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ON A FAMILY OF $(n+1)$–ARY EQUIVALENCE RELATIONS

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Abstract. The notion of a partition of type $n(n \in N)$ was introduced by J. Hartmanis in [1] as a generalization of the notion of an ordinary partition of a set. It is a well–known fact that partitions of Q (of type 1) correspond in a one–one way to equivalence relations on Q. In this article we introduce an analogous family of relations $(\mathcal{F}_n(Q))$ for partitions of type n. Furthermore, for $\rho \in \mathcal{F}_n(Q)$ the following statements hold: $\circ(\overset{n+1}{\rho}) = \rho$ and $(\overset{n}{\rho})^{-1} = \rho$; for $n = 1$: $\sim \circ \sim = \sim$ and $\sim^{-1} = \sim$ (cf. [3]). A similar family of relations for partitions of type n was described by H. E. Pickett in [2] point out the differences.

1. Preliminaries

J. Hartmanis has introduced in [1] the notion of a partition of type n $(n \in N)$, for sets having at least n distinct elements, by means of the following definition:

1.1. Definition: *Let $|Q| \geq n$, $n \in N$ and let*

$$Q^{(n)} \overset{def}{=} \{\{a_1^n\} | \{a_1^n\} \subseteq Q \wedge |\{a_1^n\}| = n\}.$$

Then, we say that $\mathcal{P}_n(Q)$ is a **partition** *of Q of the* **type** *n iff the following statements hold:*

$H1$ For each $C \in \mathcal{P}_n(Q)$ there is at least one $\{a_1^n\} \in Q^{(n)}$ such that $\{a_1^n\} \subseteq C$; and

$H2$ For each $\{a_1^n\} \in Q^{(n)}$ there is exactly one $C \in \mathcal{P}_n(Q)$ such that $\{a_1^n\} \subseteq C$.

The partitions of type 1 are the ordinary partitions of a set, and the partitions of type 2 are **incidence geometries**.

An analogous family of relations for partitions of type n was described by H. E. Pickett in [2], in the following way:

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$(\forall a_i \in Q)_1^n (a_1^n, a_1) \in \sim_n;$
$(\forall a_i \in Q)_1^{n+1} (\forall \alpha \in \{1,\ldots,n+1\}!)((a_1^{n+1}) \in \sim_n \Longrightarrow (a_{\alpha(1)},\ldots,a_{\alpha(n+1)}) \in \sim_n);$
and
$$(\forall a_i \in Q)_1^{n+2} (\{a_2^{n+1}\} \in Q^{(n)} \wedge (a_1^{n+1}) \in \sim_n \wedge (a_2^{n+2}) \in \sim_n \Longrightarrow \Longrightarrow (a_1^n, a_{n+2}) \in \sim_n). \square$$

2. Main results

2.1. Definition: *Let* $|Q| \geq n$, $n \in N$, $\mathcal{L}(Q^{n+1}) \overset{def}{=} \{(a_1^{n+1}) | (a_1^{n+1}) \in Q^{n+1} \wedge \{a_1^n\} \in Q^{(n)}\}$, *and let* $\rho \subseteq \mathcal{L}(Q^{n+1})$. *Then, we say that* ρ *is an* $(n+1)-$*ary left full* **equivalence relation** *(briefly:* $(n+1)-LFE$*-relation) on* Q *iff the following statements hold:*

$\overset{n-1}{S}$ $(\forall \{a_1^n\} \in Q^{(n)})(\forall b \in Q)((a_1^n, b) \in \rho \Longrightarrow (\forall \alpha \in \{1,\ldots,n\}!)(a_{\alpha(1)}^{\alpha(n)}, b) \in \rho);$

Rn $(\forall \{a_1^n\} \in Q^{(n)})(a_1^n, a_1) \in \rho;$

Sn $(\forall \{a_1^n\} \in Q^{(n)})(\forall \{b_1^n\} \in Q^{(n)})(\bigwedge_{i=1}^{n} (a_1^n, b_i) \in \rho \Longrightarrow \bigwedge_{i=1}^{n} (b_1^n, a_i) \in \rho);$ *and*

Tn $(\forall \{a_1^n\} \in Q^{(n)})(\forall \{c_1^n\} \in Q^{(n)})(\forall b \in Q)(\bigwedge_{i=1}^{n} (a_1^n, c_i) \in \rho \wedge \wedge (c_1^n, b) \in \rho \Longrightarrow (a_1^n, b) \in \rho).$

2.2. Theorem: *Let* $|Q| \geq n$, $n \in N$ *and let* $\rho (\subseteq \mathcal{L}(Q^{n+1}))$ *be an* $(n+1)-LFE$*-relation on* Q*. Also let for each* $(a_1^n) \in Q^n$ *with* $|\{a_1^n\}| = n$*, and for each* $b \in Q$

(0) $$b \in C_{(a_1^n)} \overset{def}{\Longleftrightarrow} (a_1^n, b) \in \rho.$$

Then $\{C_{(a_1^n)} | (a_1^n) \in Q^n \wedge \{a_1^n\} \in Q^{(n)}\}$ *is a partition of* Q *of the type* n.

Proof. 1) By $\overset{n-1}{S}$ from 2.1, we conclude that the following equality holds $C_{(a_1^n)} = C_{(a_{\alpha(1)},\ldots,a_{\alpha(n)})}$ for all $\alpha \in \{1,\ldots,n\}!$. Therefore, instead of $C_{(a_1^n)}$, we write $C_{\{a_1^n\}}$ (briefly: $C_{a_1^n}$), with $\{a_1^n\} \in Q^{(n)}$.

2) By Rn and $\overset{n-1}{S}$, we conclude that the statement H1 holds.

3) The statement H2 holds.

Sketch of the proof.

a) Let $\{a_1^n\}$, $\{b_1^n\} \in Q^{(n)}$ and let $\bigwedge_{i=1}^{n} (b_i \in C_{a_1^n})$.

b)

$$\left(c \in C_{b_1^n} \wedge \bigwedge_{i=1}^{n}(b_i \in C_{a_1^n})\right) \overset{(0),1)}{\Longleftrightarrow} \left((b_1^n, c) \in \rho \wedge \bigwedge_{i=1}^{n}(a_1^n, b_i) \in \rho\right) \overset{Tn}{\Longrightarrow}$$

$$\left((a_1^n, c) \in \rho \overset{(0),1)}{\Longleftrightarrow} c \in C_{a_1^n}\right), \text{ i.e. } C_{b_1^n} \subseteq C_{a_1^n}.$$

c)

$$\left(c \in C_{a_1^n} \wedge \bigwedge_{i=1}^{n}(b_i \in C_{a_1^n})\right) \overset{(0),1)}{\Longleftrightarrow} \left((a_1^n, c) \in \rho \wedge \bigwedge_{i=1}^{n}(a_1^n, b_i) \in \rho\right) \overset{Sn}{\Longrightarrow}$$

$$\left((a_1^n, c) \in \rho \wedge \bigwedge_{i=1}^{n}(b_1^n, a_i) \in \rho\right) \overset{Tn}{\Longrightarrow} \left((b_1^n, c) \in \rho \overset{(0),1)}{\Longleftrightarrow} c \in C_{b_1^n}\right),$$

i.e. $C_{a_1^n} \subseteq C_{b_1^n}$.

d) Let $\{a_1^n\}, \{b_1^n\}, \{c_1^n\} \in Q^{(n)}$ and let $\{c_1^n\} \subseteq C_{a_1^n} \cap C_{b_1^n}$.
Then, by a)–c), we conclude that the following equalities hold

$$C_{a_1^n} = C_{c_1^n} \text{ and } C_{c_1^n} = C_{b_1^n}, \text{ i.e. } C_{a_1^n} = C_{b_1^n}. \quad \square$$

By 1.1 and 2.1, we conclude that the following proposition holds:

2.3. Theorem: *Let $|Q| \geq n, n \in N$, let $\mathcal{P}_n(Q)$ be a partition of Q of type n, and let $\rho \subseteq \mathcal{L}(Q^{n+1})$. Let also for each $\{a_1^n\} \in Q^{(n)}$ and for each $b \in Q$*

$$(\bar{0}) \ (a_1^n, b) \in \rho \overset{def}{\Longleftrightarrow} (\exists C \in \mathcal{P}_n(Q)))(\{a_1^n\} \subseteq C \wedge b \in C).$$

Then ρ is an $(n+1)$–LFE–relation in Q. $\square$

3. Two more propositions

3.1. Definitions: *Let $|Q| \geq n$, $n \in N$ and let $\rho_1^n, \rho \in \mathcal{L}(Q^{n+1})$. Then:*
a) we say that $\circ(\rho_1^n, \rho)$ is a **composition** *of relations ρ_1^n, ρ iff for each $\{x_1^n\} \in Q^{(n)}$ and for each $y \in Q$ the following statement holds:*

$$\text{(a)} \quad ((x_1^n, y) \in \circ(\rho_1^n, \rho)) \overset{def}{\Longleftrightarrow} \left((\exists \{z_1^n\} \in Q^{(n)})(\bigwedge_{i=1}^{n}(x_1^n, z_i) \in \rho_i \wedge (z_1^n, y) \in \rho)\right);$$

b) we say that $(\rho_1^n)^{-1}$ is an **inverse relation** *of the relations ρ_1^n iff for each $\{b_1^n\} \in Q^{(n)}$ and for each $a_n \in Q$ the following statement holds:*

$$\text{(b)} \quad ((b_1^n, a_n) \in (\rho_1^n)^{-1}) \overset{def}{\Longleftrightarrow} \left((\exists a_1^{n-1} \in Q^{n-1})(\bigwedge_{j=1}^{n}(a_1^n, b_j) \in \rho_j)\right); \text{ and}$$

*c) we say that $\rho^\alpha, \alpha \in \{1, \ldots, n\}!$, is a $\alpha-$***inverse relation** *of the relation ρ iff for each $\{a_1^n\} \in Q^{(n)}$ and for each $b \in Q$ the following statement holds:*

(c) $\qquad (a_1^n, b) \in \rho^\alpha \overset{def}{\Longleftrightarrow} (a_{\alpha(1)}, \ldots, a_{\alpha(n)}, b) \in \rho.$

3.2. Theorem: *Let $|Q| \geq n$, $n \in N$ and let ρ $(\subseteq \mathcal{L}(Q^{n+1}))$ be an $(n+1)$–LFE–relation in Q. Then the following equalities hold:*

$$\circ(\overset{n+1}{\rho}) = \rho, \ (\overset{n}{\rho})^{-1} = \rho \text{ and } \rho^\alpha = \rho$$

for all $\alpha \in \{1, \ldots, n\}!$.

Sketch of the proof.

$1_1)$ $(x_1^n, y) \in \circ(\overset{n+1}{\rho}) \overset{(a)}{\Longrightarrow} (\exists\{z_1^n\} \in Q^{(n)})(\bigwedge_{i=1}^{n} (x_1^n, z_i) \in \rho \wedge (z_1^n, y) \in \rho)$
$\overset{Tn}{\Longrightarrow} (x_1^n, y) \in \rho;$

$1_2)$ $(x_1^n, y) \in \rho \overset{\overset{n-1}{S}, Rn}{\Longrightarrow} \left(\bigwedge_{i=1}^{n} (x_1^n, x_i) \in \rho \wedge (x_1^n, y) \in \rho\right) \overset{(a)}{\Longrightarrow} \left((x_1^n, y) \in \circ(\overset{n+1}{\rho})\right);$

$2_1)$ $(x_1^n, y_n) \in (\overset{n}{\rho})^{-1} \overset{(b)}{\Longrightarrow} (\exists a_1^{n-1} \in Q^{n-1})(\bigwedge_{i=1}^{n} (y_1^n, x_i) \in \rho)$
$\overset{Sn}{\Longrightarrow} \bigwedge_{i=1}^{n} (x_1^n, y_i) \in \rho \Rightarrow (x_1^n, y_n) \in \rho;$

$2_2)$ $y \notin \{x_1^n\} \in Q^{(n)}$:

$(x_1^n, y) \in \rho \quad \overset{\overset{n-1}{S}, Rn}{\Longrightarrow} \left(\bigwedge_{j=1}^{n-1} (x_1^n, x_j) \in \rho \wedge (x_1^n, y) \in \rho\right)$
$\overset{Sn}{\Longrightarrow} \left(\bigwedge_{i=1}^{n} (x_1^{n-1}, y, x_i) \in \rho \overset{(b)}{\Rightarrow} (x_1^n, y) \in (\overset{n}{\rho})^{-1}\right);$

$2_3)$ $y \in \{x_1^n\} \in Q^{(n)}$:

$(x_1^n, x_i) \in \rho \quad \overset{\overset{n-1}{S}, Rn}{\Longrightarrow} \bigwedge_{j=1}^{i-1} (x_1^n, x_j) \in \rho \wedge \bigwedge_{j=i+1}^{n} (x_1^n, x_j) \in \rho \wedge (x_1^n, x_i) \in \rho$
$\overset{Sn}{\Longrightarrow} \bigwedge_{t=1}^{n} (x_1^{i-1}, x_{i+1}^n, x_i, x_t) \in \rho \overset{(b)}{\Rightarrow} (x_1^n, x_i) \in (\overset{n}{\rho})^{-1};$

$3_1)$ $(x_1^n, y) \in \rho^\alpha \overset{(c)}{\Longrightarrow} (x_{\alpha(1)}, \ldots, x_{\alpha(n)}, y) \in \rho \overset{\overset{n-1}{S}}{\Longrightarrow} (x_1^n, y) \in \rho;$ and

$3_2)$ $(x_1^n, y) \in \rho \overset{\overset{n-1}{S}}{\Longrightarrow} (x_{\alpha(1)}, \ldots, x_{\alpha(n)}, y) \in \rho \overset{(c)}{\Longrightarrow} (x_1^n, y) \in \rho^\alpha.$

□

3.3. Theorem: *Let $|Q| \geq n, n \in N$, let $\triangle \overset{def}{=}$ $\{(a_1^n, a_i) | \{a_1^n\} \in Q^{(n)} \wedge i \in \{1, \ldots, n\}\}$ and $\rho \subseteq \mathcal{L}(Q^{n+1})$. Suppose also that the following hold:*

$1°$ *$\rho^\alpha = \rho$ for all $\alpha \in \{1, \ldots, n\}!$;*
$2°$ *$\triangle \subseteq \rho$;*
$3°$ *$(\overset{n}{\rho})^{-1} = \rho$; and*

4° $\circ(\overset{n+1}{\rho}) = \rho$.

Then ρ is an $(n+1)-LFE$–relation in Q.

The sketch of a part of the proof.

a) The statement Tn holds:

$$\left(\bigwedge_{i=1}^{n}(x_1^n, z_i) \in \rho \wedge (z_1^n, y) \in \rho \overset{(a)}{\Longrightarrow} (x_1^n, y) \in \circ(\overset{n+1}{\rho})\right) \overset{4^\circ}{\Longrightarrow} (x_1^n, y) \in \rho$$

b) $\{b_1^n\} \in Q^{(n)} \wedge \bigwedge_{i=1}^{n}(a_1^n, b_i) \in \rho \overset{(c),1^\circ}{\Longrightarrow} \bigwedge_{i=1}^{n}(b_1^n, a_i) \in (\overset{n}{\rho})^{-1} \overset{3^\circ}{\Longrightarrow} \bigwedge_{i=1}^{n}(b_1^n, a_i) \in \rho.$

□

4. References

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