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ON MONOTONE POLYNOMIAL INTERPOLATION

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Abstract. An older demonstration of the existence of an interpolatory piecewise monotone function is revisited. A new strategy to decrease the degree of interpolation polynomial is presented.

1. Introduction

Let, as in [4] $\mathcal{P}_n$ be the set of polynomials of degree n, $n \geq 1$, $P \in \mathcal{P}_n$ $x \in R$, $P(x) \in R$. Let the sequence of real numbers

$$x_0 < x_1 < \cdots < x_n \tag{1}$$

and an arbitrary sequence of real numbers

$$y_0, y_1, \dots, y_n \tag{2}$$

with

$$y_{i-1} \neq y_i, \quad i = 1, \dots, n. \tag{3}$$

The problem of piecewise interpolation [1–3,5] asks to find a polynomial $P \in \mathcal{P}_n$ which satisfy

$$P(x_i) = y_i, \quad i = 0, 1, \dots, n \tag{4}$$

and

$$P \text{ is monotone on } [x_{i-1}, x_i], \quad i = 1, \dots, n. \tag{5}$$

Let a set of indexes $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$ so that the polynomial

$$l(x) = (x - x_{i_1}) \dots (x - x_{i_n}) \tag{6}$$

satisfies

$$l(x)(y_i - y_{i-1}) > 0, \text{ for } x \in (x_{i-1}, x_i) \text{ and } i = 1, \dots, n. \tag{7}$$

and has minimal degree.

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To solve the problem given at (1)-(4), following the idea from [2], we construct the polynomials

$$l_i(x) = 1 + \frac{(x - x_{i-1})(x_i - x)}{(x_0 - x_n)^2}; i = 1, \ldots, n \tag{8}$$

A linear combination [2] of these polynomials at convenient powers constitutes one of the solutions. Unfortunately the powers could be very high. Decreasing the powers of the polynomials which verify (1)-(4) is a serious practical challenge. We prove in the next sections that the polynomials

$$h_i(x) = 1 + (x - x_{i-1})(x_i - x) \times \begin{cases} \dfrac{1}{(x_i - x_n)(x_{i-1} - x_n)} & \text{if } \dfrac{x_{i-1} + x_i}{2} < \dfrac{x_0 + x_n}{2} \\ \dfrac{1}{(x_i - x_0)(x_{i-1} - x_0)} & \text{if } \dfrac{x_{i-1} + x_i}{2} \geq \dfrac{x_0 + x_n}{2} \end{cases} \tag{9}$$

are a better choice than (8). These polynomials can substitute the polynomials

$l_i(x)$, $i = 1, \ldots, n$, in the proof and offer a solution with a power smaller than in [2].

The next section follows the skeleton of proof from [2]. For the sake of the presentation we take lemmas 1., 4., 5., 6. and theorem 1. almost identical from [2] and we add the results concerning the use of $h_i(x)$, $i = 1, \ldots, n$. Theorem 2. states that choosing $h_i(x)$, $i = 1, \ldots, n$ is a better practical strategy. The third section gives a numerical example and the forth contains a short discussion concerning algebras of functions presented in [2].

2. Main results

Lemma 1. [2] *The polynomials $l_i(x)$, $i = 1, \ldots, n$ satisfy:*

(a) $l_i(x) > 0$, $\forall x \in [x_0, x_n]$, $i = 1, \ldots, n$;

(b) $l_i(x) \leq 1$, $\forall x \in [x_0, x_{i-1}] \cup [x_i, x_n]$, $i = 1, \ldots, n$;

(c) *there are intervals $[a_i, b_i] \subset [x_{i-1}, x_i]$, $i = 1, \ldots, n$, and numbers $c_i > 1$, $i = 1, \ldots, n$, so that $l_i(x) \geq c_i$, $\forall x \in [a_i, b_i]$, $i = 1, \ldots, n$.*

Proof. [2] The proof is elementary. The polynomial $(x - x_{i-1})(x_i - x)$ is positive on $[x_{i-1}, x_i]$ hence (c) is true.

The inequalities $|x - x_{i-1}| < x_n - x_0$ and $|x_i - x| < x_n - x_0$, $x \in [x_0, x_n]$, imply

$$|(x - x_{i-1})(x_i - x)| < (x_n - x_0)^2, \text{ therefore } \left|\frac{(x - x_{i-1})(x_i - x)}{(x_n - x_0)^2}\right| \leq 1,$$

i.e., (a).

Relation (b) is easy to prove because the second term of $l_i(x)$ from (8) is negative and $l_i(x) \geq 0$ on $x \in [x_0, x_{i-1}] \cup [x_i, x_n]$, $i = 1, \dots, n$.

Since $l_i(x_{i-1}) = l_i(x_i) = 1$ and $l_i(x) > 1$ for $x \in (x_{i-1}, x_i)$, property (c) is obvious.

An example of such a polynomial is presented in Figure 3 for $l_2(x) = 1 - \dfrac{(x-2)(x-4)}{100}$ with $x_{i-1} = 2,\ \ x_i = 4,\ \ x_0 = 0,\ \ x_n = 10.$

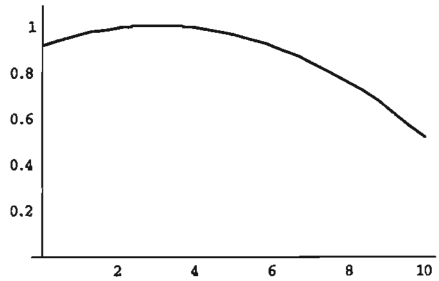

Figure 3 $\quad l_2(x) = 1 - \dfrac{(x-2)(x-4)}{100}$

Below we substitute the functions $l_i(x),\ \ i = 1, \dots, n$ with other more adequate.

The resemblance between the functions $l_i(x)$ and $h_i(x),\ \ i = 1, \dots, n$ is evident. This functions satisfy also Lemma 1.

Lemma 2. *For $x \in [x_0, x_n]$ and $i = 1, \dots, n$, the polynomials $h_i(x)$ satisfy:*

(a) $h_i(x) \geq 0, \forall x \in [x_0, x_n],\ \ i = 1, \dots, n$;

(b) $h_i(x) \leq 1, \forall x \in [x_0, x_{i-1}] \cup [x_i, x_n],\ \ i = 1, \dots, n$;

(c) *there is an interval $[a_i, b_i] \subset [x_{i-1}, x_i]$ and a real number $c_i > 1$ so that $h_i(x) \geq c_i$, for $x \in [a_i, b_i]$ and $i = 1, \dots, n$.*

Proof. Almost the same with that from Lemma 1.

The polynomials $h_i(x),\ \ i = 1, \dots, n$ vanish at least in x_0 or x_n. Figure 4 contains such a polynomial with the points as in Figure 3 i.e., $h_2(x) = 1 - \dfrac{(x-2)(x-4)}{48}$.

Figure 5 shows both graphic from figure 3 and 4. One remarks that $h_2(x)$ is under $l_2(x)$ on $[x_0, x_n] - (x_{i-1}, x_i) = [0, 10] - (2, 4) = [0, 2] \cup [4, 10]$ and $h_2(x)$ is over $l_2(x)$ on $(x_{i-1}, x_i) = (2, 4)$.

We have following lemma.

Lemma 3. *For $x \in [x_0, x_n]$ and $i = 1, \dots, n$, the polynomials $l_i(x)$ and $h_i(x)$ satisfy:*

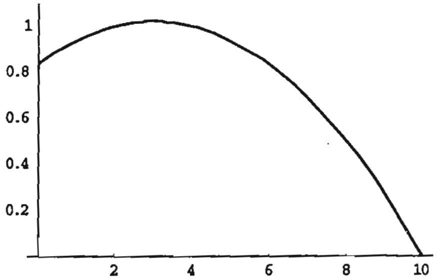

Figure 4 $h_2(x) = 1 - \dfrac{(x-2)(x-4)}{48}$

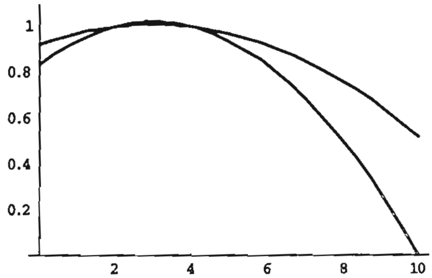

Figure 5 $l_2(x) = 1 - \dfrac{(x-2)(x-4)}{100}$; $h_2(x) = 1 - \dfrac{(x-2)(x-4)}{48}$

(a) $l_i(x) \leq h_i(x),\ \forall x \in [x_{i-1}, x_i],\ i = 1, \ldots, n;$

(b) $l_i(x) > h_i(x),\ \forall x \in [x_0, x_{i-1}) \cup (x_i, x_n]$, *for* $i = 2, \ldots, n-1$ *and* $\forall x \in (x_1, x_n]$, *for* $i = 1$ *and* $\forall x \in [x_0, x_{n-1})$ *for* $i = n$.

Proof. There are two cases. First, for $\dfrac{x_{i-1}+x_i}{2} < \dfrac{x_0+x_n}{2}$

$$h_i(x) - l_i(x) = (x - x_{i-1})(x_i - x) \times \left[\frac{1}{(x_i - x_n)(x_{i-1} - x_n)} - \frac{1}{(x_0 - x_n)^2}\right].$$

The factor from the right side is positive because

$$\frac{1}{(x_i - x_n)(x_{i-1} - x_n)} = \frac{1}{(x_n - x_i)(x_n - x_{i-1})} >$$
$$> \frac{1}{(x_n - x_0)(x_n - x_{i-1})} > \frac{1}{(x_n - x_0)^2}.$$

Hence the sign of difference is the sign of $(x - x_{i-1})(x_i - x)$ and then (a) and (b) is true.

The second case when $\frac{x_{i-1}+x_i}{2} \geq \frac{x_0+x_n}{2}$ implies

$$h_i(x) - l_i(x) = (x - x_{i-1})(x_i - x) \times \left[\frac{1}{(x_i - x_0)(x_{i-1} - x_0)} - \frac{1}{(x_n - x_0)^2}\right].$$

As before, the sign of difference is the sign of $(x - x_{i-1})(x_i - x)$ because

$$\frac{1}{(x_i - x_0)(x_{i-1} - x_0)} > \frac{1}{(x_n - x_0)(x_{i-1} - x_0)} > \frac{1}{(x_n - x_0)^2}.$$

Lemma 4. [2] *For* $k = 1, \dots, n$

$$\lim_{m\to\infty} \left| \int_{x_{k-1}}^{x_k} l(x) l_k^m(x) dx \right| = \infty.$$

Proof. [2] $\left|\int_{x_{k-1}}^{x_k} l(x) l_k^m(x) dx\right| = \int_{x_{k-1}}^{x_k} |l(x)|\, l_k^m(x) dx \geq c_k^m \int_{x_{k-1}}^{x_k} |l(x)|\, dx \to \infty$ because $c_k > 1$, $k = 1, \dots, n$.

Corollary 3. *For* $k=1, \dots, n$, *the polynomials* $h_k(x)$ *satisfy Lemma 4.*

Poof. By Lemma 3. the proof is obvious.

Lemma 5. [2] *For* $i \neq k$, $i = 1, \dots, n$, $k = 1, \dots, n$,

$$\lim_{m\to\infty} \left| \frac{\int_{x_{i-1}}^{x_i} l(x) l_k^m(x) dx}{\int_{x_{k-1}}^{x_k} l(x) l_k^m(x) dx} \right| = 0$$

Proof. [2] Let's take an arbitrary k. For $k \neq i$, by Lemma 1, therefore $\left|\int_{x_{i-1}}^{x_i} l(x) l_k^m(x) dx\right| < \int_{x_{i-1}}^{x_i} |l(x)|\, l_k^m(x) dx < \int_{x_{i-1}}^{x_i} |l(x)|\, dx$. Since $l(x)$ is continue $\int_{x_{i-1}}^{x_i} |l(x)|\, dx$, is finite. By Lemma 4, the proof is completed.

Let without proof

Lemma 6. [2] *If the numbers* a_{ik}, $i = 1, \dots, n$, $k = 1, \dots, n$, *satisfy:*
$a_{kk} = 1$ *for* $k = 1, \dots, n$, *and*
$|a_{ik}| < \frac{1}{2n}$ *if* $i \neq k$, *then exists positive real numbers* $r_1, \dots, r_n$ *such that* $\sum_{k=1}^{n} r_k a_{ik} = 1$, $i = 1, \dots, n$.

Theorem 1. *[2] There exists a polynomial P such that:*
$P(x_i) = y_i$, $i = 1, \dots, n$, *and*
$P(x)$ *is monotone on each interval* $[x_{i-1}, x_i]$, $i = 1, \dots, n$.

Proof. Without loss of generality we can take $y_0 = 0$. We search a solution in the form of

$$P(x) = \int_0^x l(t)g(t)dt, \tag{10}$$

where

$$g(t) \geq 0, \forall t \in [x_0, x_n]. \tag{11}$$

In [2] a solution of the form

$$g(t) = \sum_{k=0}^{n} r_k g_k(t) \tag{12}$$

is found, where

$$r_k \geq 0, \; k = 1, \ldots, n, \tag{13}$$

and

$$g_k(t) \geq 0, \; \forall t \in [x_0, x_n], \; k = 1, \ldots, n. \tag{14}$$

P must satisfy

$$P(x_i) = y_i, \; i = 1, \ldots, n, \tag{15}$$

that is

$$\sum_{k=1}^{n} r_k \int_0^{x_i} l(t)g_k(t)dt = y_i, \; i = 1, \ldots, n, \tag{16}$$

or equivalent

$$\sum_{k=1}^{n} r_k \int_{x_{i-1}}^{x_i} l(t)g_k(t)dt = y_i - y_{i-1}, \; i = 1, \ldots, n, \tag{17}$$

i.e.,

$$\sum_{k=1}^{n} r_k \frac{\int_{x_{i-1}}^{x_i} l(t)g_k(t)dt}{y_i - y_{i-1}} = 1, \; i = 1, \ldots, n. \tag{18}$$

To satisfy lemma 5. we need

$$\frac{\int_{x_{k-1}}^{x_k} l(t)g_k(t)dt}{y_k - y_{k-1}} = 1, \; k = 1, \ldots, n, \tag{19}$$

and

$$\left| \frac{\int\limits_{x_{i-1}}^{x_i} l(t) g_k(t) dt}{y_i - y_{i-1}} \right| < \frac{1}{2n}, \quad k = 1, \dots, n; \ i = 1, \dots, n, \ i \neq k. \tag{20}$$

To satisfy (19) we substitute $g_k(t)$ by

$$\frac{y_k - y_{k-1}}{\int\limits_{x_{k-1}}^{x_k} l(t) g_k(t) dt} l(t) g_k(t), \quad k = 1, \dots, n, \tag{21}$$

then (20) is

$$\left| \frac{y_k - y_{k-1}}{y_i - y_{i-1}} \frac{\int\limits_{x_{i-1}}^{x_i} l(t) g_k(t) dt}{\int\limits_{x_{k-1}}^{x_k} l(t) g_k(t) dt} \right| < \frac{1}{2n}, \quad i = 1, \dots, n, \ k = 1, \dots, n, \ i \neq k. \tag{22}$$

Let's take

$$g_k(t) = l_k^m(t), \tag{23}$$

where m is an integer and note that for m large enough conditions (19) and (22) are satisfied, i.e., the polynomial $g(t)$ from (12) is completely determinate.

The value of integer m from the proof is very large. We point out that the polynomials $h_i(t)$, $i = 1, \dots, n$, from Lemma 2 verify all the lemmas implied in the proof of Theorem 1. By substituting $l_i(x)$ by $h_i(x)$, for $i = 1, \dots, n$, the value of m is decreased.

Let the polynomial

$$H(x) = \int_0^x l(t) h(t) dt, \tag{24}$$

where $l(t)$ was defined before,

$$h(x) = \sum_{k=1}^{n} r_k \frac{y_k - y_{k-1}}{\int\limits_{x_{k-1}}^{x_k} l(t) h_k^m(t) dt} l(x) h_k^m(x), \tag{25}$$

and $r_1, \dots, r_n$ are positive real numbers.

Theorem 2. *There exist positive real numbers $r_1, \dots, r_n$ such that the polynomial H is piecewise monotone polynomial and its degree is less than the degree of polynomial in Theorem 1.*

Proof. First part is identic with the proof of Theorem 1 because the polynomials $h_i(x)$ and $l_i(x)$, $i = 1, \dots, n$, satisfy the same lemmas.

For the second part we take $i \neq k$ and an integer m.

As one can see that the key of the proof is the set of inequalities (22). We show that the substitution of $l_i(x)$ by $h_i(x)$ in (22) imply a smaller number in the left side. Let take the ratio

$$\rho := \frac{y_k - y_{k-1}}{y_i - y_{i-1}} \frac{\int_{x_{i-1}}^{x_i} l(t) l_k^m(t)dt}{\int_{x_{k-1}}^{x_k} l(t) l_k^m(t)dt} : \frac{y_k - y_{k-1}}{y_i - y_{i-1}} \frac{\int_{x_{i-1}}^{x_i} l(t) h_k^m(t)dt}{\int_{x_{k-1}}^{x_k} l(t) h_k^m(t)dt}, \text{ therefore}$$

(26)
$$\rho = \frac{\int_{x_{i-1}}^{x_i} l(t) l_k^m(t)dt}{\int_{x_{i-1}}^{x_i} l(t) h_k^m(t)dt} \frac{\int_{x_{k-1}}^{x_k} l(t) h_k^m(t)dt}{\int_{x_{k-1}}^{x_k} l(t) l_k^m(t)dt}$$

For $i \neq k$, from Lemma 3 we have $l_k^m(t) > h_k^m(t)$ for $t \in [x_{i-1}, x_i]$, i.e.,

(27)
$$\frac{\int_{x_{i-1}}^{x_i} l(t) l_k^m(t)dt}{\int_{x_{i-1}}^{x_i} l(t) h_k^m(t)dt} > 1.$$

From Lemma 3 too $l_k^m(t) \leq h_k^m(t)$ for $t \in [x_{k-1}, x_k]$ i.e.

(28)
$$\frac{\int_{x_{k-1}}^{x_k} l(t) h_k^m(t)dt}{\int_{x_{k-1}}^{x_k} l(t) l_k^m(t)dt} \geq 1$$

From (27) and (28), we obtain $\rho > 1$ and consequently $h_i(t)$, $i = 1, \dots, n$, satisfy condition (22) for a smaller value of m.

3. Numerical examples

The examples below are realized by a program in MATHEMATICA which can be obtained from the author. We take $n = 6$,

(29) $$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 6, x_6 = 7,$$

and

(30) $$y_0 = 0, y_1 = .25, y_2 = .38, y_3 = .23, y_4 = .44, y_5 = .58, y_6 = .62\ .$$

The first method applied for this set of data gives a solution of power $m = 220$. The corresponding matrix $\frac{y_k - y_{k-1}}{y_i - y_{i-1}} \frac{\int_{x_{i-1}}^{x_i} l(t) g_k(t) dt}{\int_{x_{k-1}}^{x_k} l(t) g_k(t) dt}$, $i = 1, \ldots, n$, $k = 1, \ldots, n$ is

1	0.077	3.19×10^{-6}	5.71×10^{-15}	3.23×10^{-39}	1.07×10^{-93}
0.077	1	0.077	3.19×10^{-6}	3.98×10^{-21}	5.01×10^{-53}
3.19×10^{-6}	0.077	1	0.77	6.52×10^{-10}	1.55×10^{-29}
5.71×10^{-15}	3.19×10^{-6}	0.077	1	0.001	5.71×10^{-15}
1.55×10^{-29}	5.71×10^{-15}	3.19×10^{-6}	0.077	1	0.077
1.07×10^{-93}	5.01×10^{-53}	1.55×10^{-29}	5.71×10^{-15}	0.001	1

We can see that the absolute values of all entries excepting the main diagonal are less than $\frac{1}{2n} = \frac{1}{12} \simeq 0.08$ as stated in theorem 1..

The second method using the functions $h_i(x)$, $i = 1, \ldots, n$ gives a matrix satisfying the conditions of theorem 2. for $m = 185$.

1	0.04	1.92×10^{-11}	6.39×10^{-60}	7.66×10^{-85}	5.97×10^{-104}
0.08	1	0.02	1.94×10^{-18}	3.05×10^{-38}	6.82×10^{-55}
3.78×10^{-6}	0.04	1	2.84×10^{-3}	4.20×10^{-16}	6.69×10^{-30}
7.19×10^{-15}	4.56×10^{-8}	0.02	1	4.93×10^{-5}	7.19×10^{-15}
9.69×10^{-30}	9.85×10^{-21}	1.92×10^{-11}	2.84×10^{-3}	4.93×10^{-5}	7.19×10^{-15}
5.97×10^{-104}	1.05×10^{-91}	2.62×10^{-77}	6.39×10^{-60}	4.93×10^{-5}	1

4. Remarks

In [2] the whole proof is built for an algebra of functions i.e., the set of polynomials are replaced by the smallest algebra $\mathcal{G}$ containing a set $\mathcal{A}$ of nondecreasing functions on $[x_0, x_n]$ possessing the property of separation for the points of the interval $[x_0, x_n]$. Without proof we present the following

Lemma 7. [2] *There exist a nondecreasing function $f \in \mathcal{G}$ such that*

$$f(x_{i-1}) < f(a_i) \leq f(b_i) < f(x_i), \tag{31}$$

where

$$0 = x_0 < a_1 < b_1 < x_1 < \ldots < x_{n-1} < a_n < b_n < x_n. \tag{32}$$

By this lemma we can construct similar functions

$$l_i(x) = 1 + \frac{(f(x) - f(x_{i-1}))(f(x_i) - f(x))}{(f(1) - f(0))^2}, \; i = 1, \ldots, n, \tag{33}$$

and $h_i(x) = 1 + (f(x) - f(x_{i-1}))(f(x_i) - f(x))\times$

$$\begin{cases} \dfrac{1}{(f(x_i) - f(x_n))\,(f(x_{i-1}) - f(x_n))} & \text{if } \dfrac{x_{i-1} + x_i}{2} < \dfrac{x_0 + x_n}{2} \\ \dfrac{1}{(f(x_i) - f(x_0))\,(f(x_{i-1}) - f(x_0))} & \text{if } \dfrac{x_{i-1} + x_i}{2} \geq \dfrac{x_0 + x_n}{2} \end{cases}$$

$$\text{for} \quad i = 1, \dots, n.$$

All lemmas of paragraph 1 and Theorems 8 and 9 are maintained.

With some supplementary conditions one can obtain better estimates of the degree of the monotone interpolation polynomials [1, 4].

The result is general and offer us a simple solution with an important decrease of the degree of polynomial used in the proof of Theorem 1. In our example it was a decrease with 16%, from 220 to 185.

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