B be a class of sequences $\{B_n\}$ from L. The class consisting of sequences defined by*

$$u_n = B_n + \sum_{k=1}^{n}\frac{B_k}{k} + u_o \tag{6}$$

for all nonnegative integers n, is the class of all regularly generated sequences $\{u_n\}$ by the class B and it is denoted by $U(B)$. For instance, if B is the class of all bounded slowly oscillating sequences, then $U(B)$ is the classical class of all slowly oscillating sequences.

In [3], for $\lambda > 1$, the de la Vallee Poussin means of $\{u_n\}$ are defined as follows

$$\tau_{n,[\lambda n]}(u) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} u_k. \tag{7}$$

A sequence $\{u_n\}$ is summable to K in the sense of the above means if

$$\lim_{\lambda \to 1+} \overline{\lim_{n}} \left| \tau_{n,[\lambda n]}(u) - K \right| = 0.$$

Consider a slowly oscillating sequence $\{u_n\}$. Then

$$\tau_{n,[\lambda n]}(u) - u_n = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (u_k - u_n) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta u_j$$

and $\left|\tau_{n,[\lambda n]}(u) - u_n\right| \leq \max_{n+1 \leq k \leq [\lambda n]} \left|\sum_{k=n+1}^{k} \Delta u_j\right|$.

Taking limsup in n of both sides of the above inequality and then taking limit of both sides as $\lambda \to 1+$, we have

$$\lim_{\lambda \to 1+} \overline{\lim_{n}} \left|\tau_{n,[\lambda n]}(u) - u_n\right| \leq \lim_{\lambda \to 1+} \overline{\lim_{n}} \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0,$$

for $\{u_n\}$ is slowly oscillating. In the sense of the de la Vallee Poussin means, slowly oscillating sequences (not necessarily convergent) are as close as possible to their means in the limiting case. It is shown in [3] that if $\{u_n\}$ is $(C,1)$-summable to K, then it is summable to K in the sense of the de la Vallee Poussin means. Indeed, for $\lambda > 1$, since

$$\tau_{n,[\lambda n]}(u) = \frac{([\lambda n] + 1)\sigma^{(1)}_{[\lambda n]}(u) - (n+1)\sigma^{(1)}_n(u)}{[\lambda n] - n},$$

we have

$$\lim_n \tau_{n,[\lambda n]}(u) = \frac{\lambda}{\lambda - 1} \lim_n \sigma^{(1)}_{[\lambda n]}(u) - \frac{1}{\lambda - 1} \lim_n \sigma^{(1)}_n(u) = \frac{\lambda}{\lambda - 1} K - \frac{1}{\lambda} K = K.$$

We will now establish the two important identities [3, 12] which will be indispensable and often used in the crucial steps of the various proofs in this dissertation. Consider for $\lambda > 1$,

$$u_n = \tau_{n,[\lambda n]}(u) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}(u_k - u_n) =$$

$$\tau_{n,[\lambda n]}(u) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j$$

$$= (\tau_{n,[\lambda n]}(u) - \sigma_n^{(1)}(u)) + \sigma_n^{(1)}(u) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j.$$

We now compute the first term in the last expression as follows. From

$$([\lambda n]+1)\sigma_{[\lambda n]}^{(1)}(u) - (n+1)\sigma_n^{(1)}(u) = \sum_{k=n+1}^{[\lambda n]} u_k,$$

after multiplying both sides by $\frac{1}{[\lambda n]-n}$, we have

$$\frac{([\lambda n]+1)\sigma_{[\lambda n]}^{(1)}(u) - (n+1)\sigma_n^{(1)}(u)}{[\lambda n]-n} = \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} u_k = \tau_{n,[\lambda n]}(u).$$

Therefore

$$\tau_{n,[\lambda n]}(u) - \sigma_n^{(1)}(u) = \frac{([\lambda n]+1)\sigma_{[\lambda n]}^{(1)}(u) - (n+1)\sigma_n^{(1)}(u)}{[\lambda n]-n} - \sigma_n^{(1)}(u)$$

$$= \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right).$$

Finally

$$u_n = \sigma_n^{(1)}(u) + \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right) - \tag{8}$$

$$\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j, \quad \lambda > 1,$$

which is the first important identity.

On the other hand, for $1 < \lambda < 2$, using the following version of the de la Vallee Poussin means $\tau_{2n-[\lambda n],n}(u) = \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n} u_k$, we will obtain the second important identity. We begin with the following identities.

$$u_n = \tau_{2n-[\lambda n],n}(u) + (u_n - \tau_{2n-[\lambda n],n}(u)) =$$

$$\tau_{2n-[\lambda n],n}(u) + \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n}(u_n - u_k)$$

$$= \tau_{2n-[\lambda n],n}(u) + \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n}\sum_{j=k+1}^{n}\Delta u_j$$

$$= (\tau_{2n-[\lambda n],n}(u) - \sigma_{2n-[\lambda n]}(u)) + \sigma^{(1)}_{2n-[\lambda n]}(u) + \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n}\sum_{j=k+1}^{n}\Delta u_j.$$

The first term on the right hand side of the last expression can be computed as follows $(n+1)\sigma^{(1)}_n(u) - (2n-[\lambda n]+1)\sigma^{(1)}_{2n-[\lambda n]}(u) = \sum_{k=2n-[\lambda n]+1}^{n} u_k$, and multiplying both sides by $\frac{1}{[\lambda n]-n}$, we get

$$\frac{(n+1)\sigma^{(1)}_n(u) - (2n-[\lambda n]+1)\sigma^{(1)}_{2n-[\lambda n]}(u)}{[\lambda n]-n} = \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n} u_k.$$

Hence

$$\tau_{2n-[\lambda n],n}(u) - \sigma^{(1)}_{2n-[\lambda n]}(u) = \frac{n+1}{[\lambda n]-n}(\sigma^{(1)}_n(u) - \sigma^{(1)}_{2n-[\lambda n]}(u)).$$

Finally

$$(9)\qquad \begin{aligned} u_n &= \sigma^{(1)}_{2n-[\lambda n]}(u) + \frac{n+1}{[\lambda n]-n}(\sigma^{(1)}_n(u) - \sigma^{(1)}_{2n-[\lambda n]}(u)) \\ &+ \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n}\sum_{j=k+1}^{n}\Delta u_j, \quad 1<\lambda<2. \end{aligned}$$

From (8) it follows that if$\{u_n\}$ is slowly oscillating and $\lim_n \sigma^{(1)}_n(u)$ exists, then $\lim_n u_n = \lim_n \sigma^{(1)}_n(u)$.

The identity (9) was obtained by the author in the Graduate Research Seminar, University of Missouri-Rolla, Spring 1999.

1.2 A brief survey of classical results

In most classical Tauberian theorems the sequence $\{\omega^{(o)}_n(u)\}$ plays an important role in obtaining Tauberian conditions. For instance, [15], for a real sequence $\{u_n\}$, if

$$(10)\qquad \omega^{(o)}_n(u) = n\Delta u_n = o(1), \quad n\to\infty$$

and the limit (1) exists, then $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$. Indeed, [3], choose a positive integer $N(x)$ in such a way that $N(x) \to \infty$ as $x \to 1-$ and

$$\sum_{n=0}^{\infty} \Delta u_n x^n - \sum_{n=0}^{N(x)} \Delta u_n = o(1), \quad x \to 1-. \tag{11}$$

Since $n\Delta u_n = o(1)$, $n \to \infty$, given $\varepsilon > 0$ there is an integer $n_o(\varepsilon)$ such that $n|\Delta u_n| < \varepsilon$ whenever $n \geq n_o(\varepsilon)$. Rewrite the expression on the left side of (11) as

$$\sum_{n=0}^{\infty} \Delta u_n x^n - \sum_{n=0}^{N(x)} \Delta u_n = \sum_{n=0}^{N(x)} \Delta u_n x^n + \sum_{m=N(x)+1}^{\infty} \Delta u_n x^n - \sum_{n=0}^{N(x)} \Delta u_n$$

$$= \sum_{m=N(x)+1}^{\infty} \Delta u_n x^n - \sum_{n=0}^{N(x)} \Delta u_n (1 - x^n) = S_1 - S_2.$$

It remains to estimate each of the above sums S_1 and S_2. The estimate for S_1 in $(0,1)$ is

$$|S_1| \leq \sum_{m=N(x)+1}^{\infty} |\Delta u_n| x^n < \varepsilon \sum_{m=N(x)+1}^{\infty} \frac{x^n}{n} < \varepsilon \frac{1}{N(x)+1} \frac{1}{1-x} < \varepsilon$$

for $N(x) = \left[\dfrac{1}{1-x}\right]$. The estimate for S_2 is obtained in a similar way for $n \geq n_o(\varepsilon)$

$$|S_2| \leq \sum_{n=0}^{N(x)} |\Delta u_n|(1 - x_n) \leq (1-x) \sum_{n=1}^{N(x)} n|\Delta u_n|$$

$$= \frac{1}{\left[\dfrac{1}{1-x}\right]} \sum_{n=1}^{N(x)} n|\Delta u_n| \leq \frac{1}{\left[\dfrac{1}{1-x}\right]} \sum_{n=1}^{N(x)} n|\Delta u_n| = \frac{1}{N(x)} \sum_{n=1}^{N(x)} n|\Delta u_n| < \varepsilon.$$

This completes the proof that (10) and the existence of the limit (1) imply (2). For a different approach to this proof see also [5, 16, 17].

There is an immediate generalization of (10) due to Tauber [15]. It can be shown that it suffices to assume $\sigma_n^{(1)}(\omega^{(o)}(u)) = V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$ and the existence of the limit (1) to obtain (2).

Theorem 1. (Tauber [15]) *For a real sequence $\{u_n\}$, let the limit (1) exist. If*

$$V_n^{(o)}(\Delta u) = o(1), \quad n \to \infty, \tag{12}$$

then $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

In order to prove this theorem, the original Tauber theorem, the Tauberian condition (12) is reduced to the Tauberian condition (10), [5, 16, 17].

Proof. Let $v_n = \sum_{k=1}^{n} k\Delta u_k = (n+1)V_n^{(o)}(\Delta u)$, $n \geq 1$, and $v_o = 0$, then we have $\Delta u_n = \dfrac{1}{n}(v_n - v_{n-1})$, $n \geq 1$, and

$$f(x) = u_o + \sum_{n=1}^{\infty} \frac{v_n}{n} x^n - \sum_{n=1}^{\infty} \frac{v_{n-1}}{n} x^n =$$

$$u_o + (1-x)\sum_{n=1}^{\infty} \frac{v_n}{n} x^n + \sum_{n=1}^{\infty} \frac{v_n}{n(n+1)} x^{n+1}.$$

From (12) it follows that $v_n = o(n)$, $n \to \infty$ and $(1-x)\sum_{n=1}^{\infty} \dfrac{v_n}{n} x^n \to 0$, as $x \to 1-$. Hence $\lim_{x\to 1-} f(x) = u_0 + \lim_{x\to 1-} \sum_{n=1}^{\infty} \dfrac{v_n}{n(n+1)} x^{n+1}$, where $\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(1-x)\sum_{n=1}^{\infty} u_n x^n = s$. Hence we have $\sum_{n=1}^{\infty} \dfrac{v_n}{n(n+1)} x^n$ $\to s - u_o$, as $x \to 1-$ by the previous result of Tauber since $\dfrac{v_n}{n(n+1)} = o\left(\dfrac{1}{n}\right)$, $n \to \infty$, the condition (12). Observe that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{v_n}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} v_n \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$\lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{v_n - v_{n+1}}{n} - \frac{v_N}{N+1} \right\} = \lim_{N\to\infty} \sum_{n=1}^{N} \Delta u_n.$$

This completes the proof of the original Tauber theorem.

Later a substantial generalization of the condition (10) was found by Littlewood [18].

Theorem 2. *For a real sequence $\{u_n\}$, let the limit (1) exist and $n\Delta u_n = O(1)$, $n \to \infty$. Then $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.*

In Theorems 1 and 2 although the sequence $\{u_n\}$ is assumed to be real, these theorems can be proved for complex $\{u_n\}$. In the case of complex sequences we need to consider the limit (1) where $z \to 1-$ must be taken along any path in the unit disk passing through $z = 1$. However, as in [17, 19], in order to prove Theorems 1 and 2 we have to choose the curve $\rho = 2\dfrac{C^2\cos\theta - C}{C^2 - 1}$ which has two branches through $z = 1$. Each branch passing through $z = 1$ of the above curve makes an angle $\arccos\left(\dfrac{1}{C}\right)$ with the real axis.

The following is the Littlewood theorem for complex sequences $\{u_n\}$.

Theorem 3. *Let* $\lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty} u_n z^n$ *exist, that is, the limit exists as* $z \to 1-$ *along a curve as above. If* $n\Delta u_n = O(1)$, $n \to \infty$, *then* $\lim_n u_n = \lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty} u_n z^n$.

Schmidt [2] redefined Landau's concept of slow oscillation in the following way. A sequence $\{u_n\}$ is slowly oscillating if $\lim_{\substack{N>M\to\infty \\ \frac{N}{M}\to 1}} (u_N - u_M) = 0$.

The proof of the Tauberian theorem that Schmidt proposed [2] using his definition of slow oscillation contained some minor errors. This theorem is also known as the generalized Littlewood theorem. Vijayaraghavan [20] gave the corrected proof of the Schmidt Tauberian theorem. See [9] for an interesting proof of the Schmidt Tauberian theorem.

Theorem 4. *Let the limit (1) exist and* $\lim_{\substack{N>M\to\infty \\ \frac{N}{M}\to 1}} (u_N - u_M) = 0$. *Then (2) holds.*

In the previous section we showed that if (5) holds, then $\{u_n\}$ is slowly oscillating. Consequently if the limit (1) exists and (5) holds, then clearly (2) holds.

The one-sided boundedness of $\{n\Delta u_n\}$ was first introduced by Landau [1]. A real sequence $\{u_n\}$ is one-sidedly bounded if $u_n \geq -C$ for some $C \geq 0$ and for all nonnegative integers n. In the real case the Littlewood condition, $n\Delta u_n = O(1)$, $n \to \infty$, may be generalized as $n\Delta u_n \geq -C$ for some $C \geq 0$ and for all nonnegative integers n. Landau proved the following Tauberian theorem.

Theorem 5. *For a real sequence* $\{u_n\}$, *let* $\lim_n \sigma_n^{(1)}(u)$ *exist and* $n\Delta u_n \geq -C$ *for some* $C \geq 0$ *and for all nonnegative integers* n. *Then* $\lim_n u_n = \lim_n \sigma_n^{(1)}(u)$.

By assuming the existence of the limit (1), Hardy and Littlewood [6] later proved the following generalization of Theorem 5.

Theorem 6. *Let the limit (1) exist and* $n\Delta u_n \geq -C$ *for some* $C \geq 0$ *and for all nonnegative integers* n. *Then (2) holds.*

The proofs of Theorems 3, 5, and 6 remained very complicated until the ingenious method of Karamata [21]. A truly profound proof of the corollary to Karamata's Hauptsatz [21] not only reduced the previous proofs to their essentials, more importantly it also opened new avenues for obtaining various Tauberian theorems involving the general moduli of the oscillatory behavior of $\{u_n\}$.

Theorem 7. (Corollary to Karamata's Hauptsatz [21]). *For a real sequence $\{u_n\}$, let the limit (1) exist and $u_n \geq -C$ for some $C \geq 0$ and for all nonnegative integers n. Then* $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

Proof. The proof [3, 19, 21] of the corollary to Karamata's Hauptsatz depends on the well-known theorem of Weierstrass, that we can approximate uniformly any continuous function on a closed set by a sequence of polynomials. For instance, let g be a continuous function on $[0,1]$. Then given any $\varepsilon > 0$, there are two polynomials p and P such that on $[0,1]$

$$p(x) \leq g(x) \leq P(x), \tag{13}$$

and

$$\int_0^1 \{g(x) - p(x)\}dx \leq \varepsilon, \qquad \int_0^1 \{P(x) - g(x)\}dx\} \leq \varepsilon. \tag{14}$$

This is clearly true if p and P differ by at most $\frac{1}{2}\varepsilon$ from $g(x) - \frac{1}{2}\varepsilon$ and $g(x) + \frac{1}{2}\varepsilon$ respectively. We may even consider a function g on $[0,1]$ with a discontinuity at $x = c \in (0,1)$ of the first kind and construct polynomials satisfying (13) and (14). For example, let $g(c-0) < g(c+0)$ and let $f(x) = g(x) + \frac{1}{2}\varepsilon$ for $x < c - \delta$ and for $x > c$; for $c - \delta \leq x \leq c$, define $f(x) = \max\left\{l(x), g(x) + \frac{1}{4}\varepsilon\right\}$, where l is the linear function of x such that $l(c - \delta) = g(c-\delta) + \frac{1}{2}\varepsilon$, $l(c) = g(c+0) + \frac{1}{2}\varepsilon$. Then we see that f is continuous, and $f(x) > g(x)$. Hence for small enough δ, a polynomial P approximating sufficiently closely f has the required properties. Similarly we may construct p. The first step in the proof is to show that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n P(x^n) = (\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n)\int_0^1 P(t)dt \tag{15}$$

holds for any polynomial P. It is sufficient to consider the case $P_m(x) = x^m$. Since

$$(1-x)\sum_{n=0}^{\infty} u_n x^{n+mn} = \frac{1-x}{1-x^{m+1}}\left\{(1-x^{m+1})\sum_{n=0}^{\infty} u_n (x^{m+1})^n\right\},$$

taking the limit $x \to 1-$ of both sides we get

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^{n+mn} =$$

$$\lim_{x^{m+1}\to 1-} \frac{1-x}{1-x^{m+1}} \lim_{x^{m+1}\to 1-} \left\{ (1-x^{m+1}) \sum_{n=0}^{\infty} u_n x^{(m+1)n} \right\}$$

$$= \frac{1}{m+1}\left\{\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n\right\} = \left(\int_0^1 x^m dx\right) \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

Since by the Weierstrass approximation theorem any continuous function g on $[0,1]$ can be approximated by a sequence of polynomials uniformly, it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n g(x^n) = \left\{\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n\right\} = \int_0^1 g(t)dt \tag{16}$$

for any continuous function g on $[0,1]$ or any function g with a discontinuity at $x = c \in (0,1)$ of the first kind.

The second step in the proof of the theorem is the choice of the function g. Let

$$g(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < e^{-1} \\ \dfrac{1}{t} & \text{for} \quad e^{-1} \le t \le 1 \end{cases}$$

Then clearly

$$\int_0^1 g(t)dt = \int_{e^{-1}}^1 \frac{dt}{t} = 1. \tag{17}$$

Furthermore $g(x^n) = 0$ if $x^n < \frac{1}{\varepsilon}$, that is, if $n \ln\left(\frac{1}{x}\right) > 1$ or $n > \dfrac{1}{\ln\left(\frac{1}{x}\right)}$.

Hence from (16) we get

$$\lim_{x\to 1-}(1-x) \sum_{n\le \frac{1}{\ln\left(\frac{1}{x}\right)}} u_n x^n \frac{1}{x^n} = \lim_{x\to 1-}(1-x) \sum_{n\le \frac{1}{\ln\left(\frac{1}{x}\right)}} u_n =$$

$$\lim_{x\to 1-}(1-x) \sum_{n=0}^{\left[\frac{1}{\ln\left(\frac{1}{x}\right)}\right]} u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

Next step is the choice of the function $N(x)$ such that $N(x) \to \infty$ as $x \to 1-$. Letting $N(x) = \dfrac{1}{\ln\left(\frac{1}{x}\right)}$ or $x = e^{-\frac{1}{N(x)}}$, since $1 - e^{-\frac{1}{N(x)}} \sim \dfrac{1}{N(x)} \sim \dfrac{1}{N(x)+1}$ as $N(x) \to \infty$, we have

$$\lim_{N(x)\to\infty} \left(1 - e^{-\frac{1}{N(x)}}\right) \sum_{n=0}^{[N(x)]} u_n = \lim_{N(x)\to\infty} \frac{1}{N(x)} \sum_{n=0}^{[N(x)]} u_n =$$
$$\lim_{N(x)\to\infty} \frac{1}{N(x)+1} \sum_{n=0}^{\infty} u_n = \lim_{x\to 1-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

This completes the proof of the corollary to Karamata's Hauptsatz.

2. Intrisic Tauberian conditions

2.1 Introduction

In the Tauberian theory, the conditions for the convergence recovery of the sequence $\{u_n\}$ out of the existence of the limit (1) were essentially based on the classical modulo $\omega_n^{(0)}(u) = n\Delta u_n$ that controls the oscillatory behavior of $\{u_n\}$. These conditions were restricting the order of the magnitude of the sequence $\{\Delta u_n\}$ both in the classical sense and Landau's sense [1]. For instance, as proved on page 65, if

$$\omega_n^{(0)}(u) = n\Delta u_n = o(1), \quad n \to \infty \tag{18}$$

and the limit (1) exists, then $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x_n$. From the immediate generalization, that is, $\sigma_n^{(1)}(\omega^{(0)}(u)) = V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, of (18) we see that it is natural to set conditions on $(C,1)$-means of the classical modulo $\omega_n^{(0)}(u)$ of $\{u_n\}$ instead of setting conditions on itself, $\omega_n^{(0)}(u)$. This indicates that some weaker conditions could be found for the convergence recovery of the sequence $\{u_n\}$ out of the existence of the limit (1) provided that other moduli of oscillatory behavior of $\{u_n\}$ are defined. Recall that we already introduced on page 62 the general control modulo of the oscillatory behavior of order m [12, 13]. For $m = 1$, we have

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta u_n - V_n^{(0)}(\Delta u) =$$
$$n(V_n^{(0)}(\Delta u) - V_{n-1}^{(0)}(\Delta u)).$$

Now we will generalize the Littlewood condition, $\omega_n^{(0)}(u) = O(1)$, $n \to \infty$ [18], by assuming that $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory.

Theorem 8. *For a real sequence $\{u_n\}$, let the limit (1) exist. If $\{n\Delta u_n\}$ is moderately oscillatory, then* $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

Proof. Since $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, we have

$$\omega_n^{(1)}(u) = n\Delta u_n - V_n^{(0)}(\Delta u) = O(1), \quad n \to \infty$$

and

$$\{\sigma_n^{(1)}(\omega^{(0)}(u))\} = \{V_n^{(0)}(\Delta u)\}$$

is slowly oscillating. Taking $(C,1)$-means of both sides and of all corresponding terms of the previous asymptotic equality, we obtain

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n(V_n^{(1)}(\Delta u) - V_{n-1}^{(1)}(\Delta u)) = O(1), \quad n \to \infty.$$

Hence for some $C \geq 0$ and for all nonnegative integers n, we have

$$n\Delta V_n^{(1)}(\Delta u) \geq -C. \tag{19}$$

On the other hand, from the existence of the limit (1) it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(u_n - \sigma_n^{(1)}(u))x^n = 0.$$

Therefore $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = 0$.

From the corollary to Karamata's Hauptsatz, we have

$$\frac{1}{n+1}\sum_{k=0}^{n} k\Delta V_k^{(1)}(\Delta u) = o(1), \quad n \to \infty.$$

But this is the original Tauber condition on $\{V_n^{(1)}(\Delta u)\}$. Hence $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Since $\{V_n^{(0)}(\Delta u)\}$ is slowly oscillating, from the identity (8)

$$V_n^{(0)}(\Delta u) = V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u))$$
$$-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}(V_j^{(0)}(\Delta u) - V_{j-1}^{(0)}(\Delta u))$$

it follows that $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Therefore from the original Tauber theorem we have $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

We could have obtained this conclusion as follows using the corollary to the original Tauber theorem. From $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ we first show that $\lim_n \sigma_n^{(1)}(u)$ exists and $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$. Notice that from the Kronecker identity we have

$$u_n - \sigma_n^{(1)}(u) = n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) = V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty.$$

Since the existence of the limit (1) implies that $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ exists, the corollary to the original Tauber theorem yields $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n$. Finally it follows from the Kronecker identity that $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n$.

In the above proof, after obtaining $V_n^{(1)}(\Delta u) = o(1)$, $n\to\infty$, one could use. Theorem 4 and get immediately $V_n^{(0)}(\Delta u) = o(1)$, $n\to\infty$, because $\{V_n^{(0)}(\Delta u)\}$ is slowly oscillating and $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}V_n^{(0)}(\Delta u)x^n = 0$. However, this dissertation is written around Karamata's Hauptsatz and its corollary because Karamata's method is the most efficient tool and is compatible with my usage of the general control modulo of higher order. We could also utilize the Littlewood theorem or the Hardy-Littlewood theorem to conclude that $V_n^{(1)}(\Delta u) = o(1)$, $n\to\infty$ in the above proof, but the only intelligent proof of the Littlewood theorem or Hardy-Littlewood theorem is the corollary to Karamata's Hauptsatz. Therefore we often prefer using the corollary to Karamata's Hauptsatz in this dissertation.

The proof of Theorem 8. suggests that the existence of the limit (1) can be weakened by assuming that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(0)}(u)x^n \tag{20}$$

exists and using higher order $\{V_n^{(m)}(\Delta u)\}$, in particular $\{V_n^{(2)}(\Delta u)\}$.

Theorem 9. *For a real sequence $\{u_n\}$, let the limit (20) exist. If $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Proof. It is sufficient to show that the limit (1) exists from the conditions of Theorem 9. Since $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, $n\Delta u_n - V_n^{(0)}(\Delta u) = O(1)$, $n\to\infty$. Taking $(C,1)$-means of both sides and of all corresponding terms of the above asymptotic equality, we get $V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = O(1)$, $n\to\infty$ and again $V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n(V_n^{(2)}(\Delta u) - V_{n-1}^{(2)}(\Delta u)) = O(1)$, $n\to\infty$. Hence for some $C\geq 0$ and for all nonnegative integers n, we have $n\Delta V_n^{(2)}(\Delta u)\geq -C$.

On the other hand, the existence of the limit (20) implies that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}V_n^{(1)}(\Delta u)x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(\sigma_n^{(1)} - \sigma_n^{(2)}(u))x^n = 0.$$

Therefore $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(2)}(\Delta u)x^n = 0$. Thus it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u))x^n = 0.$$

Hence by the corollary to Karamata's Hauptsatz, we have:

$$\frac{1}{n+1}\sum_{k=0}^{n} k(V_k^{(2)}(\Delta u) - V_{k-1}^{(2)}(\Delta u)) = o(1), \quad n\to\infty.$$

Consequently by the original Tauber theorem $V_n^{(2)}(\Delta u) = o(1), n\to\infty$. Therefore we have:

$$\sigma_n^{(2)}(u) - \sigma_n^{(3)}(u) = n(\sigma_n^{(3)}(u) - \sigma_{n-1}^{(3)}(u)) = V_n^{(2)}(\Delta u) = o(1), \quad n\to\infty.$$

Since this is a special case of the original Tauber condition and since the existence of the limit (20) implies $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(3)}(u)x^n$ exists, we get

$$\lim_n \sigma_n^{(3)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Since $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, the sequence $\{V_n^{(0)}(\Delta u)\}$ is slowly oscillating. Therefore $\{V_n^{(1)}(\Delta u)\}$ is also slowly oscillating. But $\{V_n^{(1)}(\Delta u)\}$ is $(C,1)$-summable to zero since $V_n^{(2)}(\Delta u) = o(1), n\to\infty$

Thus from the identity (8) we have $V_n^{(1)}(\Delta u) = o(1), n\to\infty$. From $\sigma_n^{(2)}(u) = \sigma_n^{(3)}(u) + V_n^{(2)}(\Delta u)$, we get

$$\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$$

and from

$$\sigma_n^{(1)}(u) = \sigma_n^{(2)}(u) + V_n^{(1)}(\Delta u)$$

it follows that $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$. But the existence of the limit $\lim_n \sigma_n^{(1)}(u)$ implies the existence of the limit (1), which is proved in [3]. Finally

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Therefore from Theorem 8. it follows that

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n$$

Since every slowly oscillating sequence is moderately oscillatory, we have the following important corollary to Theorem 9.

Corollary 3. *Let the limit (20) exist. If $\{\omega_n^{(0)}(u)\}$ is slowly oscillating, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Theorem 9. and consequently Corollary 26. are further generalizations of the Littlewood theorem.

In Theorems 8. and 9. we have assumed that the sequence $\{u_n\}$ is real. However, if we do not use the corollary to Karamata's Hauptsatz but rather the Littlewood theorem, we can have both theorems for complex $\{u_n\}$. Indeed, for a complex $\{u_n\}$, let the limit (20) exist. If $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, then

$$\lim_n u_n = \lim_{x\to 1-}(1-z)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)z^n.$$

The proof is relatively short. Since $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory, we have $\omega_n^{(1)}(u) = n\Delta u_n - V_n^{(0)}(\Delta u) = O(1)$, $n\to\infty$.

Taking $(C,1)$-means of both sides and of all corresponding terms of the above asymptotic equation we get

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n(V_n^{(1)}(\Delta u) - V_{n-1}^{(1)}(\Delta u)) = O(1), \quad n\to\infty$$

Also from the existence of the limit (20) it follows that

$$\lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty}V_n^{(1)}(\Delta u)z^n = \lim_{x\to 1-}(1-z)\sum_{n=0}^{\infty}(\sigma_n^{(1)}(u)-\sigma_n^{(2)}(u))z^n = 0.$$

Hence by the Littlewood theorem $V_n^{(1)}(\Delta u) = o(1)$, $n\to\infty$. Therefore $\left\{V_n^{(0)}(\Delta u)\right\}$ is $(C,1)$-summable to zero. But $\{V_n^{(0)}(\Delta u)\}$ is slowly oscillating since $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory. Thus from the identity (8) it follows that , $V_n^{(0)}(\Delta u) = o(1)$, $n\to\infty$. Hence as in Theorem 8. it follows that $\lim_n u_n = \lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)z^n$.

In the above argument if the existence of the limit (1) is assumed, the proof becomes even shorter.

2.2 Intrinsic Tauberian theorems for sequences with moderately oscillatory control moduli

In the rest of this thesis, we will continue exploring Tauberian theorems by using the general control moduli $\omega_n^{(m)}(u)$ of order m of the oscillatory behavior of $\{u_n\}$ as defined on page 62.

Notice that the control modulo of the oscillatory behavior of order one $\omega_n^{(1)}(u)$ can be obtained from $u_n - \sigma_n^{(1)} = V_n^{(0)}(\Delta u)$ by taking backward differences in n of each term in both sides of the Kronecker identity. That is,

$$(u_n - u_{n-1}) - (\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) = V_n^{(0)}(\Delta u) - V_{n-1}^{(0)}(\Delta u)$$

After multiplying by n both sides of the above identity, we obtain

$$n(u_n - u_{n-1}) - n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) = n\Delta u_n - V_n^{(0)}(\Delta u) = nV_n^{(0)}(\Delta u).$$

Thus

$$\omega_n^{(1)}(u) = n\Delta u_n - V_n^{(0)}(\Delta u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u)) = V_n^{(0)}(\Delta \omega^{(0)}(u))$$

Later in this section we will apply this procedure in which we have obtained $\omega_n^{(1)}(u)$ to get Tauberian theorems for $\{\sigma_n^{(m)}(u)\}$, $m \geq 0$.

In Theorem 8. it is assumed that $\{\omega_n^{(0)}(u)\}$ is moderately oscillatory to obtain convergence of $\{u_n\}$ out of the existence of the limit (1). In the following theorem we see that the condition $\{\omega_n^{(0)}(u)\}$ being moderately oscillatory can be further weakened and convergence of $\{u_n\}$ still can be obtained out of the existence of the limit (1).

Theorem 10. *For a real sequence $\{u_n\}$, let the limit (1) exist. If $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory, then* $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

Proof. Since $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory, $\{\sigma_n^{(1)}(\omega^{(1)}(u)\}$ is slowly oscillating. Therefore

$$\sigma_n^{(1)}(\omega^{(1)}(u)) - \sigma_n^{(2)}(\omega^{(1)}(u)) =$$
$$n(\sigma_n^{(2)}(\omega^{(1)}(u)) - \sigma_{n-1}^{(2)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty$$

Observe that

$$\sigma_n^{(1)}(\omega^{(1)}(u)) = V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)$$

and

$$\sigma_n^{(2)}(\omega^{(1)}(u)) = V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u).$$

Hence for some $C \geq 0$ and for all nonnegative integers n, we have

$$n(\sigma_n^{(2)}(\omega^{(1)}(u)) - \sigma_{n-1}^{(2)}(\omega^{(1)}(u))) \geq -C \tag{21}$$

Also from the existence of the limit (1) it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(\omega^{(1)}(u))x^n =$$
$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u))x^n = 0$$

Therefore

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(\sigma_n^{(1)}(\omega^{(1)}(u))-\sigma_n^{(2)}(\omega^{(1)}(u)))x^n=0. \tag{22}$$

From (21) and (22), the corollary to Karamata's Hauptsatz yields

$$\frac{1}{n+1}\sum_{k=0}^{n}k(\sigma_k^{(2)}(\omega^{(1)}(u))-\sigma_{k-1}^{(2)}(\omega^{(1)}(u)))=o(1),\quad n\to\infty.$$

But this is the original Tauber condition. Hence

$$\sigma_n^{(2)}(\omega^{(1)}(u))=V_n^{(1)}(\Delta u)-V_n^{(2)}(\Delta u)=o(1),\quad n\to\infty.$$

Since $\{\sigma_n^{(1)}(\omega^{(1)}(u)\}$ is $(C,1)$-summable to zero and is slowly oscillating, it follows from the identity (8) that

$$\sigma_n^{(1)}(\omega^{(1)}(u))=V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)=o(1),\quad n\to\infty, \tag{23}$$

i.e., $n(V_n^{(1)}(\Delta u)-V_{n-1}^{(1)}(\Delta u))=o(1), n\to\infty$. By the corollary to the original Tauber theorem we have $V_n^{(1)}(\Delta u)=o(1), n\to\infty$. Thus from (23) it follows that $V_n^{(0)}(\Delta u)=o(1), n\to\infty$. Hence by the original Tauber theorem we have

$$\lim_n u_n=\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_nx^n.$$

Theorem 8. can be proved for complex $\{u_n\}$ by using the Littlewood theorem rather than the corollary to Karamata's Haupsatz as in the previous section.

We also consider the one sided-boundedness of the control modulo $\omega_n^{(1)}(u)$ of order one to recover convergence of $\{u_n\}$ out of the existence of the limit (1). Notice that

$$\begin{aligned}\omega_n^{(1)}(u)&=\omega_n^{(0)}(u)-\sigma_n^{(1)}(\omega^{(0)}(u))=\\ n\Delta u_n-V_n^{(0)}(\Delta u)&=n\Delta V_n^{(0)}(\Delta u)\geq -C\end{aligned} \tag{24}$$

for some $C\geq 0$ and for all nonnegative integers n. However, in this case, $\{u_n\}$ must be strictly real.

Theorem 11. *For a real sequence $\{u_n\}$, let the limit (1) exist. If for some $C\geq 0$ and for all nonnegative integers n, $n\Delta V_n^{(0)}(\Delta u)\geq -C$, then*

$$\lim_n u_n=\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_nx^n.$$

Proof. Taking $(C,1)$-means of both sides of (24), we have $V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) \geq -C$ for all nonnegative integers n and for some $C \geq 0$. Also from the existence of the limit (1) it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u))x^n = 0.$$

Therefore by the corollary to Karamata's Hauptsatz, we have $\lim_n(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)) = 0$, i.e, $V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n\Delta V_n^{(2)}(\Delta u) = o(1)$, $n \to \infty$. Hence from the corollary to the original Tauber theorem we have $V_n^{(2)}(\Delta u) = o(1)$, $n \to \infty$, for the existence of the limit (1) implies that $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(2)}(\Delta u)x^n = 0$. Therefore $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. It remains to show that $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. To prove this we need the following identities

$$V_n^{(0)}(\Delta u) = V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}(V_j^{(0)}(\Delta u) - V_{j-1}^{(0)}(\Delta u)), \tag{25}$$

for $\lambda > 1$ and

$$\begin{aligned} V_n^{(0)}(\Delta u) &= V_{2n-[\lambda n]}^{(1)}(\Delta u) + \frac{n+1}{[\lambda n]-n}(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u)), \\ &+ \frac{1}{[\lambda n]-n}\sum_{k=2n-[\lambda n]+1}^{n}\sum_{j=n+1}^{k}(V_j^{(0)}(\Delta u) - V_{j-1}^{(0)}(\Delta u)), \end{aligned} \tag{26}$$

for $1 < \lambda < 2$.

Since $\{V_n^{(0)}(\Delta u)\}$ satisfies $-(V_j^{(0)}(\Delta u) - V_{j-1}^{(0)}(\Delta u)) \leq \dfrac{C}{j}$ for all positive integers j, from the identity (25) we obtain the following inequality

$$\begin{aligned} V_n^{(0)}(\Delta u) &\leq V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - \\ &\quad V_n^{(1)}(\Delta u)) + \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{C}{j} \\ &\leq V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) + C_1\sum_{k=n}^{[\lambda n]}\frac{1}{k} \\ &\leq V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) + C_1 \lg\left(\frac{[\lambda n]}{n}\right). \end{aligned}$$

Taking limsup in n of both sides of the last inequality, we get

$$\overline{\lim_n} V_n^{(0)}(\Delta u) \leq \overline{\lim_n} V_n^{(1)}(\Delta u) + \frac{\lambda}{\lambda - 1} \overline{\lim_n} (V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) + C_1 \lg \lambda.$$

Since $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$, the first two terms on the right-hand side of the above inequality vanish. Thus $\overline{\lim}_n V_n^{(0)}(\Delta u) \leq C_1 \lg \lambda$. Taking the limit of both sides as $\lambda \to 1+$, we obtain

$$\overline{\lim}_n V_n^{(0)}(\Delta u) \leq 0. \tag{27}$$

Applying a similar procedure to the identity (26), we have the following inequality

$$V_n^{(0)}(\Delta u) \geq V_{2n-[\lambda n]}^{(1)}(\Delta u) + \frac{n+1}{[\lambda n] - n}(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u))$$
$$- \frac{1}{[\lambda n] - n} \sum_{k=2n-[\lambda n]+1}^{n} \sum_{j=k+1}^{n} \frac{C}{j}$$
$$\geq V_{2n-[\lambda n]}^{(1)}(\Delta u) + \frac{n+1}{[\lambda n] - n}(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u)) - C_2 \sum_{k=2n-[\lambda n]+1}^{n} \frac{1}{k}$$
$$\geq V_{2n-[\lambda n]}^{(1)}(\Delta u) + \frac{n+1}{[\lambda n] - n}(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u)) - C_2 \sum_{k=2n-[\lambda n]+1}^{[\lambda n]} \frac{1}{k}$$
$$\geq V_{2n-[\lambda n]}^{(1)}(\Delta u) + \frac{n+1}{[\lambda n] - n}(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u)) - C_2 \lg\left(\frac{[\lambda n]}{2n - [\lambda n] + 1}\right).$$

Taking the liminf in n of both sides term by term of the last inequality, we have:

$$\underline{\lim}_n V_n^{(0)}(\Delta u) \geq \underline{\lim}_n V_{2n-[\lambda n]}^{(1)}(\Delta u) +$$
$$\frac{1}{\lambda - 1} \underline{\lim}_n \left(V_n^{(1)}(\Delta u) - V_{2n-[\lambda n]}^{(1)}(\Delta u)\right) - C_2 \lg\left(\frac{\lambda}{2 - \lambda}\right).$$

Since $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$, the first two terms on the right-hand-side of the above inequality vanish as $n \to \infty$. Hence $\underline{\lim}_n V_n^{(0)} \geq -C_2 \lg\left(\frac{\lambda}{2 - \lambda}\right)$. Finally, after taking the limit of both sides of the last inequality as $\lambda \to 1$, we have

$$\underline{\lim}_n V_n^{(0)}(\Delta u) \geq 0. \tag{28}$$

Hence from (27) and (28) it follows that $\lim_n V_n^{(0)}(\Delta u) = 0$. Finally we have $\lim_n u_n = \lim_{x \to 1-} (1 - x) \sum_{n=0}^{\infty} u_n x^n$.

The method used in Theorem 9. to show that $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ from the existence of $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$, works fine in the situations where one-sided boundedness of the sequences $\{u_n\}$, $\{\omega_n^{(0)}(u)\}$, or others, is involved, even though it is a long process. However, we can give a much shorter proof of Theorem 9. as follows. Since $n\Delta V_n^{(0)}(\Delta u) \geq -C$ for some $C \geq 0$ and for all nonnegative integers n and since the existence of the limit (1) implies $\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = 0$, from the Hardy-Littlewood theorem we have $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Therefore convergence of $\{u_n\}$ follows from the original Tauber theorem.

In Theorem 8. we can weaken the condition that the limit (1) exists by assuming the existence of the limit (20).

Theorem 12. *Let the limit (20) exist. If $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory, then* $\lim_n u_n = \lim_{x\to 11-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n$.

Proof. Since $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory, it follows that $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is slowly oscillating. Hence

$$\sigma_n^{(1)}(\omega^{(1)}(u)) - \sigma_n^{(2)}(\omega^{(1)}(u)) =$$
$$n(\sigma_n^{(2)}(\omega^{(1)}(u)) - \sigma_{n-1}^{(2)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty.$$

Since the limit (20) exists and since

$$\sigma_n^{(2)}(\omega^{(1)}(u)) = V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u),$$

it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(\omega^{(1)}(u))x^n =$$
$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u))x^n = 0.$$

Hence from the Littlewood theorem we have $\lim_n \sigma_n^{(2)}(\omega^{(1)}(u)) = 0$. That is,

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n(V_n^{(2)}(\Delta u) - V_{n-1}^{(2)}(\Delta u)) = o(1), \quad n \to \infty. \tag{29}$$

Since the existence of the limit (20) implies that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(2)}(\Delta u)x^n = 0,$$

we obtain from the corollary to the original Tauber theorem that $V_n^{(2)}(\Delta u) = o(1)$, $n \to \infty$. Thus from (29) we get

$$V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty. \tag{30}$$

Therefore from the fact that $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is slowly oscillating, that is, $\{V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\}$ is slowly oscillating, we have $\{V_n^{(0)}(\Delta u)\}$ is slowly oscillating. Hence from (30) and the identity (8), we have $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. From

$$u_n - \sigma_n^{(1)}(u) = n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) = V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty,$$

it follows by the corollary to the original Tauber theorem that $\lim_n \sigma_n^{(1)} = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$, which completes the proof.

In the proof of Theorem 10. first we could have shown that the limit (1) exists from the conditions of the theorem by showing that $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ from $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$ and used Theorem 8. to complete the proof.

Theorem 9. is also generalized as in Theorem 10. and is proved in a similar way.

Theorem 13. *For a real sequence $\{u_n\}$, let the limit (20) exist. If for some $C \geq 0$ and for all nonnegative integers n, $\omega_n^{(1)}(u) = n\Delta u_n - V_n^{(0)}(\Delta u) \geq -C$, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

We will give a detailed proof of Theorem 11. in the next chapter.

If the condition that $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory is replaced with some stronger conditions in the above theorems, the following corollaries are obtained.

Corollary 4. *Let the limit (20) exist. If $\{\omega_n^{(1)}(u)\}$ is slowly oscillating, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Corollary 5. *Let the limit (20) exist. If $\omega_n^{(1)}(u) = O(1)$, $n \to \infty$, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

When the condition that $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory is replaced by a weaker condition that $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is slowly oscillating, in Theorem 10. convergence of $\{u_n\}$ is still recovered out of the existence of the limit (20).

Theorem 14. *Let the limit (20) exist. If* $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ *is slowly oscillating, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

The proof is similar to the proof of Theorem 10. Since we have had Tauberian theorems for the classical control modulo of the oscillatory behavior, $\omega_n^{(0)}(u)$, and the control modulo of the oscillatory behavior of order one, $\omega_n^{(1)}(u)$, in a very similar way we can have a generalization of Theorem 10. by conditioning the higher order control modulo of the oscillatory behavior of $\{u_n\}$, in particular, $\{\omega_n^{(2)}(u)\}$. It is proved in the same way, but the proof is longer. By assuming the existence of the limit (1) instead of the existence of the limit (20), the proof can be made shorter.

Theorem 15. *Let the limit (20) exist. If* $\{\omega_n^{(2)}(u)\}$ *is moderately oscillatory, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

If we replace $\omega_n^{(0)}(u)$ by the iterated modulo $\omega_n^{(0)}(\omega^{(0)}(u)) = n\Delta\omega_n^{(0)}(u)$, then we have the following theorem.

Theorem 16. *Let the limit (1) exist. If* $\{\omega_n^{(0)}(\omega^{(0)}(u))\}$ *is moderately oscillatory, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n.$$

Proof. Since $\{\omega_n^{(0)}(\omega^{(0)}(u))\}$ is moderately oscillatory and since

$$\sigma_n^{(1)}(\omega^{(0)}(\omega^{(0)}(u))) = V_n^{(0)}(\Delta\omega^{(0)}(u)) = \omega_n^{(1)}(u),$$

$\{\omega_n^{(1)}(u)\}$, is slowly oscillating. Hence from the corollary 26 it follows that

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n.$$

The modulo $n\Delta\omega_n^{(0)}(u)$ can also be used in one-sided fashion for real sequences $\{u_n\}$.

Theorem 17. *For a real sequence* $\{u_n\}$*, let the limit (1) exist. If* $\omega_n^{(0)}(\omega^{(0)}(u)) \geq -C$ *for some* $C \geq 0$ *and for all nonnegative integers* n*, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}u_n x^n.$$

Proof. Since $\{\omega_n^{(0)}(\omega^{(0)}(u))\}$ implies that

$$\sigma_n^{(1)}(\omega^{(0)}(\omega^{(0)}(u))) = V_n^{(0)}(\Delta\omega^{(0)}(u)) = \omega_n^{(1)}(u) \geq -C,$$

from Theorem 11. it follows that

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

In Theorem 10. replacing $\{\omega_n^{(1)}(u)\}$ by $\{\omega_n^{(1)}(\omega^{(0)}(u))\}$, we can still recover convergence of $\{u_n\}$ out of the existence of the limit (1).

Theorem 18. *Let the limit (1) exist. If* $\{\omega_n^{(1)}(\omega^{(0)}(u))\}$ *is moderately oscillatory, then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

Proof. Since $\{\omega_n^{(1)}(\omega^{(0)}(u))\}$ is moderately oscillatory,

$$\sigma_n^{(1)}(\omega^{(1)}(\omega^{(0)}(u))) = V_n^{(0)}(\Delta\omega^{(0)}(u)) - V_n^{(1)}(\Delta\omega^{(0)}(u)) =$$
$$\omega_n^{(1)}(u) - \sigma_n^{(1)}(\omega^{(1)}(u)) = \omega_n^{(2)}(u)$$

is slowly oscillating. Hence by Theorem 13. we obtain

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

The existence of the limit (1) can be replaced by the existence of the limit (20) in all the above theorems. A generalization of Theorem 16. can also be given for real $\{u_n\}$, that is, if we assume one-sided boundedness of the iterated modulo $\omega_n^{(1)}(\omega^{(0)}(u)) = n\Delta V_n^{(0)}(\Delta\omega^{(0)}(u))$ and the existence of the limit (1) or (20), then convergence of $\{u_n\}$ is recovered.

We will finish this section by generalizing some of the previous theorems to obtain Tauberian theorems for $\{\sigma_n^{(m)}(u)\}$, for any integer $m \geq 0$.

Theorem 19. *For some integer and for real* $\{u_n\}$*, let*

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u)x^n \tag{31}$$

exist. If

$$\{V_n^{(0)}(\Delta\sigma^{(m)}(u)) - V_n^{(1)}(\Delta\sigma^{(m)}(u))\} \tag{32}$$

is moderately oscillatory, then

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(m)}(u)x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u)x^n.$$

Proof. Observe that

$$V_n^{(0)}(\Delta\sigma^{(m)}(u)) - V_n^{(1)}(\Delta\sigma^{(m)}(u)) = V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)$$

because

$$\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(0)}(\Delta\sigma^{(m)}(u)) = V_n^{(m)}(\Delta u).$$

Since $\{V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)\}$ is moderately oscillatory, $(C,1)$-means of this sequence is slowly oscillating, that is,

$$\{V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u)\} \tag{33}$$

is slowly oscillating, and

$$\begin{gathered}(V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)) - \\ \sigma_n^{(1)}(V^{(m)}(\Delta u) - V^{(m+1)}(\Delta u)) = O(1),\ n \to \infty\end{gathered} \tag{34}$$

Also

$$\begin{gathered}(V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u)) - \\ \sigma_n^{(1)}(V^{(m+1)}(\Delta u) - V^{(m+2)}(\Delta u)) = O(1),\ n \to \infty\end{gathered} \tag{35}$$

and (31) implies that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u))x^n = 0$$

From (35) it follows that

$$n\Delta\sigma_n^{(1)}(V^{(m+1)}(\Delta u) - V^{(m+2)}(\Delta u)) \geq -C$$

for some $C \geq 0$ and for all nonnegative integers n. Thus applying the corollary to Karamata's Hauptsatz we obtain

$$\frac{1}{n+1}\sum_{k=0}^{n} k\Delta\sigma_k^{(1)}(V^{(m+1)}(\Delta u) - V^{(m+2)}(\Delta u)) = o(1),\ n \to \infty,$$

but this is the original Tauber condition. Therefore

$$\sigma_n^{(1)}(V^{(m+1)}(\Delta u) - V^{(m+2)}(\Delta u)) = o(1), \quad n \to \infty$$

Hence from (33) it follows that

$$V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u)) = n(\Delta V^{(m+2)}(\Delta u)) = o(1), \quad n \to \infty \tag{36}$$

Since

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(m+1)}(\Delta u)x^n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} V_n^{(m+2)}(\Delta u)x^n = 0,$$

by the corollary to the original Tauber Theorem, we have

$$V_n^{(m+2)}(\Delta u) = o(1), \quad n \to \infty. \tag{37}$$

Hence from (36) we get

(38) $$V_n^{(m+1)}(\Delta u) = o(1), \quad n \to \infty.$$

Since $\{V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)\}$ is moderately oscillatory, we have from (33), (37), and (38) $V_n^{(m)}(\Delta u) = O(1)$, $n \to \infty$ and

$$\lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} V_n^{(m)}(\Delta u) x^n = 0.$$

Therefore $\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(m)}(\Delta u)$ implies that

$$\lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m)}(\Delta u) x^n = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m+1)}(\Delta u) x^n.$$

In the above theorem $\{u_n\}$ does not have to be real as in the previous theorems. By the Littlewood theorem, we can prove it for complex sequences $\{u_n\}$.

Theorem 20. *For some integer $m \geq 0$ and for real $\{u_n\}$, let the limit (31) exist. If $\{n\Delta\sigma_n^{(m)}(u) - V_n^{(0)}(\Delta\sigma^{(m)}(u))\}$ is moderately oscillatory, then*

$$\lim_n \sigma_n^{(m)}(u) = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u) x^n.$$

Proof. From Theorem 17. it follows that

$$\lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m)}(u) x^n = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u) x^n.$$

Since $\{n\Delta\sigma_n^{(m)}(u) - V_n^{(0)}(\Delta\sigma^{(m)}(u))\}$ is moderately oscillatory, $(C,1)$-means of this sequence is slowly oscillating. That is,

$$\{V_n^{(0)}(\Delta\sigma_n^{(m)}(u)) - V_n^{(1)}(\Delta\sigma^{(m)}(u))\}$$

is slowly oscillating. But observe that

$$V_n^{(0)}(\Delta\sigma_n^{(m)}(u)) - V_n^{(1)}(\Delta\sigma^{(m)}(u)) = V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)$$

because

$$\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(0)}(\Delta\sigma^{(m)}(u)) = V_n^{(m)}(\Delta u).$$

Hence we have

(39) $$(V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)) - \sigma_n^{(1)}(V^{(m)}(\Delta u) - V^{(m+1)}(\Delta u)) = O(1), \; n \to \infty$$

Therefore

(40) $$n\Delta\sigma_n^{(1)}(V^{(m)}(\Delta u) - V^{(m+1)}(\Delta u)) \geq -C$$

for some $C \geq 0$ and for all nonnegative integers n. Since (31) implies that

$$\lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} (V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u)) x^n = 0,$$

applying the corollary to Karamata's Hauptsatz, we get

$$\frac{1}{n+1} \sum_{k=0}^{n} k \Delta \sigma_k^{(1)} (V^{(m)}(\Delta u) - V^{(m+1)}(\Delta u)) = o(1), \; n \to \infty,$$

But this is the original Tauber condition. Therefore

$$V_n^{(m+1)}(\Delta u) - V_n^{(m+2)}(\Delta u) = o(1), \quad n \to \infty. \tag{41}$$

Since $\{V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)\}$ is slowly oscillating, we have from (41) and the identity (8) for $\{V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)\}$ that

$$V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u) = o(1), \quad n \to \infty. \tag{42}$$

That is, $n \Delta V_n^{(m+1)}(\Delta u) = o(1)$, $n \to \infty$. Since we have

$$\lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} (V_n^{(m)}(\Delta u) - V_n^{(m+1)}(\Delta u)) x^n = 0,$$

it follows from the corollary to the original Tauber theorem that $V_n^{(m+1)}(\Delta u) = o(1)$, $n \to \infty$. Thus (42) implies that $V_n^{(m)}(\Delta u) = o(1)$, $n \to \infty$. Hence from the identity

$$\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(0)}(\Delta \sigma^{(m)}(u)) = V_n^{(m)}(\Delta u) = o(1), \; n \to \infty \tag{43}$$

we have $n \Delta \sigma_n^{(m+1)}(u) = o(1)$, $n \to \infty$. Again from the corollary to the original Tauber theorem we have

$$\lim_n \sigma_n^{(m+1)}(u) = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u) x^n = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m)}(u) x^n.$$

Finally from (43) we obtain $\lim_n \sigma_n^{(m)}(u) = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(m+1)}(u) x^n$.

3. Tauberian theorems for moderately oscillatory regularly generated sequences

3.1 Introduction

Regularly generated sequences will be studied using Tauberian concepts and techniques. In the classical Tauberian theory, the common generator sequence of is the sequence of $\{u_n\}$ the Kronecker sums $\{V_n^{(0)}(\Delta u)\}$ of $\{u_n\}$. In [12, 14] it is shown that if instead of the sequence the sequence of the

Kronecker sums $\{V_n^{(0)}(\Delta u)\}$ we consider any sequence with some Tauberian-like conditions, then we could obtain certain non-classical Tauberian conditions. This motivated the concept of regularly generated sequences defined on page 63.

Let L be any linear space and let β be a class of sequences $\{\beta_n\}$ from L. The class of all sequences $\{u_n\}$ from L such that for some $\{\beta_n\}$ from β and for all nonnegative integers n

$$u_n = \beta_n + \sum_{k=1}^{n} \frac{\beta_k}{k} + u_o, \tag{44}$$

is called the class of all regularly generated series $\mathcal{U}(\beta)$ by the class β.

For instance, if β is the class of all slowly oscillating sequences, then the class $\mathcal{U}(\beta)$ is not a classical class, but it is easy to show that if the limit (1) exists for a sequence $\{u_n\}$ from $\mathcal{U}(\beta)$, then (2) holds. Indeed, from (44), for all nonnegative integers n, it follows that

$$n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta\beta_n \tag{45}$$

Taking $(C,1)$-means of both sides in all corresponding terms of (45), we have

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = V_n^{(0)}(\Delta\beta). \tag{46}$$

Due to the existence of the limit (1), it follows that

$$\lim_{x\to 1-} (1-x)\sum_{n=0}^{\infty} (V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u))x^n = 0 \tag{47}$$

and for the same reason

$$\lim_{x\to 1-} (1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = 0 \tag{48}$$

Since $\{\beta_n\}$ is slowly oscillating, $\left\{V_n^{(o)}(\Delta\beta)\right\}$ is bounded and slowly oscillating. Thus we have

$$V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) = V_n^{(o)}(\Delta\beta) = O(1), \quad n\to\infty \tag{49}$$

Hence from (48) and (49) we have by the Littlewood theorem

$$V_n^{(1)}(\Delta u) = o(1), \quad n\to\infty.$$

Since

$$V_n^{(o)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k = \beta_n$$

and

$$\left\{V_n^{(o)}(\Delta u)\right\}$$

is slowly oscillating, from the identity (8), it follows that

$$V_n^{(o)}(\Delta u) = o(1) \quad n \to \infty.$$

Hence the original Tauberian theorem

$$\lim_n u_n = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

This result can be also obtained by replacing the existence of the limit (1) with the existence of the limit (20) for real or complex sequences $\{u_n\}$ from $U(\beta)$.

Consider now a real sequence $\{u_n\}$. Observe that from (49) it follows that

(50) $$V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) \geq -C$$

for some $C \geq 0$ and for all nonnegative integers n. Now instead of using the Littlewood theorem in the previous proof we may use the corollary to Karamata's Hauptsatz to obtain

$$V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty.$$

Hence from (47) and (50) we obtain by the corollary to Karamata's Hauptsatz that

$$\frac{1}{n+1} \sum_{k=0}^{n} k\Delta V_k^{(1)}(\Delta u) = o(1), \quad n \to \infty.$$

Consequently the sequence $\left\{V_n^{(1)}(\Delta u)\right\}$ converges to zero by the original Tauber theorem. That is,

(51) $$\lim_n V_n^{(1)}(\Delta u) = \lim_{x \to 1-} (1-x) \sum_{n=0}^{\infty} V_n^{(1)}(\Delta u) x^n = 0$$

Observe that at this stage of the proof, again we could use the fact that $\left\{V_n^{(o)}(\Delta u)\right\}$ is slowly oscillating and obtain the same result as above. However, we will prove that $V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$ in a different way.

Recall that for $\lambda > 1$, de la Vallee Poussin means of $\left\{V_n^{(o)}(\Delta u)\right\}$ are defined as

$$\tau_{n,[\lambda n]}(V^{(o)}(\Delta u)) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} V_k^{(o)}(\Delta u).$$

From (51) it follows that

(52) $$\lim_{\lambda \to 1+} \overline{\lim_n} \, \tau_{n,[\lambda n]}(V^{(o)}(\Delta u)) = 0$$

and from the inequality

$$\left|V_n^{(1)}(\Delta u) - V_n^{(o)}(\Delta u)\right| \le \left|V_n^{(1)}(\Delta u) - \tau_{n,[\lambda n]}(V^{(o)}(\Delta u))\right| + \left|\tau_{n,[\lambda n]}(V^{(o)}(\Delta u)) - V_n^{(o)}(\Delta u)\right|.$$

Taking first limsup of both sides in n of the above inequality and then taking limit in $\lambda \to 1+$ we obtain

$$\lim_{\lambda\to 1+}\overline{\lim_{n}}\left|V_n^{(1)}(\Delta u) - V_n^{(o)}(\Delta u)\right| = \overline{\lim_{n}}\left|V_n^{(1)}(\Delta u) - V_n^{(o)}(\Delta u)\right|$$
$$\le \lim_{\lambda\to 1+}\overline{\lim_{n}}\left|V_n^{(1)}(\Delta u) - \tau_{n,[\lambda n]}(V^{(o)}(\Delta u))\right|$$
$$+ \lim_{\lambda\to 1+}\overline{\lim_{n}}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}(V_j^{(o)}(\Delta u) - V_{j-1}^{(o)}(\Delta u))\right|.$$

The first term on the right hand side of the last expression is zero because of the limit (52), and the second term is also equal to zero because of a more convenient definition of slow oscillation given in [3]. Consequently, we have $V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$. Finally, as before it follows that (2) holds.

Let $\{u_n\}$ be a complex sequence and let the limit (20) exist. Since $\left\{V_n^{(o)}(\Delta u)\right\} = \{\beta_n\}$ and since $\{\beta_n\}$ is slowly oscillating, $\left\{V_n^{(o)}(\Delta u)\right\}$ is slowly oscillating. Thus $\left\{V_n^{(1)}(\Delta u)\right\}$ is also slowly oscillating. From the existence of the limit (20) it follows that

$$\lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty}V_n^{(1)}(\Delta u)z^n = 0.$$

Therefore by Theorem 4. $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Hence $\left\{V_n^{(o)}(\Delta u)\right\}$ is $(C,1)$-summable to zero. Since $\left\{V_n^{(o)}(\Delta u)\right\}$ is slowly oscillating, from the identity (8), we have $V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$. Finally convergence of the sequence $\{u_n\}$ follows as in previous proofs. As an alternative way here we have utilized Theorem 4. This shows that we could have used this theorem in our previous proofs. However, as mentioned after Theorem 8 we would prefer using the original Tauber theorem and its corollary and the corollary to Karamatas Hauptsatz since these are the essentials of the classical and neoclassical Tauberian theory.

3.2 Tauberian theorems for regularly generated sequences

Let β be the class of all moderately oscillatory sequences $\{\beta_n\}$ and let β^* be the class of all sequences $\{\beta_n^*\}$ such that for some $\{\beta_n\}$ from β, and for all nonnegative integers n, $\beta_n^* = \sum\limits_{k=1}^{n} \frac{\beta_k}{k}$. The class $\mathcal{U}(\beta^*)$ is the class of regularly generated sequences by β^* if for any $\{u_n\} \in \mathcal{U}(\beta^*)$ there exists a sequence$\{\beta_n^*\} \in \beta^*$ such that for all nonnegative integers n $u_n = \beta_n^* + \sum\limits_{k=1}^{n} \frac{\beta_k^*}{k} + u_o$. After considering the class of moderately oscillatory regularly generated sequences, we will prove some theorems similar to Tauberian theorems.

Theorem 21. *Let the limit*

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n \tag{53}$$

exist. If $\{u_n\} \in \mathcal{U}(\beta^*)$, *then*

$$\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Proof. Since

$$u_n = \sum_{k=1}^{n}\frac{\beta_{k1}}{k} + \sum_{k=1}^{n}\frac{\sum\limits_{j=1}^{k}\frac{\beta_j}{j}}{k} + u_o$$

$n\Delta u_n = \beta_n + \sum\limits_{k=1}^{n} \frac{\beta_k}{k}$, and

$$V_n^{(o)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k = \sigma_n^{(1)}(\beta) + \sigma_n^{(1)}(\beta^*),$$

$$n\Delta u_n - V_n^{(o)}(\Delta u) = \beta_n + (\beta_n^* - \sigma_n^{(1)}(\beta^*)) - \sigma_n^{(1)}(\beta) = \beta_n. \tag{54}$$

Since $\{\beta_n\}$ is moderately oscillatory, $\left\{\sigma_n^{(1)}(\beta)\right\}$ is slowly oscillating. From (54) we have $V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) = \sigma_n^{(1)}(\beta)$. Hence $\left\{V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u)\right\}$ is slowly oscillating. Therefore

$$V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) - (V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)) = O(1), \quad n \to \infty,$$

that is,

$$n\Delta(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)) = O(1), \quad n \to \infty \tag{55}$$

Observe that, from the existence of the limit (20) it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u))x^n = 0 \tag{56}$$

Now from (55) and (56) the Littlewood theorem yields

(57) $$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n\Delta V_n^{(2)}(\Delta u) = o(1), \quad n \to \infty$$

Thus from the corollary to the original Tauber theorem, we have

(58) $$V_n^{(2)}(\Delta u) = o(1), \quad n \to \infty.$$

Hence from (57) and (58), we get

(59) $$V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty.$$

From the fact that $\left\{V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u)\right\}$ is slowly oscillating and (59), it is clear that $\left\{V_n^{(o)}(\Delta u)\right\}$ is slowly oscillating. Thus $V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$, from (59) and slow oscillation of $\left\{V_n^{(o)}(\Delta u)\right\}$. by the identity (8). Since $V_n^{(o)}(\Delta u) = o(1)$, $n \to \infty$, it follows that

$$\lim_n u_n = \lim_{x\to 1-} (1-x)\sum_{n=0}^{\infty} u_n x^n = \lim_{x \to 1-} (1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n.$$

From (54) we see that $\left\{\omega_n^{(1)}(u)\right\}$ is moderately oscillatory since $\{\beta_n\}$ is moderately oscillatory. Therefore we could use Theorem 10 to conclude Theorem 21.

Next we will consider the class of one-sidedly regularly generated sequences [12,14] and prove a Tauberian theorem similar to the classical one-sidedly Tauberian theorems. Let β be the class of all real sequences $\{\beta_n\}$ such that for every β_n there exists some $C_\beta \geq 0$ so that for all nonnegative integers n, $\beta_n \geq -C_\beta$. Now define $\beta_n^* = \sum\limits_{k=1}^{n} \frac{\beta_k}{k}$, for all nonnegative integers n. Denote the class of all sequences $\{\beta_n^*\}$ by β^*. The class $\mathcal{U}(\beta^*)$ is the class of all one-sidedly regularly generated sequences by β^* if for every $\{u_n\} \in \mathcal{U}(\beta^*)$, there exists a sequence $\{\beta_n^*\} \in \beta^*$ such that for all nonnegative integers n $u_n = \beta_n^* + \sum\limits_{k=1}^{n} \frac{\beta_k^*}{k}$.

In [12] it is shown that if the limit (1) exists and $\{u_n\} \in \mathcal{U}(\beta^*)$, then (2) holds. In the following theorem we will show that if the existence of the limit (1) is replaced by the existence of the limit (20) in Theorem 21 convergence of $\{u_n\} \in \mathcal{U}(\beta^*)$ is still recovered.

Theorem 22. *For a real $\{u_n\}$, let the limit (20) exist. If $\{u_n\} \in \mathcal{U}(\beta^*)$, then*

$$\lim_n u_n = \lim_{x\to 1-} (1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n.$$

Proof. Recalling (54) we see that $\omega_n^{(1)}(u) = n\Delta u_n - V_n^{(o)}(\Delta u) = \beta_n$. Since $\{\beta_n^*\}$ is one-sidedly bounded,

$$\omega_n^{(1)}(u) = \beta_n \geq -C_\beta \tag{60}$$

for some $C_\beta \geq 0$ and for all nonnegative integers n. Now if we can show that the limit (1) exists from the assumptions of Theorem 22 then we will obtain the result of this theorem from Theorem 9. Taking $(C,1)$-means of (60), we have $V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) \geq -C_\beta$ for some $C_\beta \geq 0$ and for all nonnegative integers n. If we again take $(C,1)$-means of this last inequality, we get

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n\Delta V_n^{(2)}(\Delta u) \geq -C_\beta \tag{61}$$

for some $C_\beta \geq 0$ and for all nonnegative integers n. On the other hand, from the existence of the limit (20) it follows that

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u))x^n = 0. \tag{62}$$

From (61) and (62) we obtain by the corollary to Karamata's Hauptsatz

$$\frac{1}{n+1}\sum_{k=0}^{n} k(V_k^{(2)}(\Delta u) - V_{k-1}^{(2)}(\Delta u)) = o(1), \quad .n \to \infty.$$

Thus by the original Tauber theorem it follows that

$$V_n^{(2)}(\Delta u) - V_n^{(3)}(\Delta u) = n\Delta V_n^{(3)}(\Delta u) = o(1), \quad n \to \infty. \tag{63}$$

The corollary to the original Tauber theorem yields

$$V_n^{(3)}(\Delta u) = o(1), \quad n \to \infty \tag{64}$$

From (63) and (64) we have

$$V_n^{(2)}(\Delta u) = o(1), \quad n \to \infty. \tag{65}$$

Since $\sigma_n^{(2)}(u) - \sigma_n^{(3)}(u) = n\Delta\sigma_n^{(3)}(u) = V_n^{(2)}(\Delta u) = o(1)$, $n \to \infty$, by the corollary to the original Tauber theorem, $\lim_n \sigma_n^{(3)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ which implies that the limit (1) exists [17] and equal to the limit (20).

Finally from (61) and the existence of the limit (1), we have by Theorem 9. $\lim_n u_n = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$.

We could use the Hardy-Littlewood theorem to give another proof of Theorem 22 as follows. From the existence of the limit (20) we have

$$\lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}V_n^{(1)}(\Delta u)x^n = 0. \tag{66}$$

Since

$$V_n^{(o)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) \geq -C_\beta \tag{67}$$

for some $C_\beta \geq 0$ and for all nonnegative integers n, from (66) and (67) the Hardy-Littlewood theorem yields $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Hence we have

$$\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n.$$

Therefore

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n$$

which implies that the limit (1) exists and equal to the limit (20).

In Theorems 21 and 22 letting $\beta_n^* = O(1)$, $n \to \infty$, we can easily obtain convergence of $\{u_n\}$ out of the existence of the limit (1) or (20).

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