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DESCRIPTION OF SUPER ASSOCIATIVE ALGEBRAS WITH n-QUASIGROUP OPERATIONS

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Abstract. Let Σ be a set operation over Q. Let also $w_1 = w_2$ be a law in a description of which variables $x_1, \ldots, x_s$ are included, and also operational symbols $X_1, \ldots, X_k$, whose set of lengths is a subset of the set of lengths [:arities] of operations from Σ. Then (Q, Σ) is said to be an algebra with the superidentity $w_1 = w_2$ iff for every substitution of the variables $x_1, \ldots, x_s$ with elements of Q and for every substitution of the operational symbols $X_1, \ldots, X_k$ with operations from Σ [with the corresponding lengths] $w_1 = w_2$ becomes an equality in (Q, Σ); [2]. Quasigroup algebras with associative superlaws were described by V. D. Belousov in [5]. (See also [16].) 3-quasigroup algebras with associative superlaws were primary described by Yu. M. Movsisyan [[9], p. 152-158].(Associative superlaws or hyperidentities of associativity; see also [15].) In the present paper, for n-quasigroup algebras with associative superlaws, the author was free to use the name: **super associative algebras of n-quasigroup operations** [briefly: $SAAnQ$]. In the paper, primary, **in a unique way are** described **nontrivial** $SAAnQ$ [briefly: $NetSAAnQ$] for every $n \in \mathrm{N} \setminus \{1\}$ with an exception of a case for $n = 2$. The crucial role in the mentioned description of $NetSAAnQ$ play the $\{1, n\}$-neutral and the inversing operations in an n-group. Starting with the mentioned description of $NetSAAnQ$, these algebras for $n \geq 3$ are finnaly described in terms of Hosszú-Gluskin algebras of order n.

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1. Preliminaries

1.1 About the expression a_p^q

Let $p \in \mathbf{N}$, $q \in \mathbf{N} \cup \{0\}$ and let a be a mapping of the set $\{i| \, i \in \mathbf{N} \wedge i \geq p \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence } (= \emptyset); & p > q. \end{cases}$$

For example: $X(a_1^{j-1}, Y(a_j^{j+n-1}), a_{j+n}^{2n-1})$, $j \in \{1, \ldots, n\}$, $n \in \mathbf{N} \setminus \{1, 2\}$, for $j = n$ stands for $X(a_1, \ldots, a_{n-1}, Y(a_n, \ldots, a_{2n-1}))$.
Besides, in some situations *instead of* a_p^q *we write* $(a_i)_{i=p}^q$ [briefly: $(a_i)_p^q$].
For example: $(\forall x_i \in Q)_1^q$ for $q > 1$ stands for $\forall x_1 \in Q \ldots \forall x_q \in Q$ [usually, we write: $(\forall x_1 \in Q) \ldots (\forall x_2 \in Q)$], for $q = 1$ it stands for $\forall x_1 \in Q$, and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.

1.2 About n-semigroups, n-quasigroups and n-groups

1.2.1. Definitions: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then:*
*(a) we say that (Q, A) is an n-***semigroup** *iff for every $i, j \in \{1, \ldots, n\}$, $i < j$, the following $\langle i, j \rangle$-associative law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1}) \ ;$$

*(b) we say that (Q, A) is an n-***quasigroup** *iff for every $i \in \{1, \ldots, n\}$ and for every $a_1^n \in Q$ there is* **exactly one** *$x_i \in Q$ such that the following equality holds*

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n; \ \textit{and}$$

(c) we say that (Q, A) is a Dörnte n-group [briefly: n-group] iff (Q, A) is an n-semigroup and n-quasigroup as well.

A notion of an n-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

1.2.2. Proposition [6]: *An n-semigroup (Q, A), $n \geq 2$, is an n-group iff for all there is exactly one $x \in Q$ and exactly one $y \in Q$, such that the equalities*

$$A(a_1^{n-1}, x) = a_n \textit{ and } A(y, a_1^{n-1}) = a_n$$

hold.

1.2.3. Proposition [7]: *Let (Q, A) be an n-quasigroup and $n \geq 2$. Then: (Q, A) is an n-group iff there is an $i \in \{1, \ldots, n-1\}$ such that the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^i, A(x_{i+1}^{i+n}), x_{i+n+1}^{2n-1}).$$

(See, also [8], p.p. 196.)

1.3 On the $\{i,j\}$-neutral operation in an n-groupoid

1.3.1. Definition [10]: *Let $n \geq 2$, let (Q,A) be an n-groupoid and* e *be an $(n-2)$-ary operation in Q. Let also $\{i,j\} \subseteq \{1,\dots,n\}$ and $i<j$. Then* e *is an $\{i,j\}$-neutral operation of the groupoid (Q,A) iff the following formula holds*

$$(1)\quad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)\quad (A(a_1^{i-1}, \mathrm{e}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x \;\wedge\; A(a_1^{i-1}, x, a_i^{j-2}, \mathrm{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x)^{1)}$$

1.3.2. Remark: *For $n=2$ $\mathrm{e}(a_1^{n-2})$ [$= \mathrm{e}(\emptyset) = e \in Q$] is a neutral element of the groupoid (Q,A).*

1.3.3. Proposition [10]: *Let $n \geq 2$, $\{i,j\} \subseteq \{1,\dots,n\}$ and $i<j$. Then in every n-groupoid there is* **at most one** *$\{i,j\}$-neutral operation.*

1.3.4. Proposition [10]: *In every n-group [$n \geq 2$] there is a $\{1,n\}$-neutral operation. (See, also [13].)*

1.3.5. Proposition [10]: *For $n \geq 3$, an n-semigroup (Q,A) is an n-group iff (Q,A) has a $\{1,n\}$-neutral operation.*

1.3.6. Proposition: *Let (Q,A) be an n-group,* e *its $\{1,n\}$-neutral operation and $n \geq 2$. Then the following formula holds*

$$(2)\quad (\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q)\; A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \mathrm{e}(b_1^{n-2}), b_1^{n-2}).$$

Proof: 1) for $n=2$ the formula (2) reduces to the formula
$(\forall x \in Q) A(\mathrm{e}(\emptyset), x) = A(x, \mathrm{e}(\emptyset))$; and

2) let $n \geq 3$. $F(x, b_1^{n-2}) \overset{def}{=} A(x, \mathrm{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$

$$A(F(x,b_1^{n-2}), \mathrm{e}(b_1^{n-2}), b_1^{n-2}) = A(A(x, \mathrm{e}(b_1^{n-2}), b_1^{n-2}), \mathrm{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$$

$$A(F(x,b_1^{n-2}), \mathrm{e}(b_1^{n-2}), b_1^{n-2}) = A(x, A(\mathrm{e}(b_1^{n-2}), b_1^{n-2}, \mathrm{e}(b_1^{n-2})), b_1^{n-2}) \Rightarrow$$

$$A(F(x,b_1^{n-2}), \mathrm{e}(b_1^{n-2}), b_1^{n-2}) = A(x, \mathrm{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$$

$$F(x, b_1^{n-2}) = x \Rightarrow A(x, \mathrm{e}(b_1^{n-2}), b_1^{n-2}) = A(\mathrm{e}(a_1^{n-2})), a_1^{n-2}, x).$$

1.4 On the inversing operation in an n-group

1.4.1. Proposition [11]: *Let $n \geq 2$ and (Q,A) be an n-semigroup. Then:*

a) There is at most one $(n-1)$-ary operation f in Q such that the following formulas hold

$$(1)\quad (\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)\; A(f(a_1^{n-2}, a), a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$$

and

$$(2)\quad (\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)\; A(A(x, a_1^{n-2}, a), a_1^{n-2}, f(a_1^{n-2}, a)) = x;$$

1) For $\{i,j\} = \{1,n\}$: $(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)\; (A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, x) = x \wedge A(x, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = x)$.

b) If there is an $(n-1)$-ary operation f in Q such that the formulas (1) and (2) are satisfied, then (Q, A) is an n-group; and

c) If (Q, A) is an n-group, then there is an $(n-1)$-ary operation f in Q such that the formulas (1) and (2) hold.

1.4.2. Proposition [11]: *Let (Q, A) be an n-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, f its inversing operation and $n \geq 2$. Then the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\quad A(f(a_1^{n-2}, a), a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \wedge$$
$$A(a, a_1^{n-2}, f(a_1^{n-2}, a)) = \mathbf{e}(a_1^{n-2})).$$

1.4.3. Proposition [11]: *Let (Q, A) be an n-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, f its inversing operation and $n \geq 2$. Then the formula*

$$(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q)(\forall y \in Q) A(x, b_1^{n-2}, y) =$$
$$A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))), a_1^{n-2}, y)$$

holds.

1.4.4. Remark: *As well as Proposition 1.3.4. and Proposition 1.4.1, for $n \geq 2$, e. g. the following proposition holds [14]: If the laws hold in the algebra $(Q, \{A, f, \mathbf{e}\})$ of the type $\langle n, n-1, n-2 \rangle$*
$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$, $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*
$A(a, a_1^{n-2}, f(a_1^{n-2}, a)) = \mathbf{e}(a_1^{n-2})$,
then (Q, A) is an n-group. For $n = 2$ this is the well known characterizations of groups.

1.5 On Hosszú-Gluskin algebras

1.5.1. Definition [12]: *Let $\cdot$ be a binary and φ a unary operation in Q. Let also b be a (fixed) element of the set Q, and n a (fixed) element of the set $\mathbf{N} \setminus \{1, 2\}$. We say that $(Q, \{\cdot, \varphi, b\})$ is a* **Hosszú-Gluskin algebra of order** *n [briefly: nHG-algebra] iff the following hold*

(1) $(Q, \cdot)$ is a group,

(2) $\varphi \in Aut(Q, \cdot)$,

(3) $\varphi^{n-1}(x) \cdot b = b \cdot x$ for every $x \in Q$, and

(4) $\varphi(b) = b$.

1.5.2. Hosszú-Gluskin Theorem [3-4]: *Let (Q, A) be an n-group and $n \geq 3$. Then, there is an nHG-algebra $(Q, \{\cdot, \varphi, b\})$ such that for each $x_1^n \in Q$ the equality*

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \tag{5}$$

holds.

By a simple verification we conclude that the following proposition also holds:

1.5.3. Proposition: *Let $(Q, \{\cdot, \varphi, b\})$ nHG-algebra [$n \geq 3$]. Let also*

$$A(x_1^n) \stackrel{def}{=} x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$$

for all $x_1^n \in Q$. Then (Q, A) is an n-group.

1.5.4. Definition [12]: *We say that an nHG-algebra $(Q, \{\cdot, \varphi, b\})$* **associated** *to the n-group (Q, A) iff the equality (5) holds for all $x_1^n \in Q$.*

1.5.5. Proposition [12]: *Let (Q, A) be an n-group, $n \geq 3$, $(Q, \{\cdot, \varphi, b\})$ an arbitrary nHG-algebra* **associated** *to the n-group (Q, A), $^{-1}$ the inversing operations in $(Q, \cdot)$, $k \in Q$ and for every $x, y \in Q$*

$$x \cdot_k y \stackrel{def}{=} x \cdot k \cdot y,$$

$$\varphi_k(x) \stackrel{def}{=} k^{-1} \cdot \varphi(x) \cdot \varphi(k) \text{ and}$$

$$b_k \stackrel{def}{=} k^{-1} \cdot \varphi(k^{-1}) \cdot \ldots \cdot \varphi^{n-1}(k^{-1}) \cdot b.$$

Let also

$$C_A \stackrel{def}{=} \{(Q, \{\cdot_k, \varphi_k, b_k\}) | k \in Q\}.$$

Then, C_A is a set of all nHG-algebras **associated** *to the n-group (Q, A).*

2. Introduction

2.1 On n-ary associative laws

Let $X_1, \ldots, X_4$ be n-ary operational symbols and $n \in \mathbf{N} \setminus \{1\}$. Let also Φ be a mapping of the set $\{X_1, \ldots, X_4\}$ into the set $\{X_1, \ldots, X_4\}$, $i, j \in \{1, \ldots, n\}$, $i < j$ and let $x_1, \ldots, x_{2n-1}$ be variables; $\mathcal{R}\Phi = \{\Phi X_1, \Phi X_2, \Phi X_3, \Phi X_4\} \subseteq \{X_1, \ldots, X_4\}$. Then we say that

$$\Phi X_1(x_1^{i-1}, \Phi X_2(x_i^{i+n-1}), x_{i+n}^{2n-1}) = \Phi X_3(x_1^{j-1}, \Phi X_4(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

is a n-ary associative law.

2.2 Super associative algebras with n-quasigroup operations

2.2.1. Definition: *Let (Q, Σ) be an algebra in which (Q, Z) is an n-quasigroup for every $Z \in \Sigma$. Then we say that (Q, Σ) is a* **super associative algebra with n-quasigroup operations** *[briefly: SAAnQ] iff the following statement holds: there is*

$$\Phi \in \{X_1, \ldots, X_{2n}\}^{\{X_1, \ldots, X_{2n}\}}$$

such that for **every** *$i \in \{2, \ldots, n\}$, for* **every substitution** *of the variables $x_1, \ldots, x_{2n-1}$ by elements of Q and for* **every substitution** *of the operational symbols $\Phi X_1, \ldots, \Phi X_{2n}$ by elements of Σ [keeping the same notion $x_1, \ldots, x_{2n-1}$, $\Phi X_1, \ldots, \Phi X_{2n}$] the following equality holds*

$$\Phi X_1(\Phi X_2(x_1^n), x_{n+1}^{2n-1}) = \Phi X_{2i-1}(x_1^{i-1}, \Phi X_{2i}(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

2.2.2. Remark: *The case $n = 2$ was described by V. D. Belousov in [5]. In that occasion $SAA2Q$ were said to be* **systems** *of* **quasigroups with associative superlaws**. *The case $n = 3$ was described by Yu. M. Movsisian [:the book [9]. p.p. 152-158].* □

An immediate consequence of Def. 2.2.1 and of the definition of the n-group is the following proposition:

2.2.3. Proposition: *Let $n \in \mathbf{N} \setminus \{1\}$. Then:* **if** (Q, Σ) *$SAAnQ$,* **then** (Q, Z) *is an n-group for every $Z \in \Sigma$.*

2.3 Nontrivial $SAAnQ$

If $|\Sigma| = 1$ **or** $|\mathcal{R}\Phi| = 1$, then for every set Q and for every $n \geq 2$ there is $SAAnQ$ (Q, Σ).

2.3.1. Definition: *Let (Q, Σ) be an $SAAnQ$; $n \geq 2$. Then (Q, Σ) is a* **nontrivial** *$SAAnQ$ [briefly: $NetSAAnQ$] iff the following conjuction holds*

$$|\Sigma| > 1 \wedge |\mathcal{R}\Phi| > 1.$$

2.3.2. Theorem: *Let (Q, Σ) be an $SAAnQ$. Then:* **if** (Q, Σ) *is $NetSAAnQ$,* **then** *the following statements hold:*

$$(1)\quad |\{\Phi X_1, \Phi X_2\}| = 2 \Rightarrow (\forall i \in \{2, \dots, n\})\{\Phi X_{2i-1}, \Phi X_{2i}\} = \{\Phi X_1, \Phi X_2\}; \text{ and}$$

$$(2)\quad |\{\Phi X_1, \Phi X_2\}| = 1 \Rightarrow (\forall i \in \{2, \dots, n\})(|\{\Phi X_{2i-1}, \Phi X_{2i}\}| = 1) \wedge (\exists j \in \{2, \dots, n\})\{\Phi X_{2j-1}, \Phi X_{2j}\} \neq \{\Phi X_1, \Phi X_2\}).$$

Proof. 1) Let $|\{\Phi X_1, \Phi X_2\}| = 2$.

a) Let $A, B, C, D, \bar{D}$ be arbitrary operations from Σ, $x_1, \dots, x_{2n-1}$ arbitrary elements from Q and i an arbitrary element of the set $\{2, \dots, n\}$ so that

$$A(B(x_1^n), x_{n+1}^{2n-1}) = C(x_1^{i-1}, D(x_i^{i+n-1}), x_{i+n}^{2n-1}) \text{ and}$$

$$A(B(x_1^n), x_{n+1}^{2n-1}) = C(x_1^{i-1}, \bar{D}(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

Thereby, since Σ is a set of n-quasigroup operations, we conclude that $D = \bar{D}$, i.e. that ΦX_{2i} has no free choice for the substitution with operations from the set Σ. Similary we conclude that ΦX_{2i-1} has no free choice for the substitution with operations from Σ. Hence, for every $i \in \{2, \dots, n\}$ it is true that

$$|\{\Phi X_1, \Phi X_2, \Phi X_{2i-1}, \Phi X_{2i}\}| < 4.$$

b) Let $A, B, C, \bar{C}$ be arbitrary elements from the set Σ, $x_1, \dots, x_{2n-1}$ arbitrary elements from the set Q and i an arbitrary elements from the set

$\{2,\dots,n\}$ so that

$$A(B(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{i-1},C(x_i^{i+n-1}),x_{i+n}^{2n-1}) \text{ and}$$

$$A(B(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{i-1},\bar{C}(x_i^{i+n-1}),x_{i+n}^{2n-1}); \text{ or}$$

$$A(B(x_1^n),x_{n+1}^{2n-1}) = C(x_1^{i-1},B(x_i^{i+n-1}),x_{i+n}^{2n-1}) \text{ and}$$

$$A(B(x_1^n),x_{n+1}^{2n-1}) = \bar{C}(x_1^{i-1},B(x_i^{i+n-1}),x_{i+n}^{2n-1}).$$

Hence, since Σ is a set of n-quasigroup operations, we conclude that with $\Phi X_1 = \Phi X_{2i-1}$ **or** $\Phi X_2 = \Phi X_{2i}$

$$|\{\Phi X_1, \Phi X_2, \Phi X_{2i-1}, \Phi X_{2i}\}| < 3.$$

In the same way we conclude that with $\Phi X_2 = \Phi X_{2i-1}$ **or** $\Phi X_1 = \Phi X_{2i}$ it is true that

$$|\{\Phi X_1, \Phi X_2, \Phi X_{2i-1}, \Phi X_{2i}\}| < 3$$

for every $i \in \{2,\dots,n\}$.

c) Let A, B be arbitrary elements of the set Σ, $x_1,\dots,x_{2n-1}$ arbitrary elements from Q, and i an arbitrary element of the set $\{2,\dots,n\}$ so that

$$A(B(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{i-1},Ax_i^{i+n-1}),x_{i+n}^{2n-1}) \textbf{ or}$$

$$A(B(x_1^n),x_{n+1}^{2n-1}) = B(x_1^{i-1},B(x_i^{i+n-1}),x_{i+n}^{2n-1}).$$

Hence, since (Q,A) and (Q,B) are n-groups, it follows that $A = B$, i.e. that ΦX_1 or ΦX_2 has no free choice for the substitution with operations from Σ, which is a contradiction with the assumption that the following equality holds $|\{\Phi X_1, \Phi X_2\}| = 2$.

2) Let $|\{\Phi X_1, \Phi X_2\}| = 1$.

a) Let $A, B, C, \bar{C}$ be arbitrary elements from Σ, $x_1,\dots,x_{2n-1}$ arbitrary elements from Q and i an arbitrary element from $\{2,\dots,n\}$ so that

$$A(A(x_1^n),x_{n+1}^{2n-1}) = B(x_1^{i-1},C(x_i^{i+n-1}),x_{i+n}^{2n-1}) \text{ and}$$

$$A(A(x_1^n),x_{n+1}^{2n-1}) = B(x_1^{i-1},\bar{C}(x_i^{i+n-1}),x_{i+n}^{2n-1}).$$

Hence, since Σ is a set of n-quasigroup operations, we conclude that ΦX_{2i} has no free choice for the substitution with operations in Σ. Similary we conclude that ΦX_{2i-1} also has no free choice for the substitution with operations from Σ. Thus, for every $i \in \{2,\dots,n\}$, it is true that

$$|\{\Phi X_{2i-1}, \Phi X_{2i}\}| = 1.$$

b) Finally, taking into account the proposition proved in a) and the assumption that (Q,Σ) is $NetSAAnQ$, we conclude that there is at least one $j \in \{2,\dots,n\}$ such that

$$\{\Phi X_{2j-1}, \Phi X_{2j}\} \neq \{\Phi X_1, \Phi X_2\}. \ \square$$

Theorem 2.3.2 gives the posibility to describe $NetSAAnQ$:

a) with **not more than** n operational symbols in the case $|\{\Phi X_1, \Phi X_2\}| = 1$; and

b) with **two** operational symbols in the case $|\{\Phi X_1, \Phi X_2\}| = 2$.

Consequently, having in mind Theorem 2.3.2, we induce the following two agreements:

2.3.3. Definition: *Let (Q, Σ), $|\Sigma| \geq 2$, be an algebra in which the following condition holds: (Q, Z) is an n-quasigroup, $n \geq 2$, for every $Z \in \Sigma$. Let also ω be a mapping of the set $\{1, \ldots, n\}$ into the set $\{1, \ldots, n\}$ such that it satisfies the following conditions: a) $\omega(1) = 1$, and b) $|\mathcal{R}\omega| \geq 2$. In addition, the set of all mappings ω satisfying the conditions a)-b) will be denoted by Ω. Then, we shall say that (Q, Σ) is an ω-$NetSAAnQ$ iff for every $i \in \{2, \ldots, n\}$, for every substitution of variables $x_1, \ldots, x_{2n-1}$ with elements from Q, and every substitution of n-ary operational symbols $X_{\omega(1)}, \ldots, X_{\omega(n)}$ with operations from Σ the following equality holds*

$$X_1(X_1(x_1^n), x_{n+1}^{2n-1}) = X_{\omega(i)}(x_1^{i-1}, X_{\omega(i)}(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

2.3.4. Definition: *Let (Q, Σ), $|\Sigma| \geq 2$, be an algebra in which the following proposition holds: (Q, Z) is an n-quasigroup, $n \geq 2$, for every $Z \in \Sigma$. Let also $\hat{S}$ be a subset of the set $S = \{2, \ldots, n\}$ [including $\hat{S} = \emptyset$]. Then we shall say that (Q, Σ) is an $\hat{S}$-$NetSAAnQ$ iff for every substitution of the variables $x_1, \ldots, x_{2n-1}$ with elements from Q and for every substitution of n-ary operational symbols X_1, X_2 with operations from Σ the following statement holds*

$$\bigwedge_{i \in S \setminus \hat{S}} X_1(X_2(x_1^n), x_{n+1}^{2n-1}) = X_1(x_1^{i-1}, X_2(x_i^{i+n-1}), x_{i+n}^{2n-1}) \wedge$$

$$\bigwedge_{i \in \hat{S}} X_1(X_2(x_1^n), x_{n+1}^{2n-1}) = X_2(x_1^{i-1}, X_1(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

2.3.5. Remark: *If (Q, Σ) is $NetSAAnQ$, $A \in \Sigma$ and $|\Sigma \setminus \{A\}| \geq 2$, then also $(Q, \Sigma \setminus \{A\})$ is $NetSAAnQ$.*

2.4 On $NetSAAnQ$ with $n = 2$

For $n = 2$ there are [exactly] the following cases for $NetSAAnQ$

1) $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$-$NetSAAnQ$,

2) $\emptyset$-$NetSAAnQ$ and

3) S-$NetSAAnQ$, $[S = \{2\}]$,

with which are, respectively, connected the following laws

$X_1(X_1(x_1^2), x_3) = X_2(x_1, X_2(x_2^3))$,

$X_1(X_2(x_1^2), x_3) = X_1(x_1, X_2(x_2^3))$ and

$X_1(X_2(x_1^2), x_3) = X_2(x_1, X_1(x_2^3))$.

All three cases are described by V. D. Belousov in [5]. Among others, there are the following results.

2.4.1: *Let (Q, Σ) be an NetSAAnQ. Then the following hold:*

a) (Q, A) is a group for every $A \in \Sigma$;

b) If A is an arbitrary operation from Σ, then for every $B \in \Sigma$ there is $k \in Q$ such that for all $x, y \in Q$

$$B(x, y) = A(A(x, k), y).$$

Moreover:

b_1) If (Q, Σ) is $\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right)$-NetSAAnQ, then $k \in Q$ is an element from the support of the **center of the group** *(Q, A) and k is a* **self inverse element** *in (Q, A); and*

b_2) If (Q, Σ) is $\emptyset$-NetSAAnQ, then $k \in Q$ is an element from the support of the **center of the group** *(Q, A); and*

b_3) If (Q, Σ) is an S-NetSAAnQ, then $k \in Q$ is not limited by (Q, A) [$:A = \cdot$, $B(x, y) = x \cdot k \cdot y$, $C(x, y) = x \cdot \bar{k} \cdot y$, $B(C(x, y), z) = (x \cdot \bar{k} \cdot y) \cdot k \cdot z = x \cdot \bar{k} \cdot (y \cdot k \cdot z) = C(x, B(y, z))$].

2.4.2: *If (Q, Σ) is $\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right)$-NetSAA2Q, then (Q, Σ) is $\emptyset$-NetSAA2Q.*

2.4.3: *If (Q, Σ) is $\emptyset$-NetSAA2Q, then (Q, Σ) is also an S-NetSAA2Q.*

2.4.4: *There is a NetSAA2Q (Q, Σ) such that it is $\emptyset$-NetSAA2Q and that it is not $\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right)$-NetSAAnQ.*

2.4.5: *There is a NetSAA2Q (Q, Σ) such that it is an S-NetSAA2Q and that it is not $\emptyset$-NetSAAnQ.*

Remark: *The analogon of Proposition 2.4.5 with $n \geq 3$ is not satisfied.*

3. Central operations on n-groups

3.1. Definition: *Let (Q, A) be an n-group and $n \in \mathbb{N} \setminus \{1\}$. Let also α be an $(n-2)$-ary operation in the set Q. We say that α is a* **central operation of the n-group** *(Q, A) iff the following formula holds:*

$$(\forall a_i \in Q)_1^{n-2} (\forall b_i \in Q)_1^{n-2} (\forall x \in Q)\ A(\alpha(a_1^{n-2}), a_1^{n-2}, x) =$$

$$A(x, \alpha(b_1^{n-2}), b_1^{n-2}).$$

3.2. Remarks: *a) If* $n = 2$, *then* $\alpha(c_1^{n-2})$ $[= \alpha(c_1^0) = \alpha(\emptyset) = c \in Q]$ *is a central element of the group* (Q, A); *and b) The* $\{1, n\}$*-neutral operation* $\mathbf{e}$ *of the n-group* (Q, A) *is a central operation of that n-group.*

3.3. Proposition: *Let* (Q, A) *be an n-group,* α *its central operation and* $n \in \mathbf{N} \setminus \{1\}$. *Then for every* $i \in \{1, \ldots, n\}$, *for every* $x_1^n \in Q$, *for every sequence* a_1^{n-2} *over* Q *and for every sequance* b_1^{n-2} *over* Q *the following equality holds*

$$A(\alpha(a_1^{n-2}), a_1^{n-2}, A(x_1^n)) = A(x_1^{i-1}, A(\alpha(b_1^{n-2}), b_1^{n-2}, x_i), x_{i+1}^n).$$

Proof. For $n = 2$ $\alpha(a_1^{n-2})$ $[= \alpha(\emptyset)]$ is an element of the center of the group (Q, A).

Let $n \geq 3$. Then:

1) Since, by the assumption, (Q, A) is an n-group [n-semigroup], we conclude that for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold

$$\begin{aligned} A(\alpha(a_1^{n-2}), a_1^{n-2}, A(x_1^n)) &= A(A(\alpha(a_1^{n-2}), a_1^{n-2}, x_1), x_2^n) \\ &= A(A(\alpha(b_1^{n-2}), b_1^{n-2}, x_1), x_2^n); \text{ and} \end{aligned}$$

2) Since, by the assumption, (Q, A) is an n-group and α its central operation, we conclude that for every $j \in \{1, \ldots, n-1\}$, for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following sequence of equalities hold

$$A(x_1^{j-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_j), x_{j+1}^n) =$$

$$A(x_1^{j-1}, A(x_j, \alpha(b_1^{n-2}), b_1^{n-2}), x_{j+1}^n) =$$

$$A(x_1^j, A(\alpha(b_1^{n-2}), b_1^{n-2}, x_{j+1}), x_{j+2}^n).$$

3.4. Proposition: *Let* (Q, A) *be an n-group and* $n \in \mathbf{N} \setminus \{1\}$. *Let also* α *be a mapping of the set* Q^{n-2} *into the set* Q. *Then the following statements are equaivalent:*

(a) α *is a central operation of the n-group* (Q, A);

(b) The following formula holds

$$(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall c_i \in Q)_1^{n-2}(\forall d_i \in Q)_1^{n-2}(\forall x \in Q)\,($$

$$A(\alpha(a_1^{n-2}), a_1^{n-2}, x) = A(x, \alpha(b_1^{n-2}), b_1^{n-2}) =$$

$$A(x, c_1^{n-2}, \alpha(c_1^{n-2})) = A(d_1^{n-2}, \alpha(d_1^{n-2}), x))\,;$$

(c) The following formula holds

$$(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q) \quad A(x, a_1^{n-2}, \alpha(a_1^{n-2})) =$$

$$A(b_1^{n-2}, \alpha(b_1^{n-2}), x).$$

Proof. For $n = 2$ the proposition is trivial.

Let $n \geq 3$.

I: (a) $\Leftrightarrow$ (b):

The statement (a) is an immeediate consequence of the statement (b).

By the assumption that (Q, A) is an n-group, $\boldsymbol{\alpha}$ its central operation and by Proposition 3.3, we deduce that the following sequence of implications holds

$$
\begin{aligned}
F(a_1^{n-2}, x) &\overset{def}{=} A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) \Rightarrow \\
A(F(a_1^{n-2}, x), a_1^{n-2}, z) &= A(A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, z) \Rightarrow \\
A(F(a_1^{n-2}, x), a_1^{n-2}, z) &= A(x, a_1^{n-2}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, z)) \Rightarrow \\
A(F(a_1^{n-2}, x), a_1^{n-2}, z) &= A(A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x), a_1^{n-2}, z) \Rightarrow \\
F(a_1^{n-2}, x) &= A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x),
\end{aligned}
$$

and hence we conclude that the formula

$$
\begin{aligned}
(1) \qquad & (\forall a_i \in Q)_1^{n-2} (\forall b_i \in Q)_1^{n-2} (\forall x \in Q) A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = \\
& A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)
\end{aligned}
$$

holds.

By the similar arguments, considering also the formula (1), we conclude that the following sequence of implications holds

$$
\begin{aligned}
\Phi(a_1^{n-2}, x) &\overset{def}{=} A(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x) \Rightarrow \\
A(a_1^{n-2}, z, \Phi(a_1^{n-2}, x)) &= A(a_1^{n-2}, z, A(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x)) \Rightarrow \\
A(a_1^{n-2}, z, \Phi(a_1^{n-2}, x)) &= A(a_1^{n-2}, A(z, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), x) \Rightarrow \\
A(a_1^{n-2}, z, \Phi(a_1^{n-2}, x)) &= A(a_1^{n-2}, A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, z), x) \Rightarrow \\
A(a_1^{n-2}, z, \Phi(a_1^{n-2}, x)) &= A(a_1^{n-2}, z, A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)) \Rightarrow \\
A(a_1^{n-2}, z, \Phi(a_1^{n-2}, x)) &= A(a_1^{n-2}, z, A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})) \Rightarrow \\
\Phi(a_1^{n-2}, x) &= A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}),
\end{aligned}
$$

and hence we conclude that also the formula

$$
\begin{aligned}
(2) \qquad & (\forall a_i \in Q)_1^{n-2} (\forall b_i \in Q)_1^{n-2} (\forall x \in Q) A(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}), x) = \\
& A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})
\end{aligned}
$$

holds.

Considering the formulas (1) and (2) and the definition 3.1, we conclude that the implication (a) $\Rightarrow$ (b) holds.

II: If $\boldsymbol{\alpha}$ satisfies the formula in the statement (c), then for every $i \in \{1, \ldots, n\}$, for every $x_1^n \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(A(x_1^n), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(a_1^{i-1}, A(x_i, b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2})), x_{i+1}^n).$$

Proposition II can be proved by a simple imitation of the proof of Proposition 3.3.

III: (b) $\Leftrightarrow$ (c):

The statement (c) is an immediate consequence of the statement (b).

The implication (c) $\Rightarrow$ (b) can be proved imitating the proof of the implication (a) $\Rightarrow$ (b), where instead of Proposition 3.3 Proposition II is used.

See Remark 3.11. □

A direct consequence of Proposition 3.4 is the following proposition:

3.5. Proposition: *Let (Q, A) be an n-group and $n \in \mathbf{N} \setminus \{1\}$. Let also $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the n-group (Q, A). Then for every sequence a_1^{n-2} over Q the following equality holds*

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\beta}(a_1^{n-2})) = A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})).$$

3.6. Proposition: *Let (Q, A) be an n-group, $\boldsymbol{\alpha}$ its central operation and $n \in \mathbf{N} \setminus \{1\}$. Then there is a permutation α of the set Q such that for every $x \in Q$, a_1^{n-2}, $b_1^{n-2} \in Q$ the following conjunction of equalities holds*

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = \alpha(x) \wedge A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}) = \alpha(x).$$

Proof. Let k_1^{n-2} be an arbitrary **choosen** sequence over the set Q. Then, α, where

$$\alpha(x) \stackrel{def}{=} A(x, \boldsymbol{\alpha}(k_1^{n-2}), k_1^{n-2})$$

for every $x \in Q$, is a permutation of the set Q, since (Q, A) is an n-quasigroup. Hence, by the Definition 3.1, we conclude that the proposition holds. □

An immediate consequence of Proposition 3.3 and Proposition 3.6 is the following proposition:

3.7. Proposition: *Let (Q, A) be an n-group, $\boldsymbol{\alpha}$ its central operation, α a permutation of the set Q such that for every $x, a_1^{n-2} \in Q$ the equality*

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = \alpha(x)$$

holds, and $n \in \mathbf{N} \setminus \{1\}$. Then for every $i \in \{1, \ldots, n\}$ and for every $x_1^n \in Q$ the following equality holds

$$\alpha A(x_1^n) = A(x_1^{i-1}, \alpha(x_i), x_{i+1}^n). \quad □$$

A consequence of Proposition 3.6 and of Hosszú-Gluskin theorem is the following proposition:

3.8. Lemma: *Let $n \in \mathbf{N} \setminus \{1\}$, (Q, A) an n-group and $(Q, \{\cdot, \varphi, b\})$ an nHG-algebra associated to the n-group (Q, A) [:1.5.4]. Then: if $\boldsymbol{\alpha}$ is a central operation of the n-group (Q, A), then there is exactly one* **constant** *$a \in Q$ such that for every sequence a_1^{n-2} over Q, the equality*

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_1^{n-2}) = a$$

holds.

Proof. Let c_1^{n-2} be an arbitrary [fixed] sequence over Q. Then, by Proposition 3.6, for every $x \in Q$ and for every sequence a_1^{n-2} over Q the equality

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(\boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}, x)$$

holds, from which, by Hosszú-Gluskin Theorem, we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the equality

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b \cdot x = \boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2}) \cdot b \cdot x$$

holds, i.e.,

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) = \boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2})$$

holds. Hence, since by the assumption, c_1^{n-2} is a fixed sequence over Q, by the convention that the constant $\boldsymbol{\alpha}(c_1^{n-2}) \cdot \varphi(c_1) \cdot \ldots \cdot \varphi^{n-2}(c_{n-2})$ is denoted by a, we conclude that for every sequence a_1^{n-2} over Q the following equality holds

$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) = a.$$

3.9. Proposition: *Let $n \in \mathbf{N} \setminus \{1, 2\}$, (Q, A) an n-group, $(Q, \{\cdot, \varphi, b\})$ its associated nHG-algebra and $^{-1}$ the inversing operation in the group $(Q, \cdot)$. Then:* **if**

a) $\boldsymbol{\alpha}$ is a central operation of the n-group (Q, A); and

b) α is a permutation of the set Q such that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$\alpha(x) = A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x),$$

then *there is exactly one constant $a \in Q$ such that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

(1) $\boldsymbol{\alpha}(a_1^{n-2}) = a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{-1}$;

(2) $\alpha(x) = (a \cdot b) \cdot x$;

(3) $\varphi(a) = a$ *and*

(4) $(a \cdot b) \cdot x = x \cdot (a \cdot b)$.

Proof. 1) Since $(Q, \cdot)$ is a group and $^{-1}$ its inversing operation, by Lemma 3.8, we conclude that there is exactly one constant $a \in Q$ such that for every sequence a_1^{n-2} over Q the equality (1) holds.

2) By the assumption of the proposition, by Hosszú-Gluskin Theorem, we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the equality

$$\alpha(x) = \boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b \cdot x$$

holds, and from there, by Lemma 3.8, we conclude that there is exactly one constant $a \in Q$ such that for every $x \in Q$ the equality (2) holds.

3) Considering the definition 3.1, Hosszú-Gluskin Theorem and by the fact $\varphi \in Aut(Q, \cdot)$, we conclude that for every $x \in Q, a_1^{n-2}, b_1^{n-2} \in Q$, the equality

$$\boldsymbol{\alpha}(a_1^{n-2})\cdot\varphi(a_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2})\cdot b\cdot x = x\cdot\varphi(\boldsymbol{\alpha}(b_1^{n-2})\cdot\varphi(b_1)\cdot\ldots\cdot\varphi^{n-2}(a_{n-2}))\cdot b$$

holds, and from there, by Lemma 3.8, we conclude that there is exactly one constant $a \in Q$ such that for every $x \in Q$ the equality

$$a \cdot b \cdot x = x \cdot \varphi(a) \cdot b$$

holds, i.e.,

$$\varphi(a) = a,$$

and then that for every $x \in Q$ the equality (4) holds. □

One could check that the following proposition holds:

3.10. Proposition: *Let $n \in \mathbf{N} \setminus \{1,2\}$, (Q, A) an n-group, $(Q, \{\cdot, \varphi, b\})$ its associated nHG-algebra and $^{-1}$ the inversing operation in the group $(Q, \cdot)$. Let also a be a fixed element of the set Q such that*

$$\varphi(a) = a \quad \textit{and}$$

$$(a \cdot b) \cdot x = x \cdot (a \cdot b)$$

for every $x \in Q$. Then: if for every $x \in Q, a_1^{n-2} \in Q$ we have that

$$\boldsymbol{\alpha}(a_1^{n-2}) \overset{def}{=} a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{-1} \textit{ and}$$

$$\alpha(x) \overset{def}{=} a \cdot b \cdot x,$$

then $\boldsymbol{\alpha}$ is a central operation of the n-group (Q, A) and α is a permutation of the set Q such that for every $x \in Q, a_1^{n-2}, b_1^{n-2} \in Q$, the following equalities hold

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = \alpha(x) \textit{ and } A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}) = \alpha(x).$$

3.11. Remark: *A direct consequence of Proposition 3.4 is the following proposition. Let $n \geq 3$, let (Q, A) be an n-group and let $\boldsymbol{\alpha}$ be an $(n-2)$-ary operation in the set Q,* If *for every $x \in Q, a_1^{n-2}, b_1^{n-2} \in Q$, the following equalitiy holds*

$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}),$$

then *for every $x \in Q$, every sequence a_1^{n-2} over Q and every sequence b_1^{n-2} over Q there holds*

$$A(\alpha(a_1^{n-2}), a_1^{n-2}, x) = A(x, b_1^{n-2}, \alpha(b_1^{n-2})).$$

However, the converse does not hold. For example: Let $(\{1,2,3,4\}, \cdot)$ be the Klein's group (Tab.1) and $^{-1}$ the corresponding inversing operation. Further on, let φ be the permutation of the set $\{1,2,3,4\}$ defined in the following way

$$\varphi \overset{def}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

In addition, let

$$A(x_1^3) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot x_3 \cdot 2 \text{ and}$$

$$\alpha(c) \overset{def}{=} 3 \cdot (\varphi(c))^{-1}.$$

Then: (i) $(\{1,2,3,4\}, \{\cdot, \varphi, 2\})$ is a $3HG$-algebra associated to the 3-group $(\{1,2,3,4\}, A)$; and (ii) for every $x \in \{1,2,3,4\}$ and for every $c \in \{1,2,3,4\}$ the following equalities hold

$$A(\alpha(c), c, x) = 4x, \;\; A(x, c, \alpha(c)) = 4x, \;\text{ and }\; A(x, \alpha(c), c) = 3x.$$

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Tab. 1

4. A description of ω-$NetSAAnQ$

4.1. Theorem: *Let ω be an* **arbitrary** *mapping from the set Ω. Let also (Q, Σ) be an ω-$NetSAAnQ$, $n \geq 2$, A an arbitrary operation from Σ and f_A the inversing operation in the n-group (Q, A). Then, for every $B \in \Sigma$ there is exactly one central operation α of (Q, A) such that for each $x_1^n, a_1^{n-2} \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

$$B(x_1^n) = A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)) \text{ and}$$

$$f_A(a_1^{n-2}, \alpha(a_1^{n-2})) = \alpha(a_1^{n-2}).$$

Proof. Let A and B be **arbitrary** operations from Σ. By Proposition 2.2.3, (Q, A) and (Q, B) are n-groups. By Proposition 1.3.2, (Q, A) and (Q, B) have $\{1, n\}$-neutral operations, denoted, respectively, by $\mathbf{e}_A$ and $\mathbf{e}_B$. Let also the inversing operation in (Q, A) be denoted by f_A.

1) By Definition 2.3.3 and by Proposition 2.2.3, for every $x_1^{2n-1} \in Q$ the following equality holds

$$B(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1})),$$

hence, by the substitutions $x_{n+1}^{2n-1} = a_1^{n-2}$ and $x_{2n-1} = \mathbf{e}_B(a_1^{n-2})$, where a_1^{n-2} is an arbitrary sequence over Q, we conclude that for every $x_1^n, a_1^{n-2} \in Q$ the following equality holds

$$(1) \qquad B(x_1^n) = A(x_1^{n-1}, A(x_n, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))).$$

2) Since (Q, B) is an n-group, for every $x_1^{2n-1} \in Q$ the following equality holds

$$B(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, B(x_1^{n+1}), x_{n+2}^{2n-1}),$$

hence, by the statement concerning (1), we conclude that for every x_1^{2n-1}, a_1^{n-2}, $b_1^{n-2} \in Q$, the following equality holds

$$A(A(x_1^{n-1}, A(x_n, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))), x_{n+1}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) =$$

$$A(x_1, A(x_2^n, A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2}))), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))),$$

that is, since (Q, A) is an n-semigroup, also the following equality holds

$$A(x_1, A(x_2^n, A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}), x_{n+1})), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))) =$$

$$A(x_1, A(x_2^n, A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2}))), x_{n+2}^{2n-2}, A(x_{2n-1}, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))).$$

Hence, since (Q, A) is an n-quasigroup, we conclude that for every $x_{n+1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalty holds

$$A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}), x_{n+1}) = A(x_{n+1}, b_1^{n-2}, \mathbf{e}_B(b_1^{n-2})),$$

hence by Proposition 3.4, we conclude that $\mathbf{e}_B$ is a central operation of the n-group (Q, A).

3) Starting with the statement concerning (1), with the substitutions $x_2^{n-1} = a_1^{n-2}$, $x_1 = \mathbf{e}_B(a_1^{n-2})$ and $x_n = x$, we conclude that for every $x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$x = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))),$$

hence, by Proposition 1.4.1, we conclude that for every $x, a_1^{n-2} \in Q$ the following equality holds

$$A(f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), a_1^{n-2}, x) = A(x, a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})).$$

Hence, by the substitution $x = \mathbf{e}_A(a_1^{n-2})$, we conclude that for every sequence a_1^{n-2} over Q the following equality holds

$$f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})) = \mathbf{e}_B(a_1^{n-2}).$$

4) By the assumption that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)) = A(x_1^{n-1}, A(\hat{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n),$$

and since (Q, A) is an n-quasigroup, we conclude that $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$. □

4.2. Theorem: *Let (Q, A) be an n-group, f_A its inversing operation, $n \geq 2$ and let (Q, A) has at least twoelement set of central operation $\mathbf{C}_A$ such that for every $\boldsymbol{\alpha} \in \mathbf{C}_A$ the following formula holds*

$$(1) \qquad (\forall a_i \in Q)_1^{n-2} f_A(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = \boldsymbol{\alpha}(a_1^{n-2}).$$

Let also $\hat{\mathbf{C}}_A$ be at least twoelement subset of the set $\mathbf{C}_A$ and let

$$(2) \qquad B \in \Sigma \overset{def}{\Leftrightarrow} (\exists \boldsymbol{\alpha} \in \hat{\mathbf{C}}_A)(\forall a_i \in Q)_1^{n-2}(\forall x_i \in Q)_1^n B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Then, (Q, Σ) is an ω-NetSAAnQ for every $\omega \in \Omega$.

Proof. 1) Let B be an **arbitrary** operation from Σ and $\boldsymbol{\alpha}$ its corresponding operation from $\hat{\mathbf{C}}_A$. Then, by Proposition 3.6, there is a permutation α of the set Q such that for every $x_1^n \in Q$ the following equality holds

$$(a) \qquad B(x_1^n) = A(x_1^{n-1}, \alpha(x_n)),$$

hence immediately we conclude that (Q, B) is an n-quasigroup. Further, starting with the statement concerning (a), by Proposition 3.7, we conclude that for every $x_1^{2n-1} \in Q$, the following sequence of equalities holds

$$\begin{aligned} B(B(x_1^n), x_{n+1}^{2n-1}) &= \alpha A(A(x_1^{n-1}, \alpha(x_n)), x_{n+1}^{2n-1}) \\ &= \alpha A(x_1, A(x_2^{n-1}, \alpha(x_n), x_{n+1}), x_{n+2}^{2n-1}) \\ &= B(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1}), \end{aligned}$$

i.e.

$$B(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, B(x_2^{n+1}), x_{n+2}^{2n-1}).$$

Hence, since (Q, B) is an n-quasigroup, by Proposition 1.2.3, we conclude that (Q, B) is an n-group.

2) A consequence of the condition (1) is the following statement

$$(b) \qquad (\forall x \in Q)\alpha(\alpha(x)) = x.$$

Further, starting with the statement concerning (a) and the statement (b), by Proposition 3.7, we conclude that for every $x_1^{2n-1} \in Q$ the following equality holds

$$B(B(x_1^n), x_{n+1}^{2n-1}) = A(A(x_1^n), x_{n+1}^{2n-1}).$$

Hence, since B is an arbitrary operation from Σ and since (Q, B) is an n-group, we conclude that for **every substitution** of variables $x_1, \ldots, x_{2n-1}$ with elements from Q and for **every substitution** of operational symbols

$X_{\omega(1)}, \ldots, X_{\omega(n)}$, by operations from Σ the following conjunction of equalities holds

$$\bigwedge_{i=2}^{n} X_1(X_1(x_1^n), x_{n+1}^{2n-1}) = X_{\omega(i)}(x_1^{i-1}, X_{\omega(i)}(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

Hence, by the assumption $|\hat{\mathrm{C}}_A| \geq 2$, we conclude that (Q, Σ) is an ω-*NetSAAnQ* for every $\omega \in \Omega$.

By Theorem 4.1 and by Theorem 4.2, we conclude that the following proposition holds:

4.3. Proposition: *For every $\omega \in \Omega$ and for every $\bar{\omega} \in \Omega$ the following statement holds: (Q, Σ) is an ω-NetSAAnQ [$n \geq 2$] iff (Q, Σ) is an $\bar{\omega}$-NetSAAnQ.*

4.4. Proposition: *For every $\omega \in \Omega$ and for every $\hat{S} \subseteq \{2, \ldots, n\}$ the following statement holds: If (Q, Σ) is an ω-NetSAAnQ then (Q, Σ) is an $\hat{S}$-NetSAAnQ.*

Proof. 1) Let (Q, Σ) be an ω-*NetSAAnQ* and $n \geq 2$. Let also A, B, C be **arbitrary** operations from Σ. Then, by Theorem 4.1, there are central operations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the n-group (Q, A) such that for every $x_1^n, a_1^{n-2} \in Q$ the following equalites hold

(1) $$B(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)) \text{ and}$$

(2) $$C(x_1^n) = A(x_1^{n-1}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Starting with the statement concernig (1) and (2) and by Proposition 3.6, we conclude that there are permutations α and β of the set Q such that for every $x_1^n \in Q$ the following equalities hold

$$B(x_1^n) = A(x_1^{n-1}, \alpha(x_n)) \text{ and}$$

$$C(x_1^n) = A(x_1^{n-1}, \beta(x_n)),$$

hence, by Proposition 3.7, we conclude that for every $x_1^{2n-1} \in Q$ the following equality holds

(3) $$B(C(x_1^n), x_{n+1}^{2n-1}) = \alpha(\beta(A(A(x_1^n), x_{n+1}^{2n-1}))).$$

2) The consequence of the Proposition 3.5 and Proposition 3.6 is the following statement

(4) $$(\forall x \in Q)\alpha(\beta(x)) = \beta(\alpha(x)).$$

[Sketch of the proof: $\alpha(\beta(x)) = A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \beta(x)) = A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, x)) = A(A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\beta}(a_1^{n-2})), a_1^{n-2}, x) = A(A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})), a_1^{n-2}, x) = A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x)) = A(\boldsymbol{\beta}(a_1^{n-2}), a_1^{n-2}, \alpha(x)) = \beta(\alpha(x))$.]

Finally, starting with the statement concerning (3) and (4), since (Q, A) is an n-group, we conclude that for every $i \in \{2, \ldots, n\}$ and for every $x_1^{2n-1} \in Q$ the following equalities hold

$$B(C(x_1^n), x_{n+1}^{2n-1}) = B(x_1^{i-1}, C(x_i^{i+n-1}), x_{i+n}^{2n-1}) =$$

$$= C(x_1^{i-1}, B(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

Thus, we have proved that (Q, Σ) is an $\hat{S}$-$NetSAAnQ$ for every $\hat{S} \in \{2, \dots, n\}$. □

By Theorem 4.1, Proposition 3.6 and Proposition 3.7, we conclude that also the following proposition holds:

4.5. Proposition: *Let (Q, Σ) be an ω-$NetSAAnQ$, A and B* **arbitrary** *operations from Σ, let α be a central operation of (Q, A) and α be an permutation of the set Q such that for every $x_1^n \in Q$, every $x \in Q$ and every sequence a_1^{n-2} over Q the following equalities hold*

$$B(x_1^n) = A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)) \text{ and}$$

$$\alpha(x) = A(\alpha(a_1^{n-2}), a_1^{n-2}, x).$$

Let also $n = 2m$ and $m \in \mathbf{N}$. Then α is an isomorphism from the n-group (Q, B) onto the n-group (Q, A).

5. Description of $\emptyset$-$NetSAAnQ$

5.1. Theorem: *Let (Q, Σ) be an $\emptyset$-$NetSAAnQ$, $n \geq 2$ and A an arbitrary operation from Σ. Then, for every $B \in \Sigma$ there is exactly one central operation α of the n-group (Q, A) such that for every $x_1^n, a_1^{n-2} \in Q$ the following equality holds*

$$B(x_1^n) = A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Proof. Let A and B be two **arbitrary operations** from Σ. By Proposition 1.3.4, (Q, A) and (Q, B) have $\{1, n\}$-neutral operations, which will be denoted, respectively, with $\mathbf{e}_A$ and $\mathbf{e}_B$. Let also the inversing operation in the n-group (Q, A) be denoted by f_A.

1) By Definition 2.3.4, since this is the case when $\hat{S} = \emptyset$, for every $x_1^{2n-1} \in Q$ the following equality holds

$$A(B(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{n-1}, B(x_n^{2n-1})),$$

hence, by the substitutions $x_{n+1}^{2n-2} = a_1^{n-2}$ and $x_{2n-1} = \mathbf{e}_A(a_1^{n-2})$, where a_1^{n-2} is an arbitrary sequence over Q, we conclude that for every $x_1^n, a_1^{n-2} \in Q$ the following equality holds

$$B(x_1^n) = A(x_1^{n-1}, B(x_n, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))). \tag{1}$$

Starting with the statement connected with (1), by the substitutions $x_2^{n-1} = a_1^{n-2}$ and $x_1 = \mathbf{e}_B(a_1^{n-2})$, the following equality holds

$$x_n = A(\mathbf{e}_B(a_1^{n-2}), a_1^{n-2}, B(x_n, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))),$$

hence, by Proposition 1.4.1, we conclude that for every $x_n, a_1^{n-2} \in Q$ the following equality holds

(2) $$B(x_n, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})) = A(f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), a_1^{n-2}, x_n).$$

Further, by the statements connected with (1) and (2), we conclude that for every $x_1^n, a_1^{n-2} \in Q$, the following equality holds

(3) $$B(x_1^n) = A(x_1^{n-1}, A(f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})), a_1^{n-2}, x_n)).$$

2) Let

(4) $$\alpha(a_1^{n-2}) \stackrel{def}{=} f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2}))$$

for every sequence a_1^{n-2} over Q. [By the substitutions (4), the formula (3) reduces to

$$B(x_1^n) = A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)).]$$

In the following we prove that α is a central operation of the n-group (Q, A).

By Definition 2.3.4, since this is the case $\hat{S} = \emptyset$, for every $x_1^{2n-1} \in Q$ the following equality holds

$$A(x_1^{n-2}, B(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, B(x_n^{2n-1})),$$

hence, by the statements connected with (3) and (4), we conclude that for every $x_1^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_{2n-2})), x_{2n-1}) =$$

$$A(x_1^{n-1}, A(x_n^{2n-2}, A(\alpha(b_1^{n-2}), b_1^{n-2}, x_{2n-1})),$$

i.e., since (Q, A) is an n-semigroup, the following equality also holds

$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_{2n-2})), x_{2n-1}) =$$

$$A(x_1^{n-2}, A(x_{n-1}^{2n-3}, A(x_{2n-2}, \alpha(b_1^{n-2}), b_1^{n-2})), x_{2n-1}).$$

Hence, since (Q, A) is an n-quasigroup, we conclude that $x_{2n-2} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(\alpha(a_1^{n-2}), a_1^{n-2}, x_{2n-2}) = A(x_{2n-2}, \alpha(b_1^{n-2}), b_1^{n-2}),$$

hence, by Definition 3.1, we conclude that α is a central operation of the n-group (Q, A).

3) α is uniquely determined by B: part 4) of the proof of Theorem 4.1.

5.2. Theorem: *Let (Q, A) be an n-group, $n \geq 2$ and let (Q, A) has at least two central operations. Let also $\mathbf{C}_A$ be at least twoelement subset of the set of all central operations of the n-group (Q, A), and*

$$B \in \Sigma \stackrel{def}{=} (\exists \alpha \in \mathbf{C}_A)(\forall a_i \in Q)_1^{n-2}(\forall x_i \in Q)_1^n B(x_1^n) =$$

$$= A(x_1^{n-1}, A(\alpha(a_1^{n-2}), a_1^{n-2}, x_n)).$$

Then, (Q,Σ) is an $\emptyset$-NetSAAnQ.

Proof. 1) Let B and C be arbitrary operations fron Σ. Let also $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be central operations of the n-group (Q,A) from $\mathbf{C}_A$ such that for every sequence a_1^{n-2} over Q, the following equalities hold

$$B(x_1^n)=A(x_1^{n-1},A(\alpha(a_1^{n-2}),a_1^{n-2},x_n)) \text{ and}$$

$$C(x_1^n)=A(x_1^{n-1},A(\beta(a_1^{n-2}),a_1^{n-2},x_n)).$$

Then, by Proposition 3.6, there are permutations α and β of the set Q such that for every $x_1^n\in Q$ the following equalities hold

(1) $\qquad B(x_1^n)=A(x_1^{n-1},\alpha(x_n))$ and

(2) $\qquad C(x_1^n)=A(x_1^{n-1},\beta(x_n))$,

hence immediately we conclude that (Q,B) and (Q,C) are n-quasigroups.

2) Starting with the Propositions connected with (1) and (2), by Proposition 3.7 and by assumption that (Q,A) is an n-group, we conclude that for every $i\in\{2,\dots,n\}$ and for every $x_1^n\in Q$ the following sequence of equalities holds

$$B(C(x_1^n),x_{n+1}^{2n-1})=A(A(x_1^{n-1},\beta(x_n)),x_{n+1}^{2n-2},\alpha(x_{2n-1}))=$$

$$\alpha A(\beta A(x_1^n),x_{n+1}^{2n-1})=\alpha(\beta A(A(x_1^n),x_{n+1}^{2n-1}))=$$

$$\alpha(\beta A(x_1^{i-1},A(x_i^{i+n-1}),x_{n+1}^{2n-1}))=\alpha A(x_1^{i-1},\beta A(x_i^{i+n-1}),x_{n+1}^{2n-1})=$$

$$B(x_1^{i-1},C(x_i^{i+n-1}),x_{n+1}^{2n-1}),$$

i.e., the equality

$$B(C(x_1^n),x_{n+1}^{2n-1})=B(x_1^{i-1},C(x_i^{i+n-1}),x_{n+1}^{2n-1}),$$

also holds. □

For α and β from the proof of Theorem 5.2 the following statement holds:

$$(\forall x\in Q)\alpha(\beta(x))=\beta(\alpha(x)).$$

Starting with Theorem 5.1, by the above statement and by the proof of Theorem 5.2, we conclude that the following proposition holds:

5.3. Proposition: *If (Q,Σ) is an $\emptyset$-NetSAAnQ, $n\geq 2$, then (Q,Σ) is also an $\hat{S}$-NetSAAnQ for every $\hat{S}\subseteq\{2,\dots,n\}$.*

By Theorem 5.1, Proposition 3.6 and Proposition 3.7, we conclude that the following proposition holds:

5.4. Proposition: *If (Q,Σ) is an $\emptyset$-NetSAAnQ, $n\geq 2$, then for every $A,B\in\Sigma$ there is a permutation α of the set Q such that for every $x_1^n\in Q$ the following equality holds*

$$\alpha^{n-1}B(x_1^n)=A(\alpha(x_1),\dots,\alpha(x_n)).$$

6. A description of $\hat{S}$-*NetSAAnQ*

6.1. Theorem: *Let* $n \in \mathbf{N} \setminus \{1,2\}$, $\hat{S} \subseteq \{2,\dots,n\}$ *and* $\hat{S} \neq \emptyset$. *Then, the following statement holds: if* (Q,Σ) *is an* $\hat{S}$-*NetSAAnQ, then* (Q,Σ) *is an* $\emptyset$-*NetSAAnQ*.

Proof. Let A and B be two **arbitrary** operations from Σ. By Proposition 2.2.3, (Q,A) and (Q,B) are n-groups. By Proposition 1.3.2, (Q,A) and (Q,B) have $\{1,n\}$-neutral operations, denoted, respectively, by $\mathbf{e}_A$ and $\mathbf{e}_B$. Let also the inversing operation in (Q,A) be denoted by f_A, and the inversing operation in (Q,B) be denoted by f_B.

1) By Definition 2.3.4, since $\hat{S} \neq \emptyset$, we conclude that there is an $i \in \{1,\dots,n-1\}$ such that for every $x_1^{2n-1} \in Q$ the following equality holds

$$(1) \qquad A(x_1^{i-1}, B(x_i^{i+n-1}), x_{i+n}^{2n-1}) = B(x_1^i, A(x_{i+1}^{i+n}), x_{i+n+1}^{2n-1}).$$

Starting with the statements connected with (1), with the substitutions $x_{i+1}^{i+n-2} = a_1^{n-2}$ and $x_{i+n-1} = \mathbf{e}_A(a_1^{n-2})$, where a_1^{n-2} is an arbitrary sequence over Q[2], we conclude that for every $x_1^i, a_1^{n-2}, x_{i+n}^{2n-1} \in Q$ the following equality holds

$$A(x_1^{i-1}, B(x_i, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})), x_{i+n}^{2n-1}) = B(x_1^i, x_{i+n}^{2n-1}),$$

whence, since for every $a_1^{n-2}, u, x_i \in Q$ the equivalence

$$B(x_i, a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})) = u \Leftrightarrow$$
$$x_i = B(u, a_1^{n-2}, f_B(a_1^{n-2}, \mathbf{e}_A(a_1^{n-2}))),$$

we conclude that for every $y_1^n, a_1^{n-2} \in Q$ also the following equality holds

$$(a) \qquad A(y_1^n) = B(y_1^{i-1}, B(y_i, a_1^{n-2}, \alpha_B(a_1^{n-2})), y_{i+1}^n),$$

where

$$(b) \qquad \alpha_B(a_1^{n-2}) \overset{def}{=} f_B(a_1^{n-2}, \mathbf{e}_A(a_1^{n-2})).$$

Similarly, if we put in (1) $x_{i+2}^{i+n-1} = a_1^{n-2}$ and $x_{i+1} = \mathbf{e}_B(a_1^{n-2})$, we conclude that for every $y_1^n, a_1^{n-2} \in Q$ the equality

$$(\bar{a}) \qquad B(y_1^n) = A(y_1^i, A(\alpha_A(a_1^{n-2}), a_1^{n-2}, y_{i+1}), y_{i+2}^n),$$

holds, where

$$(\bar{b}) \qquad \alpha_A(a_1^{n-2}) \overset{def}{=} f_A(a_1^{n-2}, \mathbf{e}_B(a_1^{n-2})).$$

2) Since $n \geq 3$ and since the present $\hat{S}$-*NetSAAnQ* satisfy the condition $\hat{S} \neq \emptyset$, arbitrary $A, B \in \Sigma$ satisfy not only the statement connected with (1), but also at least one of the following statements

[2] a_1^{n-2} is not an empty sequence, since $n \geq 3$.

I: There is an $j \in \{1,\dots,n-1\}$, satisfying the condition $j \le i-1$ or $j \ge i+2$, such that for every $x_1^{2n-1} \in Q$ the following holds

$$A(x_1^{j-1}, B(x_j^{j+n-1}), x_{j+n}^{2n-1}) = A(x_1^j, B(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}); \tag{2}$$

II: There is an $j \in \{1,\dots,n-1\}$, satisfying the condition $j \le i-2$ or $j \ge i+1$, such that for every $x_1^{2n-1} \in Q$ the following holds

$$B(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1}) = B(x_1^j, A(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1}); \text{ and} \tag{3}$$

III: For every $x_1^{2n-1} \in Q$ the following equalities hold

$$A(B(x_1^n), x_{n+1}^{2n-1}) = B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}) \text{ and} \tag{1_1}$$

$$B(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}) = A(x_1^2, B(x_3^{n+2}), x_{n+3}^{2n-1}) \tag{4}$$

3) In the following we prove that for an arbitrary $C \in \Sigma$, for every $D \in \Sigma$ there is a central operation α_C of the n-group (Q, C) such that for every $x_1^n \in Q$ and for every sequence a_1^{n-2} over Q the equality

$$D(x_1^n) = C(x_1^{n-1}, C(\alpha_C(a_1^{n-2}), a_1^{n-2}, x_n))^{3)}$$

holds, whence, since (Q,Σ) is an $NetSAAnQ$, by Theorem 5.2, it follows that (Q,Σ) is an $\emptyset$-$NetSAAnQ$.

Case I: Let the statement I holds together with the statement concerning (1). Starting with the statement with (2), $(\bar{a})$ and $(\bar{b})$, by the substitution $x_j^{j+n-1} = y_1^n$, we conclude that for every $x_1^{i-1}, y_1^n, x_{j+n}^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(x_1^{j-1}, A(y_1^i, A(\alpha_A(a_1^{n-2}), a_1^{n-2}, y_{i+1}), y_{i+2}^n), x_{j+n}^{2n-1}) =$$

$$A(x_1^{j-1}, y_1, A(y_2^{i+1}, A(\alpha_A(b_1^{n-2}), b_1^{n-2}, y_{i+2}), y_{i+3}^n, x_{j+n}), x_{j+n+1}^{2n-1}),$$

i.e., since (Q, A) is an n-semigroup, also the following equality holds

$$A(x_1^{j-1}, A(y_1^i, A(\alpha_A(a_1^{n-2}), a_1^{n-2}, y_{i+1}), y_{i+2}^n), x_{j+n}^{2n-1}) =$$

$$A(x_1^{j-1}, A(y_1^i, A(y_{i+1}, \alpha_A(b_1^{n-2}), b_1^{n-2}), y_{i+2}^n), x_{j+n}^{2n-1}),$$

hence, since (Q, A) is an n-quasigroup, we conclude that for every $y_{i+1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$A(\alpha_A(a_1^{n-2}), a_1^{n-2}, y_{i+1}) = A(y_{i+1}, \alpha_A(b_1^{n-2}), b_1^{n-2}).$$

Hence, by Definition 3.1, we conclude that α_A is a central operation of the n-group (Q, A). Thus, starting with statements connected with $(\bar{a})$ and $(\bar{b})$,

3) In the case I $C = A$ and $D = B$, and in the cases II and III $C = B$ and $D = A$.

by Proposition 3.3, we conclude that for every $y_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$B(y_1^n) = A(y_1^{n-1}, A(\alpha_A(a_1^{n-2}), a_1^{n-2}, y_n)).$$

Case II: Let the statement II holds together with the statement connected with (1). Starting with the statements concernig (3), (a) and (b), by the substitution $x_j^{j+n-1} = y_1^n$, similarly to the case I [using Proposition 3.4 instead of Definition 3.1], we conclude that α_B is a central operation of the n-group (Q, B) and that for every $y_1^n \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$A(y_1^n) = B(y_1^{n-1}, B(\alpha_B(a_1^{n-2}), a_1^{n-2}, y_n)).$$

Case III: Let III holds. The equality (1_1) is the equality (1) for $i = 1$. Also [for $i = 1$], te statement connected with (a) reduces to the following statement: for every $y_1^n, a_1^{n-2} \in Q$ the following equality holds

$$(a_1) \qquad A(y_1^n) = B(B(y_1, a_1^{n-2}, \alpha_B(a_1^{n-2})), y_2^n)$$
$$[= B(B(y_1, b_1^{n-2}, \alpha_B(b_1^{n-2})), y_2^n)].$$

Starting with the statements concerning (4), (a_1) and (b), by the substitution $x_2^{n+1} = y_1^n$, we conclude that for every $x_1, y_1^n, x_{n+2}^{2n-1} \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$B(x_1, B(B(y_1, a_1^{n-2}, \alpha_B(a_1^{n-2})), y_2^n), x_{n+2}^{2n-2}) =$$
$$B(B(x_1, b_1^{n-2}, \alpha_B(b_1^{n-2})), y_1, B(y_2^n, x_{n+2}), x_{n+3}^{2n-1}),$$

i.e., since (Q, B) is an n-semigroup, the following equality also holds

$$B(x_1, B(y_1, a_1^{n-2}, \alpha_B(a_1^{n-2})), B(y_2^n, x_{n+2}), x_{n+3}^{2n-1}) =$$
$$B(x_1, B(b_1^{n-2}, \alpha_B(b_1^{n-2}), y_1), B(y_2^n, x_{n+2}), x_{n+3}^{2n-1}),$$

whence, since (Q, B) is an n-quasigroup, we conclude that for every $y_1 \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equality holds

$$B(y_1, a_1^{n-2}, \alpha_B(a_1^{n-2})) = B(b_1^{n-2}, \alpha_B(b_1^{n-2}), y_1).$$

Hence, by Proposition 3.4, we conclude that α_B is a cental operation of the n-group (Q, B). Thus, starting with the statements connected with (a_1) and (b), by Proposition 3.3, for every $y_1^n \in Q$ and for every sequence a_1^{n-2} over Q, the following equality holds

$$A(y_1^n) = B(y_1^{n-1}, B(\alpha_B(a_1^{n-2}), a_1^{n-2}, y_n)).$$

7. A description $NetSAAnQ$

7.1. Theorem: *Let (Q, Σ) be an ω-$NetSAAnQ$ and $n \geq 3$. Let also A be an arbitrary operation from Σ and $(Q, \{\cdot, \varphi, b\})$ an nHG-algebra* **associated** *to the n-group (Q, A). Then, for every $B \in \Sigma$ there is exactly one $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold:*

(a) $B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n;$

(b) $(a \cdot b) \cdot x = x \cdot (a \cdot b);$

(c) $\varphi(a) = a$ *and*

(d) $(a \cdot b) \cdot (a \cdot b) = e,$

where e is a neutral element of the group $(Q, \cdot)$.

The sketch of the proof:

1)
$$\begin{aligned} B(x_1^n) &= A(x_1^{n-1}, A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n)) \\ &= x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x_n) \\ &= x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot \boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \\ &\quad \varphi^{n-2}(a_{n-2}) \cdot b \cdot x_n \\ &= x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n \end{aligned}$$
[:4.1, 1.5.2, 3.9].

2) $(\forall x \in Q)(a \cdot b) \cdot x = x \cdot (a \cdot b)$ [:4.1, 3.9].

3) $\varphi(a) = a$ [:4.1, 3.9].

4)
$$f_A(a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = \boldsymbol{\alpha}(a_1^{n-2}) \Leftrightarrow$$
$$\mathbf{e}_A(a_1^{n-2}) = A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2}));$$
$$\mathbf{e}_A(a_1^{n-2}) = b^{-1} \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{-1};$$
$$A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = \mathbf{e}_A(a_1^{n-2}) \Leftrightarrow$$
$$\boldsymbol{\alpha}(a_1^{n-2}) \cdot \varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}) \cdot b \cdot \boldsymbol{\alpha}(a_1^{n-2}) = \mathbf{e}_A(a_1^{n-2}) \Leftrightarrow$$
$$a \cdot b \cdot \boldsymbol{\alpha}(a_1^{n-2}) = \mathbf{e}_A(a_1^{n-2}) \Leftrightarrow$$
$$a \cdot b \cdot a \cdot (\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{-1} = b^{-1}(\varphi(a_1) \cdot \ldots \cdot \varphi^{n-2}(a_{n-2}))^{-1} \Leftrightarrow$$
$$a \cdot b \cdot a = b^{-1} \Leftrightarrow$$
$$(a \cdot b) \cdot (a \cdot b) = e \quad \square$$

The statements 1)-3) from the sketch of the proof of Theorem 7.1 are valid also if 6.1, 5.1, 1.5.2, 3.9; 6.1, 5.1, 3.9 and 6.1, 5.1, 3.9 are used respectively instead of propositions listed in the brackets. Thereby, with the mentioned substitutions of used propositions 1)-3) is the sketch of the proof of the following proposition:

7.2. Theorem: *Let (Q,Σ) be an $\hat{S}$-NetSAAnQ and $n \geq 3$. Let also A be an arbitraty operation from Σ and $(Q,\{\cdot,\varphi,b\})$ an nHG-algebra* **associated** *to the n-group (Q,A). Then, for every $B \in \Sigma$ there is exactly one $a \in Q$ such that for every $x, x_1^n \in Q$ the following equalities hold:*

(a) $B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n$;

(b) $(a \cdot b) \cdot x = x \cdot (a \cdot b)$ *and*

(c) $\varphi(a) = a$. □

By Proposition 3.10, Theorem 5.2 and Proposition 5.3, we conclude that the following proposition holds:

7.3. Theorem: *Let $n \geq 3$, let (Q,A) be an n-group, $(Q,\{\cdot,\varphi,b\})$ its* **associated** *nHG-algebra and let $\mathcal{A}$ be at* **least two element subset** *of the set Q, such that for every $a \in Q$ the following holds:*

$$a \in \mathcal{A} \overset{def}{\Leftrightarrow} \varphi(a) = a \wedge (\forall x \in Q)(a \cdot b) \cdot x = x \cdot (a \cdot b).$$

Let also Σ be the set of n-ary operations ($n \geq 3$) in Q such that for every B

$$B \in \Sigma \overset{def}{\Leftrightarrow} a \in \mathcal{A} \wedge (\forall x_i \in Q)_1^n B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n.$$

Then $(Q,\Sigma)$is an $\hat{S}$-NetSAAnQ for every $\hat{S} \subseteq \{2,\ldots,n\}$. □

By Proposition 3.10, part 4) of the sketch of the proof of Theorem 7.1 and by Theorem 4.2, we conclude that the following proposition holds:

7.4. Theorem: *Let $n \geq 3$, let (Q,A) be an n-group, $(Q,\{\cdot,\varphi,b\})$ its* **associated** *nHG-algebra, e the neutral element of the group $(Q,\cdot)$ and let $\mathcal{A}$ be at* **least two element subset** *of the set Q, such that for every $a \in Q$ the following holds:*

$$a \in \mathcal{A} \overset{def}{=} \varphi(a) = a \wedge (\forall x \in Q)((a \cdot b) \cdot x = x \cdot (a \cdot b) \wedge (a \cdot b) \cdot (a \cdot b) = e).$$

Let also Σ be the set of n-ary operations in Q such that for every B the following holds:

$$B \in \Sigma \overset{def}{\Leftrightarrow} B(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot a \cdot b \cdot x_n \wedge a \in \mathcal{A}.$$

Then, $(Q,\Sigma)$is an ω-NetSAAnQ for every $\omega \in \Omega$.

8. On a description of the case $n = 3$ by Yu. M. Movsisyan

Nontrivial super associative algebras with 3-quasigroup operations were described firstly by Yu. M. Movsisyan [: [9], p. 152-158].

In this section we compare one proposition of Yu. M. Movsisyan [: [9], p. 152, direction "⇒" of Theorem 2.2.37] with the corresponding proposition from 7 for $n = 3$ [:Theorem 7.2 for $n = 3$]. Therefore, we advance the following definition:

8.1: *Let $(Q, \{\circ, \beta, r, s, t\})$ be an algebra, where $\circ$ is a binary operation in Q, β is a permutation of the set Q, and r, s, t fixed elements of the set Q. Then we say that $(Q, \{\circ, \beta, r, s, t\})$ is a 3M-algebra iff the following statements hold:*
(1) $(Q, \circ)$ is a group;
(2) $\beta \in Aut(Q, \circ)$;
(3) $\beta(s \circ r) = r \circ s \circ t^{(-1)}$, where $^{(-1)}$ is the inversing operation in the group $(Q, \circ)$; and
(4) $(\forall x \in Q)\beta^2(x) \circ (\beta(r^{(-1)}) \circ s) = (\beta(r^{(-1)}) \circ s) \circ x$. □

Using Definition 8.1, Theorem of Movsisyan corresponding Theorem7.2 for $n = 3$ [: [9], p. 152, direction "$\Rightarrow$" of Theorem 2.2.37], can be formulated in the following way:

8.2: *Let (Q, Σ) be an $\hat{S}$-NetSAAnQ and $n = 3$. Then there is 3M-algebra $(Q, \{\circ, \beta, r, s, t\})$ such that the following statement holds: for every $B \in \Sigma$ there is exactly one $p \in Q$ such that for every $x, x_1^3 \in Q$ the following equalities are satisfied:*
(a) $B(x_1^3) = x_1 \circ r \circ \beta(x_2) \circ s \circ p \circ x_3$
(b) $p \circ x = x \circ p$ and
(c) $\beta(p) = t \circ p$.

Using 2.3.5, Theorem 7.2 can be formulated in the similar way for $n = 3$:

8.3: *Let (Q, Σ) be an $\hat{S}$-NetSAAnQ and $n = 3$. Then there is 3HG-algebra $(Q, \{\cdot, \varphi, b\})$ such that the following statement holds: for every $B \in \Sigma$ there is exactly one $a \in Q$ such that for every $x, x_1^3 \in Q$ the following equalities are satisfied:*
($\bar{a}$) $B(x_1^3) = x_1 \cdot \varphi(x_2) \cdot b \cdot a \cdot b \cdot x_3$,
($\bar{b}$) $(a \cdot b) \cdot x = x \cdot (a \cdot b)$ and
($\bar{c}$) $\varphi(a) = a$. □

By 8.2 and 8.3, using the statements connected with (a) and (ā), respectively, from 8.2 and 8.3, we conclude that also the following proposition holds:

8.4. Proposition: *Let (Q, Σ) be an $\hat{S}$-NetSAAnQ and $n = 3$. Let also $(Q, \{\circ, \beta, r, s, t\})$ and $(Q, \{\cdot, \varphi, b\})$ be a 3M- and 3HG-algebras, respectively, associated in the sense of 8.2 and 8.3 to the algebra (Q, Σ). Then there is exactly one $k \in Q$ such that for every $x, y \in Q$ the following equality holds*

$$x \circ y = x \cdot k \cdot y.$$

Finally, by 1.5.3, 8.4, 1.5.5, 7.2, 8.3 and 8.2, we conclude that the following proposition holds:

8.5. Proposition: *Let (Q, Σ) be an $\hat{S}$-NetSAAnQ and $n = 3$. Let also $(Q, \{\circ, \beta, r, s, t\})$ be a 3M-algebra associated to the algebra (Q, Σ) in the sense of 8.2. Then, there is a 3HG-algebra $(Q, \{\cdot, \overset{\circ}{\varphi}, \overset{\circ}{b}\})$ such that the following statements holds: for every $B \in \Sigma$ there is exactly one $p \in Q$, and exactly one*

$\overset{\circ}{a} \in Q$ *such that for every* $x, x_1^3 \in Q$, *together with (a)-(c) from 8.2, also the following equalities hold*

$$B(x_1^3) = x_1 \circ \overset{\circ}{\varphi}(x_2) \circ \overset{\circ}{b} \circ \overset{\circ}{a} \circ \overset{\circ}{b} \circ x_3,$$

$$(\overset{\circ}{a} \circ \overset{\circ}{b}) \circ x = x \circ (\overset{\circ}{a} \circ \overset{\circ}{b}), \varphi(\overset{\circ}{a}) = \overset{\circ}{a},$$

$$\overset{\circ}{b} \circ \overset{\circ}{a} \circ \overset{\circ}{b} = r \circ s \circ p \ and \overset{\circ}{\varphi}(x) = r \circ \beta(x) \circ r^{(-1)}.$$

9. References

[1] Dörnte W.: *Untersuchengen über einen verallgemeinerten Gruppenbegriff*, Math. Z. 29 (1928), 1-19.

[2] Bruck, R. H.: *A survey of binary systems*, Springer-Verlag, Berlin-Heidelberg-Göttingen 1958.

[3] Hosszú, M.: *On the explicit form of n-group operations*, Publ. math., Debrecen (10) 1-4 (1963), 88-92.

[4] Gluskin, L. M.: *Position operatives*, Mat. sb. t. (68) (110) No3 (1965), 444-472. (In Russian.)

[5] Belousov, V. D.: *Systems of quasigroups with generalized identities*, Usp. mat. nauk 20, No 1, 1965, 75-146. (In Russian.)

[6] Szász, C.: *Asupra axomelor care la baza definititei unui n-group*, "Lucrări ştint. Inst. Politehn. Braşov. Fac. Mec." 7(1965), 43-47. (Ref. Zh. Mat. 12A, 1966, 258.)

[7] Sokolov, E. I.: *On Gluskin-Hosszú theorem for Dörnte n-group*, Math. issl. 39, "Stiinca", Kishinev 1976, 187-189. (In Russian.)

[8] Dudek W.A., Glazek K., Gleichgewicht B.: *A note on the axioms of n-groups*, Coll. Math. Soc. J. Bolyai, 29. Universal Algebra, Esztergrom (Hungary), 1977, 195-202.

[9] Movsisyan, Yu. M.: *Introduction to the theory of algebras with hiperidentities*, Izdat. Erevan Univ., Erevan, 1986. (In Russian.)

[10] Ušan, J.: *Neutral operations of n-groupoids*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 18-2 (1988), 117-126. (In Russian.)

[11] Ušan, J.: *A comment of n-groups*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 24-1 (1994), 281-288.

[12] Ušan, J.: *On Hosszú-Gluskin algebras corresponding to the same n-group*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 25-1 (1995), 101-119.

[13] Ušan, J.: *On n-groups with $\{i,j\}$-neutral operation for $\{i,j\} \neq \{1,n\}$*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 25-2 (1995), 167-178.

[14] Ušan J.: n-Groups, $n \geq 2$, as Varieties of Type $\langle n, n-1, n-2 \rangle$, Algebra and Model Theory, Collection of papers edited by A.G. Pinus and K.N. Ponomaryov, Novosibirsk 1997, 182–208.

[15] Denecke K. and Wismath L.S.: Hyperidentities and clones, Goron and Breach Sci. Publ., Singapore 2000.

[16] Glazek K.: Bibliography of n-groups (polyadic groups) and some group-like n-ary systems, Proc. of the Synposium "n-ary structures" (Skopje 1982), Macedonian Academy of Sciences and Arts, Skopje 1982, 253-289.

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