Mathematica Moravica
Vol. **27**, No. **2** (2023), 91–114 doi: 10.5937/MatMor2302091S

Graph polynomials associated with Dyson–Schwinger equations

Ali Shojaei-Fard

Abstract. Quantum motions are encoded by a particular family of recursive Hochschild equations in the renormalization Hopf algebra which represent Dyson–Schwinger equations, combinatorially. Feynman graphons, which topologically complete the space of Feynman diagrams of a gauge field theory, are considered to formulate some random graph representations for solutions of quantum motions. This framework leads us to explain the structures of Tutte and Kirchhoff–Symanzik polynomials associated with solutions of Dyson–Schwinger equations. These new graph polynomials are applied to formulate a new parametric representation for large Feynman diagrams and their corresponding Feynman rules.

1. Introduction

On the one hand, Dyson–Schwinger equations are original tools for the study of quantum motions in gauge field theories. The solutions of these equations in strongly coupled theories have non-perturbative aspects. Numerical, computational and combinatorial methods together with the analytic, algebraic and geometric approaches are formulated in dealing with this important challenge in mathematical and theoretical physics [1, 4, 17–20, 26, 27, 31–38]. On the other hand, graph polynomials, such as Tutte and Kirchhoff–Symanzik polynomials, are considered to study graph invariants and characterize graphs in terms of their fundamental properties. These polynomials are at least invariant under graph isomorphisms. Tutte polynomial, as a two variables graph polynomial, has a universal property which is useful to evaluate any multiplicative graph invariant under a deletion/contraction reduction machinery. This recursive setting is useful to understand how graph polynomials can be specialized or generalized. These homogeneous polynomials, which relate graphs to objects in a polynomial

---

2020 *Mathematics Subject Classification.* 05C31, 05C63, 81Q30, 81T16.
Key words and phrases. Combinatorial Dyson–Schwinger equations; Feynman graphons; Graph polynomials; Parametric representations.
Full paper. Received 16 February 2023, accepted 21 October 2023, available online 20 November 2023.

©2023 Mathematica Moravica

ring, are formulated in terms of a chosen coordinate system with variables corresponding to the edges of the graph. The corresponding affine hypersurfaces, embedded in some projective spaces, are called graph hypersurfaces [2, 3, 13, 14, 21, 22, 25]. In this paper, we consider recent applications of infinite combinatorics to Quantum Field Theory [30, 33–35] to construct a new family of graph polynomials which encode the combinatorics of quantum motions.

1.1. **The physics behind the main idea.** Non-perturbative aspects of gauge field theories under strong coupling constants are the most difficult problems in modern theoretical and mathematical physics. These physical theories are the main tools for the study of elementary particles in High Energy Physics. For example, Quantum Chromodynamics (QCD), as the gauge field theory of the strong interaction of quarks and gluons, considers the physics of elementary particles at the length scales around 10^{-15} meter or smaller with the energy scales around $\Lambda_{\text{QCD}} \sim 0.2\text{GeV}$ or greater. The combinatorial 1PI Green's functions in QCD, which encode formal expansions of Feynman diagrams corresponding to specific propagations or interactions, are given by

$$(1) \qquad \begin{aligned} G^{e_i}(c_g) &= \mathbb{I} - \sum_{\text{res}(\Gamma)=e_i} c_g^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)}, \\ G^{v_j}(c_g) &= \mathbb{I} + \sum_{\text{res}(\Gamma)=v_j} c_g^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)}. \end{aligned}$$

The amount c_g is the running coupling constant with respect to the bare coupling constant $g \geq 1$ and $|\Gamma|$ is the loop order of Γ. The amplitudes e_i correspond to quark, gluon and ghost fields and the amplitudes v_j correspond to five types of interactions between them. Fixed point equations for these Green's functions, called Dyson–Schwinger equations, determine quantum motions in QCD. Solutions of these quantum motions are given by polynomials with respect to c_g such that the appropriate higher loop order Feynman diagrams are coefficients in these expansions. The analysis of these solutions at energies $< \Lambda_{\text{QCD}}$ encapsulates low energy QCD, where running coupling constants increase and non-perturbative aspects such as confinement do happen [1, 4, 19, 32, 33, 35, 38, 39].

Firstly, graph polynomials are useful tools for the study of Feynman integrals which contribute to the structure of 1PI Green's functions [2, 3, 9, 21, 22]. Secondly, the Connes–Kreimer renromalization Hopf algebra provided a combinatorial reformulation of Dyson–Schwinger equations [8, 17, 18, 20]. Thirdly, solutions of combinatorial Dyson–Schwinger equations are interpreted by a new class of graph functions which topologically enrich the renormalization Hopf algebra [30, 34–36]. Thanks to these background, the main idea of this research is to formulate a new family of graph polynomials for the study of non-perturbative solutions of quantum motions which have infinite number of Feynman integrals in their structure. Our study provides some new computational tools in dealing with quantum motions and their renormalization process in strongly coupled physical theories such as low energy QCD.

1.2. **Parametric representations of Feynman integrals via graph polynomials.** The action functional of a gauge field theory Φ with Lagrangian density $\mathcal{L}_\Phi = \mathcal{L}_{\Phi,0} + \mathcal{L}_{\Phi,\text{int}}$ is given by

$$S[\phi] = \int \mathcal{L}_\Phi[\phi] d^D x dt, \tag{2}$$

such that the interaction part $\mathcal{L}_{\Phi,\text{int}}$ is a polynomial with respect to the (running) coupling constants. Feynman diagrams are building blocks of Lagrangian framework. These combinatorial graphs, which encode interactions between elementary particles, are decorated by momenta information.

Definition 1. A Feynman diagram Γ is a finite weighted decorated oriented graph with the following properties.

- Γ contains a set Γ^0 of vertices which present interactions and a set Γ^1 of edges which present elementary and virtual particles.
- $\Gamma^1 = \Gamma^1_{\text{int}} \sqcup \Gamma^1_{\text{ext}}$. The objects in Γ^1_{int}, which are edges with the beginning and ending vertices, present virtual particles, while the objects in Γ^1_{ext}, which are edges with the beginning or ending vertices, present elementary particles in Φ.
- The valence of each vertex in Γ^0 is the degree of one of the monomials in $\mathcal{L}_\Phi$.
- External edges in Γ^1_{ext} obey the conservation law $\sum_{e \in \Gamma^1_{\text{ext}}} p_e = 0$ with respect to the momentum parameter. It means that the amount of momenta for input particles is the same as the amount of momenta for output particles in Γ.

Feynman rules, which encode fundamental information of the physical theory, together with Fourier transforms are applied to replace Feynman diagrams underlying the momentum space with their corresponding Feynman integrals. In this transition, each loop associates to an integrate over the corresponding momentum. The divergence or convergence of each integrate is determined by superficial degree of divergence of its corresponding Feynman diagram. Thanks to Feynman rules, Feynman diagrams restore the summation over the probability amplitudes corresponding to all possible exchanges of virtual particles which contribute to a process in the physical theory. In other words, for a quantum expectation value $\mathcal{O}(\phi)$, the interaction part of the functional integral

$$\int \mathcal{O}(\phi) \exp(i \frac{S[\phi]}{\hbar}) \mathcal{D}[\phi], \tag{3}$$

derived from the equation (2), generates a formal expansion of terms indexed by Feynman diagrams with respect to their loop numbers. Green's functions, which have a general form

$$G_N(x_1,\dots,x_N) = \frac{\int \exp(i\frac{S[\phi]}{\hbar})\phi(x_1)\cdots\phi(x_N)\mathcal{D}[\phi]}{\int \exp(i\frac{S[\phi]}{\hbar})\mathcal{D}[\phi]}, \tag{4}$$

are represented in terms of formal expansions of powers of (running) coupling constants together with Feynman diagrams as coefficients [28].

Feynman rules assign a functional of external momenta to each Feynman diagram. It is given by integrating over internal momenta as variables in terms of propagators of internal edges and momentum conservation at each vertex.

Definition 2. Up to multiplicative constants, Feynman integral corresponding to any Feynman diagram Γ with N_Γ external edges with momentum information $p_\Gamma := (p_1,\dots,p_{N_\Gamma})$ is parametrically represented by

$$U(\Gamma;p_\Gamma) := \frac{\Gamma(n_\Gamma - \frac{Dl_\Gamma}{2})}{(4\pi)^{Dl_\Gamma/2}} \int_{\sigma_{n_\Gamma}} v_{n_\Gamma} \frac{P_\Gamma(w,p_\Gamma)^{-n_\Gamma+Dl_\Gamma/2}}{\Psi_\Gamma(w)^{-n_\Gamma+D(l_\Gamma+1)/2}}. \tag{5}$$

In this formula:

- n_Γ is the number of internal edges in Γ.
- $\sigma_{n_\Gamma} := \{w = (w_1,\dots,w_{n_\Gamma}) \in \mathbb{R}_+^{n_\Gamma} : w_1 + \cdots + w_{n_\Gamma} = 1\}$ is a simplex together with the volume form v_{n_Γ}.
- l_Γ, as the first Betti number of Γ, is the loop number.
- $P_\Gamma(w,p_\Gamma)$ is the second Symanzik polynomial with respect to the external momenta variables. It is a homogeneous polynomial which is defined by cut sets of Γ.
- $\Psi_\Gamma(w)$ is the first Kirchhoff–Symanzik polynomial. It is a homogeneous polynomial of degree l_Γ.
- The integral is defined in the affine hypersurface complement $\mathbb{A}^{n_\Gamma} \setminus \hat{X}_\Gamma$ such that

$$\hat{X}_\Gamma := \{w \in \mathbb{A}^{n_\Gamma} : \Psi_\Gamma(w) = 0\}. \tag{6}$$

For more details see [25].

Definition 2, as the standard parametric representation, shows the importance of graph polynomials and motivic tools for the computation of Feynman integrals [21,22,24,25]. In this direction, some new applications of Hodge structures in the calculation of Feynman integrals underlying graph polynomials are investigated [9, 11]. A motivic version of Feynman rules is addressed on the basis of Kirchhoff–Symanzik polynomials to formulate the algebro-geometric Feynman rules characters [2, 3]. These abstract characters send each class in the Grothendieck ring of conical immersed affine varieties to an object in the Grothendieck ring of varieties spanned by classes

$[X_\Gamma]$ with respect to Feynman diagrams in the physical theory Φ. This program, which is on the basis of the deletion/contraction operators and Tutte–Grothendieck polynomial, finds some interesting interconnections between Feynman integrals and periods of algebraic varieties [24, 25]. In addition, it addresses a motivic treatment in dealing with perturbative and non-perturbative renormalization program at the level of the universal motivic Feynman rules character [24, 29].

1.3. **Random graph representations of Feynman diagrams.** Graphons or graph functions are tools in infinite combinatorics to study graph limits of sequences of weighted finite graphs where the space of finite graphs can be topologically completed with respect to cut norm [15,23]. Measure theoretic tools are applied to build suitable ground measure spaces for the construction of non-trivial graphons as the graph limits of sequences of sparse finite graphs [5–7, 10].

Definition 3. For a given probability measure space (Ω, μ_Ω),

- A graphon is a real valued symmetric bounded μ_Ω-measurable function $W : \Omega \times \Omega \to \mathbb{R}$.
- For any invertible μ_Ω-measure preserving transformation ρ on Ω, W^ρ is called a labeled graphon which is given by

$$W^\rho(x, y) := W(\rho(x), \rho(y)). \tag{7}$$

- Labeled graphons V_1, V_2 are called weakly isomorphic, if and only if there exists a graphon W together with μ_Ω-measure preserving transformations ρ_1, ρ_2 such that

$$V_1 = W^{\rho_1}, \quad V_2 = W^{\rho_2}. \tag{8}$$

- Set $[W]_\approx$ as the equivalence class of all labeled graphons which are weakly isomorphic with W. It is called an unlabeled graphon. Define the cut norm on the space of unlabeled graphons given by

$$d_{\text{cut}}([W_1]_\approx, [W_2]_\approx) := \inf_{\rho_1,\rho_2} \|W_1^{\rho_1} - W_2^{\rho_2}\|_{\text{cut}} \tag{9}$$

such that

$$\|W^\rho\|_{\text{cut}} := \sup_{S,T} \left| \int_{S\times T} W^\rho(x, y) d\mu_\Omega(x) d\mu_\Omega(y) \right| \tag{10}$$

where the supremum is on all non-trivial μ_Ω-measurable subsets of Ω.

For more details see [15, 23].

The Connes–Kreimer Hopf algebraic renormalization encodes the BPHZ perturbative renormalization in the context of a graded connected commutative non-cocommutative Hopf algebra $H_{\text{FG}}(\Phi)$ of Feynman diagrams over the field $\mathbb{Q}$. The objects of this Hopf algebra, which is freely generated by 1PI Feynman diagrams in Φ, have representations in terms of decorated rooted trees. This combinatorial version applies primitive (1PI) Feynman diagrams as a collection of decorations to represent higher loop order Feynman diagrams with nested overlapping loops in terms of (linear combinations of) non-planar decorated rooted trees. Thanks to the existence of an injective Hopf algebra homomorphism from $H_{\mathrm{FG}}(\Phi)$ to $H_{\mathrm{CK}}(\Phi)$, we send each Feynman diagram Γ to its combinatorial representation t_Γ [8, 12, 16, 39].

Lemma 1. *For a given probability measure space (Ω, μ_Ω), there exists a unique unlabeled graphon class associated with Γ.*

Proof. The vertex number n of t_Γ defines a partition $p_n : I_1, \dots, I_n$ of Ω. The adjacency matrix $A_{t_\Gamma} = (a_{ij})_{n\times n}$ of t_Γ defines a bounded μ_Ω-measurable function $W^{p_n}_{t_\Gamma} : \Omega \times \Omega \to \mathbb{R}$ given by

$$(x, y) \in I_i \times I_j \mapsto a_{ij}. \tag{11}$$

It is a labeled graphon such that thanks to Definition 3, the weakly isomorphic equivalence class $[W^{p_n}_{t_\Gamma}]_\approx$ is the unique unlabeled graphon with respect to the combinatorics of Γ. We call it an unlabeled Feynman graphon and present it with $[W_\Gamma]_\approx$. □

The space of Feynman diagrams in Φ is topologically completed with respect to the cut-distance topology (given by Definition 3), where Feynman graphons are graph limits which present the convergence of sequences of finite Feynman diagrams [30]. Choosing suitable ground measure spaces such as $[a, b] \subseteq \mathbb{R}_+$ equipped with Lebesgue or Gaussian measures and rescaling techniques are applied to generate non-trivial graph limits [35]. Feynman graphons are useful tools for the construction of a new family of random graphs and random graph processes which contribute to the structure of formal expansions of higher loop order Feynman diagrams [35, 37]. In this direction, a new theory of non-perturbative renormalization [31, 34] and a new theory of computation for the space of quantum motions [33, 36, 37] are developed.

Definition 4. Consider a probability measure space (Ω, μ_Ω) and any Feynman diagram Γ in Φ with the corresponding unlabeled Feynman graphon $[W_\Gamma]_\approx$.

- A random graph representation of Γ is a random graph R^ρ_Γ which is built in terms of choosing independently $n = |t_\Gamma|$ nodes $x_1, \dots, x_n$ from Ω such that with the probability $W^\rho_\Gamma(x_i, x_j)$ there exists an edge between x_i and x_j in R^ρ_Γ where ρ is a μ_Ω-measure preserving transformation on Ω.
- A sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams in Φ is convergent if and only if the sequence $\{\frac{[W_{\Gamma_n}]_\approx}{\|[W_{\Gamma_n}]_\approx\|_{\mathrm{cut}}}\}_{n\geq 1}$ of the unlabeled Feynman graphons converges to a non-zero Feynman graphon $[W]_\approx$ with respect to the cut-distance topology given by Definition 3.

Thanks to Definitions 3 and 4 and Lemma 1, we build a new infinite graph X_W from the unlabeled Feynman graphon $[W]_\approx$. X_W is built in terms of the adjacency matrix of the infinite tree or forest t_∞ which is the graph limit of the sequence $\{t_{\Gamma_n}\}_{n\geq 1}$ of rooted trees. We call X_W a large Feynman diagram associated with the sequence $\{\Gamma_n\}_{n\geq 1}$.

Lemma 2. *Following the notations of Definition 4, there exists a random graph representation for the graph limit* $[W]_\approx$.

Proof. For a given μ_Ω-measure preserving transformation τ on Ω, we build an infinite random graph R_W^τ in terms of choosing an infinite countable nodes $x_1, x_2, \ldots$ from Ω such that with the probability $W^\rho(x_i, x_j)$ there exists an edge between x_i and x_j in R_W^τ.

For each Feynman diagram Γ_n, random graph representations $R_{\Gamma_n}^{\rho_n}$ and $R_{\Gamma_n}^{\psi_n}$ are isomorphic since $W_{\Gamma_n}^{\rho_n}$ and $W_{\Gamma_n}^{\psi_n}$ are weakly isomorphic for any μ_Ω-measure preserving transformations ρ_n, ψ_n on Ω. Therefore up to the weakly isomorphic relation, we consider random graph representations $R_{\Gamma_1}, \ldots, R_{\Gamma_n}, \ldots$ and R_W corresponding to the unlabeled Feynman graphons $[W_{\Gamma_1}]_\approx, \ldots, [W_{\Gamma_n}]_\approx, \ldots$ and $[W]_\approx$. It is observed that the sequence $\{R_{\Gamma_n}\}_{n\geq 1}$ converges to R_W when the number of chosen nodes goes to infinity. □

Thanks to Definition 4 and Lemma 2, R_W is a random graph representation of the large Feynman diagram X_W.

1.4. **Achievements.** Graph polynomials [13, 14] provide fundamental computational tools for the study of Feynman integrals in Quantum Field Theory. Tutte and Kirchhoff–Symanzik polynomials on the space of Feynman diagrams are studied to clarify the motivic nature of Feynman integrals and renormalization process in gauge field theories [2, 3, 9, 11, 21, 22, 24, 25, 29]. Thanks to the parametric and random graph representations of Feynman diagrams (explained in Subsections 1.2 and 1.3) and combinatorial reformulation of quantum motions in terms of the Connes–Kreimer Hopf algebraic framework, the following results are obtained.

- Random graph representations of solutions of quantum motions are formulated.
- Generalizations of Tutte and Kirchhoff–Symanzik polynomials are constructed as graph polynomials which encode invariants of solutions of quantum motions.
- A new parametric representation for solutions of quantum motions is obtained.
- Feynman rules of the topological Hopf algebra of renormalization are parametrically characterized.
- An algebro-geometric representation of those Feynman rules which contribute to the structure of quantum motions are formulated.

2. Random graph representations of quantum motions

Fixed point equations of Green's functions determine quantum equations of motion in gauge field theories. These recursive integral equations, which are called Dyson–Schwinger equations, have been studied in terms of various mathematical tools. Dyson–Schwinger equations can be interpreted as a quantized version of the Euler–Lagrange equations of motion originated from the principal of the least action. Solutions of Dyson–Schwinger equations are actually polynomials of coupling constants of the physical theory. Therefore their solutions in strongly interacting physical theories address non-perturbative aspects where we need to deal with infinite formal expansions of higher loop order Feynman diagrams. However these formal expansions can be governed by perturbation methods in weakly coupled physical theories with vanishing beta function. In other words, Dyson–Schwinger equations behave linearly in the conformal part of the physical theory while in non-conformal part, these equations behave non-linearly [1,4,20,26,27,38].

The Connes–Kreimer approach encapsulates the BPHZ perturbative renormalization in terms of the co-product

$$\Delta_{\mathrm{FG}}(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \tag{12}$$

such that the sum is over all disjoint unions of superficially divergent 1PI non-trivial subgraphs of Γ such that the quotient graph Γ/γ is the result of shrinking all internal edges of each connected component of γ into a vertex or an edge. The combinatorial version of the renormalization coproduct is given by

$$\Delta_{\mathrm{CK}}(t) = t \otimes 1 + 1 \otimes t + \sum_{c} P_c(t) \otimes R_c(t). \tag{13}$$

In this formula,

– The sum is over all non-trivial admissible cuts of t. An admissible cut c is a subset of edges of t such that any path from the root r_t to any leaf of t has at most one edge of c.
– $R_c(t)$ is the subtree of t which contains r_t after applying c .
– $P_c(t)$ is the remaining forest of subtrees of t.

For more details see [8, 12]. This Hopf algebraic framework is applied to reformulate Dyson–Schwinger equations under a combinatorial setting.

Definition 5. Consider the complex $\{C_n\}_{n\geq 0}$ such that for each $n \geq 1$, C_n is the $\mathbb{Q}$-vector space of cochains of degree n generated by all linear maps $L : H_{\mathrm{FG}}(\Phi) \to H_{\mathrm{FG}}(\Phi)^{\otimes n}$ such that $C_0 = \mathbb{Q}$. The renormalization coproduct (12) defines an operator

$$\mathbf{b}L := (\mathrm{id} \otimes L)\Delta_{\mathrm{FG}} + \sum_{k=1}^{n} (-1)^k \Delta_k L + (-1)^{n+1} L \otimes \mathbb{I}, \tag{14}$$

such that

- Δ_k is the renormalization coproduct which acts on the k-th component in $H_{\rm FG}(\Phi)^{\otimes n}$.
- $\mathrm{id} : H_{\rm FG}(\Phi) \to H_{\rm FG}(\Phi)$ is the identity operator.
- $\mathbb{I}$ is the unit constant operator such that $L \otimes \mathbb{I}(\Gamma) \mapsto L(\Gamma) \otimes \mathbb{I}$.

It is a coboundary operator (i.e. $\mathbf{b}^2 = 0$) such that its corresponding Hochschild cohomology is presented by $HH^{\bullet}_{\varepsilon}(H_{\rm FG}(\Phi))$.

Remark 1.

- The operator $\mathbb{I}$ is in the kernel of any 0-coboundary.
- The one-cocycle condition is given by

$$\mathbf{b}L = 0 \quad \Leftrightarrow \quad \Delta_{\rm FG} L = (\mathrm{id} \otimes L)\Delta_{\rm FG} + L \otimes \mathbb{I}. \tag{15}$$

- For any primitive Feynman diagram γ, define $B^+_\gamma : H_{\rm FG}(\Phi) \to H_{\rm FG}(\Phi)$ as a linear homogeneous operator which sends each Γ to a linear expansion of Feynman diagrams generated by the insertion of Γ into γ in terms of types of external edges of Γ and types of vertices of γ. B^+_γ is called a grafting operator.
- For any primitive Feynman diagram γ, $\mathbf{b}B^+_\gamma = 0$ which means that grafting operators are Hochschild one-cocycle.
- Primitive Feynman diagrams in $H_{\rm FG}(\Phi)$ determine some generators of $HH^1_\varepsilon(H_{\rm FG}(\Phi))$ in terms of the grafting operators B^+_γ.
- For $n \geq 2$, $HH^n_\varepsilon(H_{\rm FG}(\Phi)) = 0$.
- The pair $(H_{\rm CK}, B^+)$ has the universal property in a category of pairs (H, L) of commutative Hopf algebras H and Hochschild one-cocycles $L : H \to H$. The grafting operator B^+ sends a forest $t_1 \cdots t_n$ of rooted trees to a new rooted tree t by adding a new root r together with n new edges from r to the roots $r_{t_1}, \dots, r_{t_n}$.

For more details see [16, 18].

Definition 6. Consider a family $\{\gamma_n\}_{n\geq 1}$ of primitive Feynman diagrams in a gauge field theory Φ with the bare coupling constant g. The recursive Hochschild equation

$$X = \mathbb{I} + \sum_{n\geq 1} (\lambda g)^n \omega_n B^+_{\gamma_n}(X^{n+1}) \tag{16}$$

in $H_{\rm FG}(\Phi)[[\lambda g]]$ is called a combinatorial Dyson–Schwinger equation under the running coupling constant λg for $0 < \lambda \leq 1$ such that ω_n are some constants.

- This equation has a unique solution $X = \sum_{n\geq 0} (\lambda g)^n X_n$ such that X_0 is the empty graph and for $n \geq 1$,

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\left(\sum_{k_1+\cdots+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1} \cdots X_{k_{j+1}} \right) \in H_{\rm FG}(\Phi). \tag{17}$$

For more details see [17, 18].

Lemma 3. *Following the notations of Definition 6, for any probability measure space* (Ω, μ_Ω)*, there exists a unique unlabeled graphon class associated with the partial sums*

$$Y_m := X_0 + (\lambda g)X_1 + \cdots + (\lambda g)^m X_m, \quad m \geq 1. \tag{18}$$

Proof. For each $m \geq 1$, consider its rooted tree representation t_{Y_m} of Y_m which is a forest of decorated non-planar rooted trees. Thanks to Lemma 1, each component X_k has a unique unlabeled Feynman graphon. Therefore Feynman graphon W_{Y_m} is a normalization of the direct sum of stretched versions of Feynman graphons associated with $X_1, \ldots, X_m$. We have

$$W_{Y_m} = \frac{W_{X_1} + \cdots + W_{X_m}}{||W_{X_1} + \cdots + W_{X_m}||_{\mathrm{cut}}}, \tag{19}$$

such that

- W_{X_k} is a stretched Feynman graphon of weight $(\lambda g)^k$ for $k = 1, \ldots, m$ which is defined on a measurable subset $I_k \subset \Omega$ with $\mu_\Omega(I_k) = (\lambda g)^k$.
- For any $i \neq j$, $I_i \cap I_j = \emptyset$ such that $\{I_1, \ldots, I_m\}$ is a partition of Ω.

Further details about the structure of W_{Y_m} is given in [34, 35]. □

Feynman graphons give us a topological enrichment of the renormalization Hopf algebra. The resulting topological Hopf algebra, presented by $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$, encodes graph limits of sequences of finite Feynman diagrams. Therefore this new space is rich enough to interpret solutions of quantum motions as graph limits of partial sums. This approach has been discussed in [30, 34, 35].

Theorem 1. *Solutions of quantum motions have random graph representations.*

Proof. We associate a random graph representation $R_{X_{\mathrm{DSE}}}$ to the solution X_{DSE} of a combinatorial Dyson–Schwinger equation DSE with the general form (16).

Choose the Lebesgue measure space $(\mathbb{R}_+, m)$ as the ground space of our Feynman graphon models. Thanks to Definition 4 and Lemma 3, we can show that the sequence $\{Y_m\}_{m\geq 1}$ of partial sums is convergent to X_{DSE} with respect to the cut-distance topology when m tends to infinity. Feynman graphon $W_{X_{\mathrm{DSE}}}$ is an infinite direct sum of stretched versions of Feynman graphons associated with $X_1, \ldots, X_m, \ldots$. We have

$$W_{X_{\mathrm{DSE}}} = \frac{W_{X_1} + \cdots + W_{X_m} + \cdots}{||W_{X_1} + \cdots + W_{X_m} + \cdots ||_{\mathrm{cut}}}, \tag{20}$$

such that

- W_{X_k} is a stretched Feynman graphon of weight $(\lambda g)^k$ for $k = 1, \ldots, m, \ldots,$ which is defined on a measurable subset $I_k \subset \mathbb{R}_+$ with $m(I_k) = (\lambda g)^k$.

– For any $i \neq j$, $I_i \cap I_j = \emptyset$ such that $\{I_1, \ldots, I_m, \ldots\}$ is a partition of $\mathbb{R}_+$.

Since the space of Feynman graphons is topologically complete, $W_{X_{\mathrm{DSE}}}$ is a well-defined Feynman graphon on $\mathbb{R}_+$. Applying rescaling methods enable us to project $W_{X_{\mathrm{DSE}}}$ to a graphon on the ground Lebesgue measure space $([0,1), m)$. Further details about the structure of $W_{X_{\mathrm{DSE}}}$ is given in [34].

Thanks to Definition 4, Lemmas 2 and 3, up to the weakly isomorphic relation on labeled graphons with respect to Lebesgue measure preserving transformations on $\mathbb{R}_+$, a random graph representation R_{Y_m} of Y_m is defined in terms of choosing independently $n_{Y_m} = |t_{Y_m}|$ nodes $x_1, \ldots, x_{n_{Y_m}}$ from $\mathbb{R}_+$ such that with the probability $W_{Y_m}(x_i, x_j)$, there exists an edge between x_i and x_j in R_{Y_m}.

Up to the weakly isomorphic relation with respect to Lebesgue measure preserving transformations on $\mathbb{R}_+$, define $R_{X_{\mathrm{DSE}}}$ as the convergent limit of the sequence $\{R_{Y_m}\}_{m \geq 1}$. $R_{X_{\mathrm{DSE}}}$ is an infinite random graph built in terms of choosing independently infinite countable nodes $x_1, x_2, \ldots$ from $\mathbb{R}_+$ such that with the probability $W_{X_{\mathrm{DSE}}}(x_i, x_j)$, there exists an edge between x_i and x_j in $R_{X_{\mathrm{DSE}}}$. We consider $R_{X_{\mathrm{DSE}}}$ as the random graph representation for X_{DSE}. □

3. Quantum motions via graph polynomials

In general, a graph invariant is a function on the space of graphs which has the same output on isomorphic graphs. Graph invariants characterize graphs in terms of some particular properties. Graph polynomials, such as Tutte polynomial, image the space of graphs to some polynomial rings. Applications of graph polynomials to Quantum Field Theory have been investigated in several research works [2, 3, 9, 11, 21, 22, 24, 25]. In this part, we show the importance of random graph representations of solutions of combinatorial Dyson–Schwinger equations in the structure of Tutte and Kirchhoff–Symanzik polynomials which encode invariants of these equations under strongly coupled coupling constants.

Definition 7. For a given finite connected graph G and any edge $e \in E(G)$,

– $G \setminus e$ is a new graph as the result of deleting an edge $e \in E(G)$. It has the same set of vertices $V(G)$ and the set of edges $E(G) - \{e\}$.
– G/e is a new graph as the result of contracting an edge e in terms of identifying the endpoints of the edge e by shrinking this edge.
– Tutte polynomial $T(G; x, y)$ is a two variables recursive polynomial with respect to the independent variables x, y such that
  - $T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y)$,
  - $T(G; x, y) = xT(G/e; x, y)$, if e is a co-loop,
  - $T(G; x, y) = yT(G \setminus e; x, y)$, if e is a loop.

Lemma 4. *Tutte polynomial of a finite connected graph G is defined in terms of spanning trees in G.*

Proof. Define a total order on the set $E(G) = \{e_1, \ldots, e_n\}$ given by

$$e_i \prec e_j \quad \Leftrightarrow \quad i > j. \tag{21}$$

For a spanning tree t in G, an edge e is called internally active in t if $e \in E(t)$, and it is the smallest edge in the cut defined by e. In dual version, an edge u is called externally active if $u \notin E(t)$, and it is the smallest edge in the cycle defined by u. Following Definition 7, Tutte polynomial of the totally ordered graph G can be defined in terms of the formal expansion

$$T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j \tag{22}$$

such that t_{ij} counts spanning trees with internal activity i and external activity j. This definition is independent of the chosen total order. □

Remark 2.

- Tutte polynomial has the universal property with respect to the invariants of graphs. This means that any multiplicative graph invariant on disjoint unions and one-point joins of graphs which is formulated via a deletion/contraction reduction process is given in terms of an evaluation of Tutte polynomial.
- There exist some well-defined generalizations of Tutte polynomial, such as Tutte–Grothendieck polynomial, which is independent of the choice of any order structure on the edge sets of graphs.

For more details see [13, 14].

Theorem 2. *There exists a generalization of Tutte polynomial which encodes invariants of non-perturbative solutions of combinatorial Dyson–Schwinger equations.*

Proof. Tutte polynomial given by Definition 7 is multiplicative over disjoint unions of finite Feynman diagrams. For a finite connected Feynman diagram Γ without any overlapping subdivergence with the corresponding rooted tree representation t_Γ, $\Gamma = \coprod_{v \in V(t_\Gamma)} \Gamma_v$ such that each Γ_v is 1PI Feynman subdiagram inserted at a vertex of t_Γ while edges of t_Γ are bridges. Then we have

$$T(\Gamma; x, y) = \prod_{j=1}^{n} T(\Gamma_{v_j}; x, y), \quad n = |t_\Gamma|. \tag{23}$$

Consider an equation DSE with the general form (16) underlying the (running) coupling constant $\lambda g = 1$, the corresponding sequence $\{Y_m\}_{m \geq 1}$ of partial sums and the solution X_{DSE}. For each $m \geq 1$, Y_m is a finite linear combination of weighted graphs which can be interpreted as a disjoint union of graphs,

$$Y_m = X_1 + \cdots + X_m \mapsto X_1 \sqcup \cdots \sqcup X_m. \tag{24}$$

Thanks to Lemma 4 and Remark 2, we have

$$T(Y_m; x, y) = \prod_{s=1}^{m} T(X_s; x, y) = \prod_{s=1}^{m} \sum_{i_s, j_s} t_{i_s j_s} x^{i_s} y^{j_s} \tag{25}$$

such that $t_{i_s j_s}$ is the number of spanning trees (or forests) in X_s with internal activity i_s and external activity j_s.

Metric structure on the space of Feynman graphons can be lifted on to the space of Feynman diagrams and combinatorial Dyson–Schwinger equations [34, 35]. Thanks to Definition 3, Lemma 3 and [30, 34], $\{Y_m\}_{m\geq 1}$ is convergent to X_{DSE} with respect to the cut-distance topology. Therefore for each $\epsilon > 0$, there exists N_ϵ such that for each $m_1, m_2 \geq N_\epsilon$,

$$d(Y_{m_1}, Y_{m_2}) := d_{\text{cut}}([W_{Y_{m_1}}]_\approx, [W_{Y_{m_2}}]_\approx) < \epsilon. \tag{26}$$

Following the proof of Lemma 3, for each $m \geq 1$, Feynman graphon W_{Y_m} is defined in terms of the rooted tree representations $t_{X_1}, \ldots, t_{X_m}$ of Feynman diagrams $X_1, \ldots, X_m$. $t_{X_1}, \ldots, t_{X_m}$ are the only spanning trees (or forests) in themselves. The relation (26) shows that spanning forests of partial sums tend to the spanning forest $t_{X_{\text{DSE}}}$ of X_{DSE}.

For the collection $\left\{ \prod_{s=1}^{m} T(t_{X_s}; x, y) \right\}_{m\geq 1}$ of Tutte polynomials corresponding to the rooted tree representations of graphs X_s, define the collection

$$\left\{ p_m : \prod_{s=1}^{\infty} T(t_{X_s}; x, y) \longrightarrow \prod_{s=1}^{m} T(t_{X_s}; x, y) \right\}_{m\geq 1} \tag{27}$$

of projection operators.

Thanks to the universal property of Tutte polynomial, for any graph invariant polynomial R satisfied in Definition 7 together with the collection

$$\left\{ f_m : R \longrightarrow \prod_{s=1}^{m} T(t_{X_s}; x, y) \right\}_{m\geq 1}, \tag{28}$$

we can define the unique map

$$\digamma : R \longrightarrow \prod_{s=1}^{\infty} T(t_{X_s}; x, y) \tag{29}$$

such that $f_m = p_m \circ \digamma$. Therefore we consider $\prod_{s=1}^{\infty} T(t_{X_s}; x, y)$ as Tutte polynomial associated with the infinite forest $t_{X_{\text{DSE}}}$.

It leads us to define Tutte polynomial associated with X_{DSE} as the infinite direct product of Tutte polynomials associated with the components X_s. We have

$$T(X_{\text{DSE}}; x, y) = \prod_{s=1}^{\infty} T(X_s; x, y). \tag{30}$$ □

Definition 8. For a finite connected Feynman diagram Γ with the loop number l_Γ, define the circuit matrix $\hat{\eta} = (\eta_{ik})_{ik}$ such that $i \mapsto e_i \in E(\Gamma)$ and k ranges over loops $\gamma_1, \ldots, \gamma_{l_\Gamma}$.

- If an edge e_i belongs to a loop γ_k with the same/reverse orientations, then $\eta_{ik} = 1$, $\eta_{ik} = -1$, respectively.
- If the edge e_i does not belong to a loop γ_k, then $\eta_{ik} = 0$.
- The arrays of the associated Kirchhoff–Symanzik $l_\Gamma \times l_\Gamma$-matrix $M_\Gamma(w)$ are given by

$$(M_\Gamma(w))_{kr} = \sum_{i=1}^{n_\Gamma} w_i \eta_{ik} \eta_{ir} \tag{31}$$

such that n_Γ is the number of internal edges in Γ. This matrix defines a function

$$M_\Gamma : \mathbb{A}^{n_\Gamma} \longrightarrow \mathbb{A}^{l_\Gamma^2}, \; w = (w_1, \ldots, w_{n_\Gamma}) \mapsto M_\Gamma(w) \tag{32}$$

on higher dimensional affine spaces.

For more details see [13, 14, 25].

Definition 9. For a finite Feynman diagram Γ with the loop number l_Γ,

- First Kirchhoff–Symanzik polynomial associated with Γ is given by

$$\Psi_\Gamma(w) = \det(M_\Gamma(w)) \tag{33}$$

which is independent of the choice of an orientation on Γ and the basis of loops.
- Ψ_Γ, as a function on $\mathbb{A}^{n_\Gamma}$, is a homogeneous polynomial of degree l_Γ which can be formulated in terms of spanning trees. We have

$$\Psi_\Gamma(w) = \sum_{t \subset \Gamma} \prod_{e \notin E(t)} w_e \tag{34}$$

such that the sum is over all spanning trees t in Γ. For each spanning tree, the product is over all edges of Γ that are not in the selected spanning tree.
- Ψ_Γ is multiplicative over connected components.

For more details see [11, 13, 14, 24, 25].

Theorem 3. *First Kirchhoff–Symanzik polynomial of the non-perturbative solution of any combinatorial Dyson–Schwinger equation is well-defined.*

Proof. Consider an equation DSE with the general form (16) underlying the (running) coupling constant $\lambda g = 1$, the corresponding sequence $\{Y_m\}_{m\geq 1}$ of partial sums and the solution X_{DSE}. Thanks to Definition 9, for each $m \geq 1$, apply the correspondence

$$Y_m = X_1 + \cdots + X_m \mapsto X_1 \sqcup \cdots \sqcup X_m \tag{35}$$

to obtain the first Kirchhoff–Symanzik polynomial of Y_m as the direct product of the polynomials associated with each of its components. We have

$$\Psi_{Y_m}(w_{(\oplus m)}) = \prod_{j=1}^{m} \Psi_{X_j}(w_{(j)}), \quad \Psi_{X_j}(w_{(j)}) = \sum_{T_j \subset X_j} \prod_{e \notin E(T_j)} w_e. \tag{36}$$

- The sum $\sum_{T_j \subset X_j}$ is taken over all spanning trees (or forests) T_j in X_j.
- The product $\prod_{e \notin E(T_j)}$ is taken over all edges in X_j which are not in the spanning tree (or forest) T_j.
- For each $j \geq 1$, $w_{(j)} := (w_1^{(j)}, \ldots, w_{n_{X_j}}^{(j)}) \in \mathbb{A}^{n_{X_j}}$ such that n_{X_j} is the number of internal edges in X_j.
- For each $m \geq 1$, $w_{(\oplus m)} := (w_{(1)}, \ldots, w_{(m)}) \in \mathbb{A}^{n_{Y_m}}$ such that $n_{Y_m} = \sum_{j=1}^{m} n_{X_j}$ is the number of internal edges in Y_m.

For each $m \geq 1$, set SF_m as the collection of all spanning forests in Y_m. Equip the collection

$$\text{SF}_{\text{DSE}} := \bigsqcup_{m=1}^{\infty} \text{SF}_m \tag{37}$$

of spanning forests in X_{DSE} with a distance function given by

$$d(\text{sf}_i, \text{sf}_j) := d(Y_{m_i}, Y_{m_j}), \ \forall \ \text{sf}_i \in \text{SF}_{m_i}, \text{sf}_j \in \text{SF}_{m_j}, \tag{38}$$

such that

$$d(Y_{m_i}, Y_{m_j}) = d_{\text{cut}}([W_{Y_{m_i}}]_{\approx}, [W_{Y_{m_j}}]_{\approx}). \tag{39}$$

Thanks to the cut-distance convergence of the sequence $\{Y_m\}_{m\geq 1}$ to X_{DSE}, for each $\epsilon > 0$, there exists N_ϵ such that for each $u, v \geq N_\epsilon$,

$$d(\text{sf}_u, \text{sf}_v) = d_{\text{cut}}([W_{Y_{m_u}}]_{\approx}, [W_{Y_{m_v}}]_{\approx}) < \epsilon. \tag{40}$$

Therefore

- Spanning forests in Y_m tend to the spanning forests in X_{DSE} when m tends to infinity.
- First Kirchhoff–Symanzik polynomial $\Psi_{X_{\text{DSE}}}(z)$ associated with the equation DSE is defined as the limit of the sequence $\{\Psi_{Y_m}(w_{(\oplus m)})\}_{m\geq 1}$ of the first Kirchhoff–Symanzik polynomials of the partial sums $\{Y_m\}_{m\geq 1}$ with respect to the metric (38) when m tends to infinity.

Formula (36) and the metric (38) show that the infinite direct product

$$\Psi_{X_{\mathrm{DSE}}}(z) = \prod_{j=1}^{\infty} \Psi_{X_j}(w_{(j)}) = \sum_{T \subset X_{\mathrm{DSE}}} \prod_{e \notin E(T)} z_e, \tag{41}$$

as a function on an infinite dimensional affine space $\mathbb{A}_{\mathrm{DSE}}$ given by

$$z = (z_1, z_2, \dots) \in \mathbb{A}_{\mathrm{DSE}} := \prod_{j=1}^{\infty} \mathbb{A}^{n_{X_j}}, \tag{42}$$

is well-defined such that

$$z = (z_1, z_2, \dots) = (w_{(1)}, w_{(2)}, \dots). \tag{43}$$

The sum $\sum_{T \subset X_{\mathrm{DSE}}}$ in the formula (41) is taken over all spanning forests $T \in \mathrm{SF}_{\mathrm{DSE}}$ such that the product $\prod_{e \notin E(T)}$ is taken over all edges of X_{DSE} which are not in T. □

Remark 3. Any combinatorial Dyson–Schwinger equation $\mathrm{DSE}(\lambda g)$ under a (running) coupling constant $\lambda g < 1$ has a perturbative solution given by the convergent limit of a geometric type series. In other words, its solution $X_{\mathrm{DSE}(\lambda g)}$ contributes to formal geometric or binomial series such as

$$\frac{\mathbb{I}}{\mathbb{I} - X_{\mathrm{DSE}(\lambda g)}} = \mathbb{I} + X_{\mathrm{DSE}(\lambda g)} + X^2_{\mathrm{DSE}(\lambda g)} + \cdots \tag{44}$$

or

$$(X_{\mathrm{DSE}(\lambda g)})^r = \sum_{n=0}^{\infty} \binom{r}{n} (X_{\mathrm{DSE}(\lambda g)} - \mathbb{I})^n \tag{45}$$

such that $\mathbb{I}$, as the unit of the algebra structure on Feynman diagrams, is the empty graph and $r \in \mathbb{R}$. Thanks to Theorems 2, 3, graph polynomials associated with $X_{\mathrm{DSE}(\lambda g)}$ are compatible with these series where formal expansions are replaced with infinite direct products. It means that

$$T\Big(\frac{\mathbb{I}}{\mathbb{I} - X_{\mathrm{DSE}(\lambda g)}}; x, y\Big) = \prod_{n=1}^{\infty} T(X^n_{\mathrm{DSE}(\lambda g)}; x, y) \tag{46}$$

with

$$\Psi_{\frac{\mathbb{I}}{\mathbb{I} - X_{\mathrm{DSE}(\lambda g)}}}(z) = \prod_{n=1}^{\infty} \Psi_{X^n_{\mathrm{DSE}(\lambda g)}}(z), \tag{47}$$

or

$$T((X_{\mathrm{DSE}(\lambda g)})^r; x, y) = \prod_{n=0}^{\infty} \binom{r}{n} T((X_{\mathrm{DSE}(\lambda g)} - \mathbb{I})^n; x, y), \tag{48}$$

with

$$\Psi_{(X_{\mathrm{DSE}(\lambda g)})^r}(z) = \prod_{n=0}^{\infty} \binom{r}{n} \Psi_{(X_{\mathrm{DSE}(\lambda g)} - \mathbb{I})^n}(z). \tag{49}$$

Theorem 4. *The first Kirchhoff–Symanzik polynomial of the solution of any combinatorial Dyson–Schwinger equation can be computed recursively in terms of the deletion and the contraction operators.*

Proof. Consider a combinatorial Dyson–Schwinger equation $\mathrm{DSE}(\lambda g)$ under a (running) coupling constant λg with the corresponding solution $X_{\mathrm{DSE}(\lambda g)}$ and the sequence $\{Y_m\}_{m\geq 1}$ of its partial sums. Deletion and contraction operators on finite Feynman diagrams are defined in [3, 25]. Thanks to the cut-distance convergence of the sequence $\{Y_m\}_{m\geq 1}$ to $X_{\mathrm{DSE}(\lambda g)}$ with respect to the cut-distance topology ([30, 34]), we extend deletion and contraction operators to large Feynman diagrams.

Consider the hypersurface

$$\hat{V}_{X_{\mathrm{DSE}(\lambda g)}} := \{z = (z_1, z_2, \dots) \in \mathbb{A}_{\mathrm{DSE}(\lambda g)} : \Psi_{X_{\mathrm{DSE}(\lambda g)}}(z) = 0\} \tag{50}$$

with the corresponding projective hypersurface

$$V_{X_{\mathrm{DSE}(\lambda g)}} := \{z = (z_1, z_2, \dots) \in \mathbb{P}_{\mathrm{DSE}(\lambda g)} : \Psi_{X_{\mathrm{DSE}(\lambda g)}}(z) = 0\} \tag{51}$$

in the infinite dimensional projective space $\mathbb{P}_{\mathrm{DSE}(\lambda g)} := \prod_{i=1}^{\infty} \mathbb{P}^{n_{X_i}-1}$. For edges $e_1, e_2, \dots, e_r, \dots$ of $X_{\mathrm{DSE}(\lambda g)}$ with the corresponding variables $z_1, z_2, \dots, z_r, \dots$ in $\mathbb{P}_{\mathrm{DSE}(\lambda g)}$ define the following generalized versions of the deletion and contraction operators.

- Deletion operator $\frac{\partial \Psi_{X_{\mathrm{DSE}(\lambda g)}}(z)}{\partial z_r} := \Psi_{X_{\mathrm{DSE}(\lambda g)} \setminus e_r}(z)$ is the result of deleting the edge e_r from $X_{\mathrm{DSE}(\lambda g)}$.
- Contraction operator $\Psi_{X_{\mathrm{DSE}(\lambda g)}}(z)|_{z_r=0} := \Psi_{X_{\mathrm{DSE}(\lambda g)}/e_r}(z)$ is the result of contracting the edge e_r to a point in $X_{\mathrm{DSE}(\lambda g)}$.

Thanks to [3, 25], for each $m \geq 1$, we have the recursive equation

$$\Psi_{Y_m}(w_{(\oplus m)}) = w_e \Psi_{Y_m \setminus e}(w_{(\oplus m)-}) + \Psi_{Y_m/e}(w_{(\oplus m)-}), \tag{52}$$

with respect to the variable $w_{(\oplus m)-} \in \mathbb{P}^{n_{Y_m}-2}$ associated with edges $f \neq e$ of Y_m. Thanks to Theorem 3, when m tends to infinity, we have the recursive equation

$$\Psi_{X_{\mathrm{DSE}(\lambda g)}}(z) = w_e \Psi_{X_{\mathrm{DSE}(\lambda g)} \setminus e}(z) + \Psi_{X_{\mathrm{DSE}(\lambda g)}/e}(z), \tag{53}$$

with respect to the infinite parameter $z \in \mathbb{P}_{\mathrm{DSE}(\lambda g)}$, for each edge e, which is not a bridge in $X_{\mathrm{DSE}(\lambda g)}$. The term $w_e \Psi_{X_{\mathrm{DSE}(\lambda g)} \setminus e}(z)$ collects those monomials corresponding to spanning forests of $X_{\mathrm{DSE}(\lambda g)}$ which do not include e. □

4. Application

In this section, we apply generalized Tutte and Kirchhoff–Symanzik polynomials constructed in Section 3 to extend the parametric representation given by Definition 2 to large Feynman diagrams. Then we parametrically formulate Feynman rules which contribute to solutions of quantum motions in terms of these new graph polynomials.

Corollary 1. *There exists a parametric representation for the solution of a combinatorial Dyson–Schwinger equation.*

Proof. Consider an equation DSE with the general form (16) underlying the (running) coupling constant $\lambda g = 1$, the corresponding sequence $\{Y_m\}_{m\geq 1}$ of the partial sums and the solution X_{DSE}. For each $m \geq 1$, thanks to Definition 2 and proof of Theorem 3, the parametric representation of the partial sum Y_m is given by

(54)
$$U(Y_m, p_{Y_m}) := \frac{\Gamma(n_{Y_m} - \frac{Dl_{Y_m}}{2})}{(4\pi)^{Dl_{Y_m}/2}} \int_{\sigma_{n_{Y_m}}} v_{n_{Y_m}} \frac{P_{Y_m}(w_{(\oplus m)}, p_{Y_m})^{-n_{Y_m}+Dl_{Y_m}/2}}{\Psi_{Y_m}(w_{(\oplus m)})^{-n_{Y_m}+D(l_{Y_m}+1)/2}},$$

such that $p_{Y_m} = (p_1, \dots, p_{N_{Y_m}})$ is the momentum information of external edges in Y_m, $n_{Y_m} = \sum_{j=1}^{m} n_{X_j}$ is the number of internal edges in Y_m, $l_{Y_m} = \sum_{j=1}^{m} l_{X_j}$ is the loop number of Y_m and

$$\sigma_{n_{Y_m}} := \{w_{(\oplus m)} = (w_{(1)}, \dots, w_{(m)}) \in \mathbb{R}_+^{n_{Y_m}} : w_{(1)} + \cdots + w_{(m)} = 1\} \tag{55}$$

is a simplex together with the volume form $v_{n_{Y_m}}$. $U(Y_m, p_{Y_m})$ is defined in the affine hypersurface complement $\mathbb{A}^{n_{Y_m}} \setminus \hat{V}_{Y_m}$ such that

$$\hat{V}_{Y_m} := \{w_{(\oplus m)} \in \mathbb{A}^{n_{Y_m}} : \Psi_{Y_m}(w_{(\oplus m)}) = 0\}. \tag{56}$$

Consider the normalized version of the above parametric representation and for each $m \geq 1$, set

$$\tilde{U}(Y_m, p_{Y_m}) := \frac{(4\pi)^{Dl_{Y_m}/2} U(Y_m, p_{Y_m})}{\Gamma(n_{Y_m} - \frac{Dl_{Y_m}}{2})}. \tag{57}$$

Define a distance function on the space of normalized parametric representations of the partial sums of X_{DSE} in terms of Feynman graphon representations of the partial sums (i.e., Lemma 3) together with the cut norm (9). For $m_1, m_2 \geq 1$, it is given by

$$d(\tilde{U}(Y_{m_1}, p_{Y_{m_1}}), \tilde{U}(Y_{m_2}, p_{Y_{m_2}})) := d_{\mathrm{cut}}([W_{Y_{m_1}}]_{\approx}, [W_{Y_{m_2}}]_{\approx}). \tag{58}$$

Thanks to Theorems 1, 3 and 4, the sequence $\{\tilde{U}(Y_m, p_{Y_m})\}_{m\geq 1}$ of normalized parametric representations of the partial sums is convergent with respect to the cut distance topology when m tends to infinity. This convergent integral function, which is an integral over the simplex

$$\sigma_{\mathrm{DSE}} := \{z = (z_1, z_2, \dots) \in \mathbb{R}_+^{\infty} : \sum_{j=1}^{\infty} z_j = 1\}, \tag{59}$$

depends on the first Kirchhoff–Symanzik polynomial $\Psi_{X_{\mathrm{DSE}}}(z)$. Thanks to equations (41), (42), (50) and (51), this convergent integral function is defined in the affine hypersurface complement $\mathbb{A}_{\mathrm{DSE}} \setminus \hat{V}_{X_{\mathrm{DSE}}}$. □

Some particular objects in the complex Lie group $\mathrm{Hom}(H_{\mathrm{FG}}(\Phi), A_{\mathrm{dr}})$ of characters of the renormalization Hopf algebra restore Feynman rules in gauge field theories. These characters enable us to associate dimensionally regularized Feynman integrals to Feynman diagrams where we relate the combinatorial versions of Green's functions and Slavnov–Taylor / Ward–Takahashi identities to their corresponding integral version. Feynman rules characters deform the renormalization antipode to extract finite values from Feynman integrals underlying the minimal subtraction scheme [12, 16, 39]. Now it is possible to characterize Feynman rules of large Feynman diagrams which contribute to the cut-distance topological completion of the renormalization Hopf algebra in terms of our generalized graph polynomials. The next result provides a parametric representation for Feynman rules characters of the enriched Hopf algebra $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$.

Corollary 2. *Let Φ be a strongly coupled gauge field theory with the bare coupling constant $g = 1$. Generalized Tutte polynomials associated with large Feynman diagrams which contribute to the non-perturbative solutions of quantum motions characterize Feynman rules characters of $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$.*

Proof. An abstract Feynman rules character on the space of finite Feynman diagrams is given by [2, 3]

$$\mathcal{U}(\Gamma) = T(\Gamma; x, y). \tag{60}$$

Following Definition 6, consider a combinatorial Dyson–Schwinger equation DSE with the solution $X_{\mathrm{DSE}} = \sum_{n\geq 0} X_n$ and the sequence $\{Y_m\}_{m\geq 1}$ of the partial sums. Thanks to Theorem 2, Tutte polynomial $\Psi_{X_{\mathrm{DSE}}}(z)$ is constructed as the infinite direct product of Tutte polynomials associated with components X_n. Thanks to the cut-distance convergence $\{Y_m\}_{m\geq 1}$ to X_{DSE}, extend $\mathcal{U}$ to the level of large Feynman diagrams. We have $\tilde{\mathcal{U}} \in \mathrm{Hom}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A_{\mathrm{dr}})$ such that

$$\begin{aligned}\tilde{\mathcal{U}}(X_{\mathrm{DSE}}) &= \lim_{m\to\infty} \mathcal{U}(Y_m) = \lim_{m\to\infty} \sum_{s=1}^{m} \mathcal{U}(X_s) \\ &= \lim_{m\to\infty} \sum_{s=1}^{m} T(X_s; x, y) = \sum_{s=1}^{\infty} T(X_s; x, y).\end{aligned} \tag{61}$$

Consider the Grothendieck ring $\tilde{\mathcal{F}}$ of immersed conical varieties generated by the equivalence classes $[\hat{V}_\Gamma]$ and $[\hat{V}_{X_{\mathrm{DSE}}}]$ up to linear changes of coordinates of varieties $\hat{V}_\Gamma \subset \mathbb{A}^{|\Gamma^1_{\mathrm{int}}|}$ and $\hat{V}_{X_{\mathrm{DSE}}} \subset \mathbb{A}_{\mathrm{DSE}}$. These changes are defined by homogeneous ideals with the inclusion-exclusion relation

$$[\hat{V}_{X_{\mathrm{DSE}}}] = [\hat{R}] + [\hat{V}_{X_{\mathrm{DSE}}} \setminus \hat{R}] \tag{62}$$

for a closed embedding. The algebro-geometric version of $\tilde{\mathcal{U}}$ is the abstract character $\bar{\mathcal{U}}$ such that

– $\bar{\mathcal{U}}$ sends X_{DSE} to $\tilde{I}([\mathbb{A}_{\mathrm{DSE}} \setminus \hat{V}_{X_{\mathrm{DSE}}}])$ such that $[\mathbb{A}_{\mathrm{DSE}} \setminus \hat{V}_{X_{\mathrm{DSE}}}] \in \tilde{\mathcal{F}}$.
– $\bar{\mathcal{U}}$ sends Γ to $\tilde{I}([\mathbb{A}^{|\Gamma^1_{\mathrm{int}}|} \setminus \hat{V}_\Gamma])$ such that $[\mathbb{A}^{|\Gamma^1_{\mathrm{int}}|} \setminus \hat{V}_\Gamma] \in \tilde{\mathcal{F}}$.
– $\tilde{I} : \tilde{\mathcal{F}} \longrightarrow A_{\mathrm{dr}}$ is a ring homomorphism such that A_{dr} is the commutative unital regularization algebra of Laurent series with finite pole parts. □

The algebro-geometric Feynman rules characters are constructed in terms of a polynomial invariant originated from the Chern–Schwartz–MacPherson characteristic classes of singular varieties [2,3,24,25]. Thanks to the formulation of the first Kirchhoff–Symanzik polynomial associated with solutions of combinatorial Dyson–Schwinger equations, now it is possible to define another graph invariant for these solutions.

Corollary 3. *There exists an extension of the Chern–Schwartz–MacPherson homomorphism for solutions of quantum motions.*

Proof. This is a direct result of [2,3] together with Theorems 3, 4, Remark 3 and Corollary 2.

We consider a combinatorial Dyson–Schwinger equation DSE under the (running) coupling constant $\lambda g = 1$. The existence of the CSM-homomorphism $I^{\mathrm{DSE}}_{\mathrm{CSM}}$ is supported by the completed topological space of Feynman graphons. The first Kirchhoff–Symanzik polynomial $\Psi_{X_{\mathrm{DSE}}}(z)$ of the unique solution X_{DSE} determines the hypersurface $\hat{V}_{X_{\mathrm{DSE}}} \subset \mathbb{A}_{\mathrm{DSE}}$. For each $m \geq 1$, there exists a transformation

$$\begin{aligned} \nu_{\oplus m} : 1_{\hat{V}_{Y_m}} \mapsto a_0^{(\oplus m)}[\mathbb{P}^0] + a_1^{(\oplus m)}[\mathbb{P}^1] \\ + a_2^{(\oplus m)}[\mathbb{P}^2] + \cdots + a_{n_{Y_m}}^{(\oplus m)}[\mathbb{P}^{n_{Y_m}}], \end{aligned} \tag{63}$$

such that $\hat{V}_{Y_m} \subset \mathbb{A}^{n_{Y_m}}$ and $n_{Y_m} = \sum_{i=1}^{m} n_{X_i}$ is the number of internal edges in Y_m. Define

$$\begin{aligned} I^{Y_m}_{\mathrm{CSM}} : \tilde{\mathcal{F}} \longrightarrow \mathbb{Z}[y], \\ [\hat{V}_{Y_m}] \mapsto a_0^{(\oplus m)} + a_1^{(\oplus m)} y + a_2^{(\oplus m)} y^2 + \cdots + a_m^{(\oplus m)} y^m. \end{aligned} \tag{64}$$

Thanks to the convergence of the sequence $\{Y_m\}_{m \geq 1}$ to X_{DSE} with respect to the cut-distance topology, the sequence $\{\nu_{\oplus m}\}_{m \geq 1}$ of transformations converges to a new transformation

$$\begin{aligned} \nu_{\mathrm{DSE}} : 1_{\hat{V}_{X_{\mathrm{DSE}}}} \mapsto a_0^{(\mathrm{DSE})}[\mathbb{P}^0] + a_1^{(\mathrm{DSE})}[\mathbb{P}^1] + a_2^{(\mathrm{DSE})}[\mathbb{P}^2] \\ + \cdots + a_n^{(\mathrm{DSE})}[\mathbb{P}^n] + \cdots . \end{aligned} \tag{65}$$

Now define a new map

$$\begin{aligned} I^{\mathrm{DSE}}_{\mathrm{CSM}} : \tilde{\mathcal{F}} \longrightarrow \mathbb{Z}[y], \\ [\hat{V}_{X_{\mathrm{DSE}}}] \mapsto a_0^{(\mathrm{DSE})} + a_1^{(\mathrm{DSE})} y + a_2^{(\mathrm{DSE})} y^2 + \cdots + a_n^{(\mathrm{DSE})} y^n + \cdots , \end{aligned} \tag{66}$$

and then extend it linearly to obtain our promising group homomorphism. □

5. Conclusion

This research addressed some new applications of infinite combinatorics to Quantum Field Theory where we developed a new theory of random and parametric representations for the study of non-perturbative solutions of combinatorial Dyson–Schwinger equations in strongly coupled gauge field theories.

5.1. **Summary of the results.** We associated some random graph representations to solutions of Dyson–Schwinger equations in terms of Feynman graphon models of these equations (i.e. Theorem 1). This Feynman graphon setting led us to explain the structure of graph polynomials extracted from the combinatorial information of solutions of quantum motions (i.e. Theorems 2, 3 and 4). These new generalized graph polynomials determined a new parametric representation of large Feynman diagrams which contribute to solutions of quantum motions (i.e. Corollary 1). In addition, we characterized Feynman rules characters on the topological Hopf algebra of renormalization. Formulating these particular characters, which act on large Feynman diagrams, in terms of a substructure of generalized Tutte polynomials clarified a new algebro-geometric setting (i.e., Corollaries 2 and 3).

5.2. **A physical application.** Renormalization of solutions of quantum motions generates some non-perturbative data of strongly coupled physical systems [4,19,20]. Theory of Feynman graphons provided a non-perturbative generalization of the BPHZ renormalization program for quantum motions underlying the Connes–Kreimer theory and the Riemann–Hilbert correspondence [31, 34]. In this direction, Corollaries 1 and 2 are new computational tools for non-perturbative data extracted from the topological Hopf algebra of renormalization. In other words, for an equation DSE with the solution X_{DSE} and Feynman graphon representation $W_{X_{\mathrm{DSE}}}$ in a gauge field theory Φ, the functional

$$X_{\mathrm{DSE}} \mapsto \left(S^{\tilde{\mathcal{U}}}_{R_{\mathrm{ms}}} * \tilde{\mathcal{U}}\right)(X_{\mathrm{DSE}}) \tag{67}$$

determines the renormalized value corresponding to the renormalization of X_{DSE}. We have

$$X_{\mathrm{DSE}} = \lim_{m\to\infty} Y_m \Leftrightarrow W_{X_{\mathrm{DSE}}} = \lim_{m\to\infty} W_{Y_m}, \tag{68}$$

with respect to the metric (9). The renormalization coproduct (12) and Definition 4 show that $\Delta_{\mathrm{FG}}(X_{\mathrm{DSE}})$ is well-defined such that

$$\Delta_{\mathrm{FG}}(X_{\mathrm{DSE}}) = \lim_{m\to\infty} \Delta_{\mathrm{FG}}(Y_m) \Leftrightarrow \Delta(W_{X_{\mathrm{DSE}}}) = \lim_{m\to\infty} \Delta(W_{Y_m}), \tag{69}$$

where for each $m \geq 1$,

$$\Delta(W_{Y_m}) = W_{Y_m} \otimes W_{\mathbb{I}} + W_{\mathbb{I}} \otimes W_{Y_m} + \sum_{\gamma_m} W_{\gamma_m} \otimes W_{Y_m/\gamma_m}. \tag{70}$$

Therefore

$$\Big(S^{\tilde{\mathcal{U}}}_{R_{\rm ms}} * \tilde{\mathcal{U}}\Big)(X_{\rm DSE}) = \lim_{m\to\infty} \Big(S^{\mathcal{U}}_{R_{\rm ms}} * \mathcal{U}\Big)(Y_m) \tag{71}$$

is well-defined such that $S^{\tilde{\mathcal{U}}}_{R_{\rm ms}}$ is a deformed or twisted version of the antipode of the Hopf algebra of renormalization $H^{\rm cut}_{\rm FG}(\Phi)$ topologically completed by the metric (9).

5.3. **Some perspectives for future works.** Graph polynomials provided a fundamental bridge between Quantum Field Theory and the theory of motives in Algebraic Geometry, where (multiple) zeta values are extracted from the computational processes of Feynman integrals [2, 3, 9, 11, 29].

- Our new parametric representation of solutions of quantum motions addresses a new approach to search for motives associated with combinatorial Dyson–Schwinger equations.
- According to the basic observation of Kontsevich, if any hypersurface $X_\Gamma \subset \mathbb{P}^{n_0-1}$, $n_0 = \#E(\Gamma)$ is mixed Tate, then there exists a polynomial P_Γ with $\mathbb{Z}$-coefficients such that for any finite field $\mathbb{F}_q$, we have $\#X(\mathbb{F}_q) = P_\Gamma(q)$ [9]. Theorems 2, 3 and 4 are new tools to search for some combinatorial Dyson–Schwinger equations DSE such that their solutions could satisfy a generalized version of Kontsevich's condition in the infinite dimensional projective space $\mathbb{P}_{\rm DSE} = \prod_{i=1}^{\infty} \mathbb{P}^{n_{X_i}-1}$.

Acknowledgments

The author is grateful to Max Planck Institute for Mathematics, Bonn, Germany for the support and hospitality during the work on the initial steps of this research.

References

[1] D. Antonov, Modern approaches to non-perturbative QCD and other confining gauge theories, Universe, 2022.

[2] P. Aluffi, M. Marcolli, *Algebro-geometric Feynman rules*, International Journal of Geometric Methods in Modern Physics, 8 (1) (2011), n203–237.

[3] P. Aluffi, M. Marcolli, *Feynman motives and deletion-contraction relations*, pp. 21–64, In: J.I. Cogolludo-Agustín, E. Hironaka, Editors (Eds), Topology of algebraic varieties and singularities, Contemporary Mathematics, 538, 2011.

[4] A. Bashir, *Non-perturbative aspects of Schwinger–Dyson equations*, AIP Conference Proceedings, 670 (1) (2003), 145–152.

[5] C. Borgs, J.T. Chayes, H. Cohn, N. Holden, *Sparse exchangeable graphs and their limits via graphon processes*, Journal of Machine Learning Research, 18 (210) (2018), 1–71.

[6] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, *An L^p theory of sparse graph convergence II: LD convergence, quotients and right convergence*, Annals of Probability, 46 (1) (2018), 337–396.

[7] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, *An L^p theory of sparse graph convergence I: Limits, sparse random graph models, and power law distributions*, Transactions of the American Mathematical Society, 372 (5) (2019), 3019–3062.

[8] D.J. Broadhurst, D. Kreimer, *Towards cohomology of renormalization: bigrading the combinatorial Hopf algebra of rooted trees*, Communications in Mathematical Physics, 215 (2000), 217–236.

[9] S. Bloch, H. Esnault, D. Kreimer, *On motives associated to graph polynomials*, Communications in Mathematical Physics, 267 (1) (2006), 181–225.

[10] B. Bollobas, O. Riordan, *Metrics for sparse graphs*, Surveys in Combinatorics 2009, LMS Lecture Notes Series, 365 (2009), 211–287.

[11] C. Bogner, S. Weinzierl, *Periods and Feynman integrals*, Journal of Mathematical Physics, 50 (4) (2009), Article ID: 042302, 16 pages.

[12] A. Connes, D. Kreimer, *From local perturbation theory to Hopf and Lie algebras of Feynman graphs*, Letters in Mathematical Physics, 56 (1) (2001), 3–15.

[13] J.A. Ellis-Monaghan, C. Merino, *Graph polynomials and their applications I: the Tutte polynomial*, pp. 219–255, In: M. Dehmer (Eds), Structural Analysis of Complex Networks, Birkhäuser, Boston, 2011.

[14] J.A. Ellis-Monaghan, C. Merino, *Graph polynomials and their applications II: interrelations and interpretations*, pp. 257–292. In: M. Dehmer (Eds), Structural Analysis of Complex Networks, Birkhäuser, Boston, 2011.

[15] S. Janson, Graphons, cut norm and distance, couplings and rearrangements, NYJM Monographs, 4 (2013), 76 pages.

[16] D. Kreimer, *Structures in Feynman graphs: Hopf algebras and symmetries*, In: Mikhail Lyubich, Leon Takhtajan (Eds), Graphs and patterns in mathematics and theoretical physics, Proceedings of Symposia in Pure Mathematics, 73 (2005), 43–78.

[17] D. Kreimer, *Anatomy of a gauge theory*, Annals of Physics, 321 (12) (2006), 2757–2781.

[18] D. Kreimer, *Dyson–Schwinger equations: from Hopf algebras to number theory*, In: Ilia Binder, Dirk Kreimer (Eds), Universality and renormalization: Universality and Renormalization: From Stochastic Evolution to Renormalization of Quantum Fields, Fields Institute Communications, 50 (2007), 225–248.

[19] J. Kuti, *Nonperturbative quantum chromodynamics*, Nuclear Physics A, 416 (1984), 25–54.

[20] O. Kruger, D. Kreimer, *Filtrations in Dyson–Schwinger equations: next-to$^{\{j\}}$-leading log expansions systematically*, Annals of Physics, 360 (2015), 293–340.

[21] T. Krajewski, V. Rivasseau, A. Tanasa, Zhituo Wang, *Topological graph polynomials and Quantum Field Theory, Part I: Heat Kernel Theories*, Journal of Noncommutative Geometry, 4 (1) (2010), 29–82.

[22] T. Krajewski, V. Rivasseau, F. Vignes-Tourneret, *Topological graph polynomials and Quantum Field Theory, Part II: Mehler kernel theories*, Annales Henri Poincaré, 12 (3) (2011), 483–545.

[23] L. Lovasz, Large networks and graph limits, Series: Colloquium Publications, 60 (2012), American Mathematical Society, Providence, RI.

[24] M. Marcolli, *Motivic renormalization and singularities*, Quanta of maths, Clay Mathematics Proceedings, 11 (2010), 409–458.

[25] M. Marcolli, Feynman motives, World Scientific, 2010.

[26] M. Marino, *Lectures on non-perturbative effects in large N gauge theories, matrix models and strings*, Fortschritte der Physik, 62 (5-6) (2014), 455–540.

[27] M. Marino, Instantons and large N: an introduction to non-perturbative methods in Quantum Field Theory, Cambridge University Press, 2015.

[28] V. P. Nair, Quantum Field Theory: a modern perspective, Graduate Texts in Contemporary Physics, Springer, 2005.

[29] A. Shojaei-Fard, *Motivic Dyson–Schwinger equations*, International Journal of Modern Physics A, 28 (2) (2013), Article ID: 1350102, 19 pages.

[30] A. Shojaei-Fard, *Graphons and renormalization of large Feynman diagrams*, Opuscula Mathematica, 38 (3) (2018), 427–455.

[31] A. Shojaei-Fard, *Non-perturbative β-functions via Feynman graphons*, Modern Physics Letters A, 34 (14) (2019), Article ID: 1950109, 10 pages.

[32] A. Shojaei-Fard, *Formal aspects of non-perturbative Quantum Field Theory via an operator theoretic setting*, International Journal of Geometric Methods in Modern Physics, 16 (12) (2019), Article ID: 1950192, 23 pages.

[33] A. Shojaei-Fard, *The complexities of nonperturbative computations*, Russian Journal of Mathematical Physics, 28 (3) (2021), 358–376.

[34] A. Shojaei-Fard, *The dynamics of non-perturbative phases via Banach bundles*, Nuclear Physics B, 969 (2021), Article ID: 115478, 39 pages.

[35] A. Shojaei-Fard, *The analytic evolution of Dyson–Schwinger equations via homomorphism densities*, Mathematical Physics, Analysis and Geometry, 24 (2) (2021), Article ID: 18, 28 pages.

[36] A. Shojaei-Fard, *Non-perturbative graph languages, halting problem and complexity*, Forum Mathematicum, 34 (5) (2022), 1159–1185.

[37] A. Shojaei-Fard, *Halting problem in Feynman graphon processes derived from the renormalization Hopf algebra*, Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science, 2(64) (1) (2022), no. 1, 139–157.

[38] E. Shuryak, Nonperturbative topological phenomena in QCD and related theories, Series: Lecture Notes in Physics, Volume 977, Springer Cham, 2021.

[39] W.D. van Suijlekom, *Renormalization of gauge fields: a Hopf algebra approach*, Communications in Mathematical Physics, 276 (3) (2007), 773–798.

ALI SHOJAEI-FARD
PH.D. INDEPENDENT SCHOLAR,
1461863596 MARZDARAN BLVD.
TEHRAN
IRAN
E-mail address: shojaeifa@yahoo.com