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# Frames generated by double sequences in Hilbert spaces

Narendra Biswas*, Chetna Mehra, Mahesh C. Joshi

Abstract. In this paper, we introduce frames generated by double sequences ($d$-frame) in Hilbert spaces and describe some of their properties. Furthermore, we discuss frame operators, alternate dual frames and stability for $d$-frames.

## 1. Introduction

Throughout this paper, the notations $\mathcal{H}$ and $\mathbb{F}$ will intend for an infinite dimensional Hilbert space and scalar field of real and complex numbers, respectively. $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ have their usual meanings.

**Definition 1** ([1])**.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist positive constants $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq \lambda_2\|x\|^2, \quad x \in \mathcal{H}. \tag{1}$$

The positive constants $\lambda_1$ and $\lambda_2$ are called the lower and upper frame bounds respectively. If $\lambda_1 = \lambda_2$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be a tight frame and if $\lambda_1 = \lambda_2 = 1$, then $\{x_n\}_{n\in\mathbb{N}}$ is called Parseval frame.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying the upper frame condition, i.e.,

$$\sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq \lambda_2\|x\|^2$$

is called a Bessel sequence with Bessel bound $\lambda_2$.

Here, we recall that every Bessel sequence in a Hilbert space is not necessarily a frame. But by including some additional elements or by sparsing the elements of such sequences one can convert these sequences into frames. Considering this fact, recently Sharma et al. [8] tried to construct frames

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for the Hilbert spaces using such Bessel sequences which are not frames for the given spaces. In fact, they gave the following definition.

**Definition 2** ([8]). Let $\mathcal{H}$ be a Hilbert space and $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$, where $\{m_n\}$ be an increasing sequence of positive integers. Then, $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is called an approximative frame for $\mathcal{H}$ if there exist positive constants $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1\|x\|^2 \leq \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2 \leq \lambda_2\|x\|^2, \quad x\in\mathcal{H}. \tag{2}$$

The positive constants $\lambda_1$ and $\lambda_2$ are called the lower and upper approximative frame bounds, respectively. If $\lambda_1=\lambda_2$, then $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is a tight approximative frame and if $\lambda_1=\lambda_2=1$, then it is called a Parseval approximative frame. A sequence $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is said to be an approximative Bessel sequence if righthand side of inequality (2) is satisfied.

Again, Sharma et. al. published a corrigendum [9] on the same paper [8] with following statement:

"The paper [8] requires some clarifications, throughout the paper [8], $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is a sequence of special index in $\mathcal{H}$ such that $x_{n,i}=x_{n+1,i}$, $i=1,2,\dots,m_n; n\in\mathbb{N}$ and $\{\alpha_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is a sequence of special index in $\mathbb{F}$ such that $\alpha_{n,i}=\alpha_{n+1,i}$, $i=1,2,\dots,m_n; n\in\mathbb{N}$."

The idea of construction of approximative frame is quite interesting, but there is some ambiguity in the definition of approximative frame and examples discussed in [8] and corrigendum [9]. We have the following dissensions on the Definition 2 and examples given in [8]:

(*i*) Considering $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ as a sequence then it contains same terms repeated infinitely many times. Hence, $\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2$ becomes unbounded.

(*ii*) As considered by the authors in examples [8], $n$ a fixed number and $m_n$, a function of $n$ and the claim that the sequence satisfies the equation (2) is contradictory.

(*iii*) Since, $\{m_n\}$ is an increasing sequence of natural numbers and from the Definition 2, it is clear that $m_n$ is dependent on $n$ such that $m_n\geq n$, hence in equation (2), $\lim_{n\to\infty}$ implies $m_n\to\infty$. Which is also a contradictory statement in case of a sequence.

(*iv*) Considering $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ a sequence of finite sequences, $m_n$ an increasing sequence depending on $n$, then the statement $x_{n,i}=x_{n+1,i}, i=1,2,\dots,m_n; n\in\mathbb{N}$ given in corrigendum [9] is again contradictory because $n^{th}$ term, i.e., $\{x_{n,i}\}_{i=1,2,\dots,m_n}$ contains $m_n$ terms while $(n+1)^{th}$ term i.e. $\{x_{n+1,i}\}_{i=1,2,\dots,m_{n+1}}$ contains $m_{n+1}$ terms.

Now, we discuss above infusions with the help of Examples (3.6) and (3.2) of [8] and [9] in details.

**Example 3.6 ([8]):** "Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Define a sequence $\{x_n\}$ by $x_n = \frac{e_n}{\sqrt{n}}, n \in \mathbb{N}$. Then it is elementary to observe that $\{x_n\}$ is not a frame for $\mathcal{H}$. Define a sequence $\{x_{n,i}\}_{\substack{i=1,2,\dots,\frac{n(n+1)}{2}\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ by

$$\begin{aligned}
x_{n,1} &= e_1,\\
x_{n,2} &= x_{n,3} = \frac{e_2}{\sqrt{2}},\\
x_{n,4} &= x_{n,5} = x_{n,6} = \frac{e_3}{\sqrt{3}},\\
&\vdots\\
x_{n,\frac{n(n-1)}{2}+1} &= x_{n,\frac{n(n-1)}{2}+2} = \cdots = x_{n,\frac{n(n+1)}{2}} = \frac{e_n}{\sqrt{n}}, \quad n \in \mathbb{N}."
\end{aligned}$$

(i) Considering sequence $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$. As given $m_n = \frac{n(n+1)}{2}$, hence for $n = 1$, $i = 1$; for $n = 2$, $i = 1, 2, 3$; for $n = 3$, $i = 1, 2, 3, 4, 5, 6$ and so on. So the construction of the sequence is as:

$$\begin{aligned}
&\Big\{x_{1,1}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, x_{3,5}, x_{3,6}, \dots,\\
&\qquad x_{n,1}, x_{n,2}, x_{n,3}, \dots, x_{n,\frac{n(n-1)}{2}+1}, \dots, x_{n,\frac{n(n+1)}{2}}, \dots\Big\}\\
&= \Big\{e_1, e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots,\\
&\qquad e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots, \frac{e_n}{\sqrt{n}}, \dots, \frac{e_n}{\sqrt{n}}, \dots\Big\}.
\end{aligned}$$

Here, the terms $e_1, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \dots$ are repeated infinitely many times. So, we can not find upper bound condition of equation (2). Hence, the given sequence $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ is not even an approximative Bessel sequence.

(ii) If we consider $\{x_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}$ as a sequence of sequences, i.e.,

$$\begin{aligned}
\{x_{n,i}\}_{\substack{i=1,2,\dots,\frac{n(n+1)}{2}\\ n\in\mathbb{N}}} &= \Big\{\{e_1\}, \Big\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}\Big\}, \dots,\\
&\qquad \Big\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \dots, \frac{e_n}{\sqrt{n}}\Big\}, \dots\Big\},
\end{aligned}$$

then the $(n+1)^{th}$ term of the above sequence will have $(n+1)$ more terms than the total terms of $n^{th}$ term. Hence the statement given in [9] is erroneous. Further, it cannot be claimed that such sequence is an approximative frame while only the infine'th term of sequence, i.e., $\left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \dots, \frac{e_n}{\sqrt{n}}, \dots, \frac{e_n}{\sqrt{n}}, \frac{e_{n+1}}{\sqrt{n+1}}, \dots\right\}$ is a tight frame for infinite dimensional Hilbert space, which is already well known as a frame.

Hence the claim by authors [8] that $\{x_{n,i}\}_{\substack{i=1,2,\dots,\frac{n(n+1)}{2}\\ n\in\mathbb{N}}}$ is a tight approximative frame is not a valid statement.

**Example 3.2 ([9]):** "Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Define a sequence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ by

$$y_{n,i} = \begin{cases} e_i, & \text{if } \ i = 1, 2, \dots, n \\ e_r, & \text{if } \ i > n \ \text{ and } \ r \equiv i \bmod n. \end{cases}$$

Then the sequence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ gives rise to a sequence $\{x_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ such that

$$\begin{aligned} x_{n,1} &= x_{n,2} = e_1, \\ x_{n,3} &= x_{n,4} = e_2, \\ &\vdots \\ x_{n,2n-1} &= x_{n,2n} = e_n, \ n \in \mathbb{N}." \end{aligned}$$

Here, the claim by the authors [9] that $\{x_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$, hence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ is an approximative frame with bounds $A = 1$ and $B = 2$, has following dissensions.

Case 1. Considering a sequence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$. The construction of sequence is as:

$$\begin{aligned} \{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}} = \{&y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, \dots, y_{n,1}, y_{n,2}, \dots, \\ &y_{n,2n}, y_{n+1,1}, y_{n+1,2}, \dots, y_{n+1,2n+2}, \dots\} \\ = \{&e_1, e_1, e_1, e_2, e_1, e_2, \dots, e_1, e_2, \dots, e_n, e_1, e_2, \dots, e_{n+1}, \dots\}. \end{aligned}$$

So, as discussed in previous example, the sequence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ is neither a Bessel sequence nor an approximative Bessel sequence.

Case 2. If we consider $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$ as a sequence of sequences, then the $(n+1)^{th}$ term will have 2 more terms than the total terms of $n^{th}$ term. Which is again a contradiction to the statement given in corrigendum [9]. Similarly, as in previous example, only the infine'th term of the sequence $\{y_{n,i}\}_{\substack{i=1,2,\dots,2n\\ n\in\mathbb{N}}}$, i.e.,

$$\{e_1, e_1, e_2, e_2, e_3, e_3, \dots, e_n, e_n, e_{n+1}e_{n+1}, \dots\}$$

is a frame for infinite dimensional Hilbert space $\mathcal{H}$.

The same is the case for Example 3.2, Example 3.3, Example 3.5 and Example 3.6 in [8].

Considering the above discussions, it is concluded that the authors have not been able to construct the synthesis operator, analysis operator, frame operator for the entire terms of the sequence i.e., the results given in [8] are erroneous. From the above discussion, if we consider limit for both the suffices, then it becomes a case of double sequences.

Now, we define a new generalization of frame with the help of double sequences and named it as $d$-frame.

The theory of double sequence and double series is an extension of single or ordinary sequences and series. In 1900, Pringsheim [6] introduced the concept of real double sequences and their convergence. A double sequence is a function or mapping $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ and denoted by $\{x_{ij}\}_{i,j\in\mathbb{N}}$ or $\{x_{ij}\}$. The infinite sum of double sequence $\sum\limits_{i,j\in\mathbb{N}} x_{ij}$ is known as double series. We use following definition and concepts to define $d$-frame and to prove the results on the properties of $d$-frame and frame operators.

**Definition 3** ([4],[3])**.** A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be convergent to $l$ in the Pringsheim's sense if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{ij} - l| < \epsilon$ whenever $i, j \geq N_\epsilon$, where $l$ is called the Pringsheim limit of $\{x_{ij}\}_{i,j\in\mathbb{N}}$.

A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be Cauchy sequence if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{pq} - x_{ij}| < \epsilon$ for all $p \geq i \geq N_\epsilon, q \geq j \geq N_\epsilon$.

The sequence of partial sums of double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is defined by

$$S = \{S_{mn}\}_{m,n\in\mathbb{N}}, \quad \text{where} \quad S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}, \quad \text{for all } m, n \in \mathbb{N}.$$

If $\lim\limits_{m,n\to\infty} S_{mn} = l$, then the double series $\sum\limits_{i,j\in\mathbb{N}} x_{ij}$ is said to be convergent and vice versa. Also,

$$\lim_{m,n\to\infty} S_{mn} = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} x_{ij} = \sum_{i,j\in\mathbb{N}} x_{ij}.$$

If no such limit exists then the double series is divergent.

If every $x_{ij}$ is non-negative then $\sum\limits_{i,j\in\mathbb{N}} x_{ij}$ is convergent if and only if $\{S_{mn}\}_{m,n\in\mathbb{N}}$ is bounded above. For more details about double sequences and series, see [5, 7] and references therein.

## 2. $d$-FRAMES

**Definition 4.** The double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a $d$-frame for $\mathcal{H}$ if there exist positive constants $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1\|x\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, x_{ij}\rangle|^2 \leq \lambda_2\|x\|^2, \quad \text{for all } x\in\mathcal{H}. \tag{3}$$

The constants $\lambda_1$ and $\lambda_2$ are called lower and upper $d$-frame bounds respectively. If $\lambda_1 = \lambda_2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called tight $d$-frame, and if $\lambda_1 = \lambda_2 = 1$, then it is called Parseval $d$-frame.

**Remark 1.** A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in Hilbert space $\mathcal{H}$ is called $d$-Bessel sequence if it satisfies upper $d$-frame inequality i.e.

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, x_{ij}\rangle|^2 \leq \lambda_2\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

**Remark 2.** Let $\{y_i\}_{i\in\mathbb{N}}$ is a frame for Hilbert space $\mathcal{H}$ with lower and upper frame bounds $\lambda_1$ and $\lambda_2$, respectively. Then, we define a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ as

$$x_{ij} = \begin{cases} y_i, & i=j,\\ 0, & \text{otherwise,}\end{cases}$$

which is a $d$-frame for $\mathcal{H}$ with the same bounds $\lambda_1$ and $\lambda_2$.

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Following examples vindicate the Definition 4.

**Example 1.** Define a sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ by

$$x_{ij} = \begin{cases} e_i, & i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Parseval $d$-frame for $\mathcal{H}$.

**Example 2.** Define a sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ by

$$x_{ij} = \begin{cases} e_i, & i=j+1,\\ e_j, & j=i+1,\\ e_i, & i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-frame for $\mathcal{H}$ with $d$-frame bounds $\lambda_1 = 1, \lambda_2 = 3$.

We know that every Bessel sequence is not a frame always. One can construct a double sequence from a given Bessel sequence, which becomes a $d$-frame.

**Example 3.** Given a sequence $\{x_n\}$ such that $x_n = \frac{e_n}{\sqrt{n}}$, for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is a Bessel sequence but not a frame for $\mathcal{H}$ because it does not satisfy the lower condition of frame. Define a sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ by

$$\begin{aligned} x_{11} &= e_1, \\ x_{21} &= x_{22} = \frac{e_2}{\sqrt{2}}, \\ x_{31} &= x_{32} = x_{33} = \frac{e_3}{\sqrt{3}}, \\ &\vdots \\ x_{n1} &= x_{n2} = x_{n3} = \cdots = x_{nn} = \frac{e_n}{\sqrt{n}}, \\ &\vdots \\ x_{ij} &= 0, \quad \text{for all } i < j. \end{aligned}$$

Then, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Parseval $d$-frame.

**Example 4.** The sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n = e_n + e_{n+1}$, for all $n \in \mathbb{N}$ is a Bessel sequence for $\mathcal{H}$, but not a frame for $\mathcal{H}$. Define a sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ by

$$x_{ij} = \begin{cases} e_i + e_j, & i = j \text{ or } i = j+1 \text{ or } j = i+1, \\ 0, & \text{otherwise,} \end{cases}$$

which is a $d$-frame for $\mathcal{H}$ with lower and upper $d$-frame bounds $\lambda_1 = 4$ and $\lambda_2 = 8$ respectively.

For the rest part of this paper, we define the space as

$$\ell^2(\mathbb{N}\times\mathbb{N}) = \left\{ \{\alpha_{ij}\}_{i,j\in\mathbb{N}} : \ \alpha_{ij} \in \mathbb{F}, \ \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\alpha_{ij}|^2 < \infty \right\}.$$

Then $\ell^2(\mathbb{N}\times\mathbb{N})$ is a Hilbert space with the norm induced by the inner product which is given by

$$\left\langle \{\alpha_{ij}\}_{i,j\in\mathbb{N}}, \{\beta_{ij}\}_{i,j\in\mathbb{N}} \right\rangle = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \alpha_{ij}\overline{\beta_{ij}},$$

for all $\{\alpha_{ij}\}_{i,j\in\mathbb{N}}, \{\beta_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$.

Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a $d$-Bessel sequence. Define operator $\mathcal{T} : \ell^2(\mathbb{N}\times\mathbb{N}) \to \mathcal{H}$ as

$$\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) \quad = \quad \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \alpha_{ij}x_{ij}, \quad \text{for all } \{\alpha_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N}).$$

If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-frame then operator $\mathcal{T}$ is called pre d-frame (synthesis) operator and the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is called analysis operator for $d$-frame.

**Theorem 1.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-Bessel sequence with d-Bessel bound $\lambda_2$ if and only if the operator $\mathcal{T}$ is linear, well defined and bounded with $\|\mathcal{T}\| \le \sqrt{\lambda_2}$.*

*Proof.* From the definition of $\mathcal{T}$, it is obvious that $\mathcal{T}$ is linear. Let $\{\alpha_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$. For any $m,n,m',n' \in \mathbb{N}$ with $m > m', n > n'$, we have

$$\begin{aligned}
\left\| \sum_{i,j=1}^{m,n} \alpha_{ij}x_{ij} - \sum_{i,j=1}^{m',n'} \alpha_{ij}x_{ij} \right\| &= \sup_{\|y\|=1} \left( \left| \left\langle \sum_{i,j=1}^{m,n} \alpha_{ij}x_{ij} - \sum_{i,j=1}^{m',n'} \alpha_{ij}x_{ij}, y \right\rangle \right| \right) \\
&\le \sup_{\|y\|=1} \left( \sum_{i,j=1}^{m,n} |\alpha_{ij}\langle x_{ij}, y\rangle| + \sum_{i,j=1}^{m',n'} |\alpha_{ij}\langle x_{ij}, y\rangle| \right) \\
&\le \sup_{\|y\|=1} \left( \left( \sum_{i,j=1}^{m,n} |\alpha_{ij}|^2 \right)^{1/2} \left( \sum_{i,j=1}^{m,n} |\langle x_{ij}, y\rangle|^2 \right)^{1/2} \right. \\
&\quad \left. + \left( \sum_{i,j=1}^{m',n'} |\alpha_{ij}|^2 \right)^{1/2} \left( \sum_{i,j=1}^{m',n'} |\langle x_{ij}, y\rangle|^2 \right)^{1/2} \right) \\
&\le \sqrt{\lambda_2} \left( \left( \sum_{i,j=1}^{m,n} |\alpha_{ij}|^2 \right)^{1/2} + \left( \sum_{i,j=1}^{m',n'} |\alpha_{ij}|^2 \right)^{1/2} \right),
\end{aligned}$$

implies that $\lim\limits_{m,n\to\infty} \sum\limits_{i,j=1}^{m,n} \alpha_{ij}x_{ij}$ exists. Hence, $\mathcal{T}$ is well defined.
Further,

$$\begin{aligned}
\|\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\| &= \sup_{\|x\|=1} \left| \left\langle \mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}), x \right\rangle \right| \\
&= \sup_{\|x\|=1} \lim_{m,n\to\infty} \left| \left\langle \sum_{i,j=1}^{m,n} \alpha_{ij}x_{ij}, x \right\rangle \right| \\
&\le \sqrt{\lambda_2} \lim_{m,n\to\infty} \left( \sum_{i,j=1}^{m,n} |\alpha_{ij}|^2 \right)^{1/2}.
\end{aligned}$$

This implies that $\|\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\| \le \sqrt{\lambda_2}\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\|$, hence $\mathcal{T}$ is bounded operator with $\|\mathcal{T}\| \le \sqrt{\lambda_2}$.

Conversely, for any $x \in \mathcal{H}$, we have

$$\left\langle x, \mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) \right\rangle = \left\langle x, \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \alpha_{ij}x_{ij} \right\rangle$$

$$
\begin{aligned}
&= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \overline{\alpha}_{ij} \langle x, x_{ij}\rangle \\
&= \big\langle \{\langle x, x_{ij}\rangle\}_{i,j\in\mathbb{N}}, \{\alpha_{ij}\}_{i,j\in\mathbb{N}} \big\rangle.
\end{aligned}
$$

Hence,

$$
\mathcal{T}^*(x) = \{\langle x, x_{ij}\rangle\}_{i,j\in\mathbb{N}}, \quad \text{for all } x \in \mathcal{H}. \tag{4}
$$

Thus,

$$
\begin{aligned}
\|\mathcal{T}^*(x)\|^2 &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \\
&\leq \sqrt{\lambda_2}\|x\|^2.
\end{aligned}
$$

Hence, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-Bessel sequence with bound $\lambda_2$. $\square$

Now, define $d$-frame operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ for $d$-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ by

$$
\begin{aligned}
\mathcal{S}(x) &= \mathcal{T}\mathcal{T}^*(x) \\
&= \mathcal{T}(\{\langle x, x_{ij}\rangle\}_{i,j\in\mathbb{N}}) \\
&= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}, \quad \text{for all } x \in \mathcal{H}.
\end{aligned}
$$

Since $\mathcal{T}$ and $\mathcal{T}^*$ both are linear, so $\mathcal{S}$ is also linear.

**Theorem 2.** *$\mathcal{S}$ is bounded, self adjoint, positive and invertible operator.*

*Proof.* $\|\mathcal{S}\| = \|\mathcal{T}\mathcal{T}^*\| \leq \|\mathcal{T}\|^2 \leq \lambda_2$ and $\mathcal{S}^* = (\mathcal{T}\mathcal{T}^*)^* = \mathcal{T}\mathcal{T}^* = \mathcal{S}$. Hence, $\mathcal{S}$ is bounded and self adjoint operator.

For $x \in \mathcal{H}$,

$$
\begin{aligned}
\langle \mathcal{S}(x), x\rangle &= \left\langle \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}, x \right\rangle \\
&= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2.
\end{aligned}
$$

Using definition of $d$-frame, we have

$$
\lambda_1\langle \mathcal{I}(x), x\rangle \leq \langle \mathcal{S}(x), x\rangle \leq \lambda_2\langle \mathcal{I}(x), x\rangle, \quad \text{for all } x \in \mathcal{H}.
$$

Hence,

$$
\lambda_1.\mathcal{I} \leq \mathcal{S} \leq \lambda_2.\mathcal{I}. \tag{5}
$$

Thus, $\mathcal{S}$ is a positive operator. Further,

$$
\mathcal{I} - \lambda_2^{-1}\mathcal{S} \leq \frac{\lambda_2 - \lambda_1}{\lambda_2}\mathcal{I},
$$

implies $\|\mathcal{I} - \lambda_2^{-1}\mathcal{S}\| < 1$. Hence, $\mathcal{S}$ is invertible operator. $\square$

Generalizing Theorem 1, if we take an extra condition i.e., operator $\mathcal{T}$ is surjective then following theorem gives a necessary and sufficient condition for a double sequence to be a frame

**Theorem 3.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-frame for $\mathcal{H}$ if and only if the operator $\mathcal{T}$ is well defined, bounded, linear and surjective.*

*Proof.* It is clear from Theorem 1 that, the operator $\mathcal{T}$ is well defined, bounded and linear. Since $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-frame hence the $d$-frame operator $\mathcal{S} = \mathcal{T}\mathcal{T}^*$ is invertible(bijective) which implies $\mathcal{T}$ is also surjective.

Conversely, let $\mathcal{T}$ is well defined, bounded, linear and surjective.
From Theorem 1 it is already clear that $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Bessel sequence. Now we prove the lower $d$-frame inequality.

Since $\mathcal{T}$ is surjective and $\mathcal{T}^*$ is one-one operator, then the operator $S = \mathcal{T}\mathcal{T}^*$ is invertible and positive.
For any $a, b \in \mathcal{H}$, $|\langle a, b\rangle| \le \|a\|\,\|b\|$ (Cauchy Schwarz inequality).

Consider $a = \mathcal{S}^{-\frac{1}{2}}(x)$ and $b = \mathcal{S}^{\frac{1}{2}}(x)$, then

$$|\langle \mathcal{S}^{-\frac{1}{2}}(x), \mathcal{S}^{\frac{1}{2}}(x)\rangle| \le \|\mathcal{S}^{-\frac{1}{2}}(x)\|\;\;\|\mathcal{S}^{\frac{1}{2}}(x)\|$$

implies $|\langle \mathcal{S}^{\frac{1}{2}}\mathcal{S}^{-\frac{1}{2}}(x), x\rangle| \le \langle \mathcal{S}^{-\frac{1}{2}}(x), \mathcal{S}^{-\frac{1}{2}}(x)\rangle^{\frac{1}{2}}\;\langle \mathcal{S}^{\frac{1}{2}}(x), \mathcal{S}^{\frac{1}{2}}(x)\rangle^{\frac{1}{2}}$, from where we get

$$\|x\|^2 \le \langle \mathcal{S}^{-1}(x), x\rangle^{\frac{1}{2}}\;\langle \mathcal{S}(x), x\rangle^{\frac{1}{2}}.$$

Squaring both side and using Cauchy Schwarz inequality for $\langle \mathcal{S}^{-1}x, x\rangle$ we have

$$\|x\|^4 \le \|\mathcal{S}^{-1}(x)\|\,\|x\|\;\langle \mathcal{S}(x), x\rangle.$$

Hence, since $\mathcal{S}$ is bounded,

$$\|x\|^4 \le \|\mathcal{S}^{-1}\|\;\|x\|^2\,\langle \mathcal{S}(x), x\rangle.$$

Finally,

$$\begin{aligned}\frac{1}{\|\mathcal{S}^{-1}\|}\|x\|^2 \le \langle \mathcal{S}(x), x\rangle = \langle \mathcal{T}\mathcal{T}^*(x), x\rangle \;&=\; \left\langle \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x, x_{ij}\rangle x_{ij}, x\right\rangle \\ &=\; \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, x_{ij}\rangle|^2.\end{aligned}$$

Hence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-frame for $\mathcal{H}$. $\square$

Now, we establish following result to characterize $d$-frames in terms of bounded linear operators.

**Theorem 4.** *The image of a d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ under a linear bounded operator $\mathbb{T}$ on $\mathcal{H}$ is again a d-frame for $\mathcal{H}$ if and only if there exist a positive constant $\lambda$ such that the adjoint operator $\mathbb{T}^*$ satisfies*

$$\|\mathbb{T}^*(x)\|^2 \geq \lambda\|x\|^2, \quad \textit{for all } x \in \mathcal{H}.$$

*Proof.* Since $\mathbb{T}$ is a linear bounded operator hence $\mathbb{T}^*$ is also linear bounded. Taking $\mathbb{T}^*(x) \in \mathcal{H}$ and using the definition of $d$-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$,

$$\lambda_1\|\mathbb{T}^*(x)\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle\mathbb{T}^*(x), x_{ij}\rangle|^2 \leq \lambda_2\|\mathbb{T}^*(x)\|^2.$$

By the given condition, we get

$$\begin{aligned}\lambda\lambda_1\|x\|^2 &\leq \lambda_1\|\mathbb{T}^*(x)\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, \mathbb{T}(x_{ij})\rangle|^2\\ &\leq \lambda_2\|\mathbb{T}^*(x)\|^2 \leq \lambda_2\|\mathbb{T}^*\|^2\|x\|^2.\end{aligned}$$

Thus $\{\mathbb{T}(x_{ij})\}_{i,j\in\mathbb{N}}$ is a $d$-frame for $\mathcal{H}$. Converse is obvious by the definition of $d$-frame. □

**Remark 3.** From Theorem 4, it is clear that image of a $d$-frame under a linear bounded operator is always a $d$-Bessel sequence.

In the following theorem, we prove that one can also construct a $d$-frame with the help of $d$-frame operator.

**Theorem 5.** *For a d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ with bounds $\lambda_1$ and $\lambda_2$ respectively and d-frame operator $\mathcal{S}$, the double sequence $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ is again a d-frame.*

*Proof.* From equation (5) of Theorem 2, we have

$$\lambda_1.\mathcal{I} \leq \mathcal{S} \leq \lambda_2.\mathcal{I},$$

which implies

$$\lambda_2^{-1}.\mathcal{I} \leq \mathcal{S}^{-1} \leq \lambda_1^{-1}\mathcal{I}.$$

Taking inner product with $x$, we get

$$\lambda_2^{-1}\|x\|^2 \leq \left\langle\mathcal{S}^{-1}x, x\right\rangle \leq \lambda_1^{-1}\|x\|^2. \tag{6}$$

Now,

$$\begin{aligned}\left\langle\mathcal{S}^{-1}(x), x\right\rangle &= \left\langle\mathcal{S}^{-1}\mathcal{S}\mathcal{S}^{-1}(x), x\right\rangle\\ &= \left\langle\mathcal{S}^{-1}\Big(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle\mathcal{S}^{-1}x, x_{ij}\right\rangle x_{ij}\Big), x\right\rangle\\ &= \left\langle\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle\mathcal{S}^{-1}(x_{ij}), x\right\rangle\end{aligned}$$

$$= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle\right|^2.$$

Hence, by equation (6), $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ is a $d$-frame for $\mathcal{H}$ with lower and upper bound $\lambda_2^{-1}$ and $\lambda_1^{-1}$ respectively i.e.,

$$\lambda_2^{-1}\|x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle\right|^2 \leq \lambda_1^{-1}\|x\|^2. \tag{7}$$ □

**Remark 4.** In above theorem, equations (6) and (7) show that $\mathcal{S}^{-1}$ is a $d$-frame operator for the $d$-frame $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$. And for any $x \in \mathcal{H}$,

$$x = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle x_{ij} = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle \mathcal{S}^{-1}(x_{ij}).$$

**Corollary 1.** *For a d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and operator $\mathcal{S}$, $\{\mathcal{S}^{-1/2}(x_{ij})\}_{i,j\in\mathbb{N}}$ is Parseval d-frame, where $\mathcal{S}^{-1/2}$ is square root of $\mathcal{S}^{-1}$.*

## 3. Alternate dual $d$-frames

In this section, we study alternate/canonical dual $d$-frame and its properties.

**Definition 5.** Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a $d$-frame for Hilbert space $\mathcal{H}$. A $d$-frame $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ is called alternate dual $d$-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$, if for all $x \in \mathcal{H}$

$$x = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, \tilde{x}_{ij}\rangle x_{ij},$$

or

$$x = \sum_{i,j\in\mathbb{N}} \langle x, \tilde{x}_{ij}\rangle x_{ij}.$$

**Remark 5.** $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ is a special type of dual $d$-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$, called canonical dual $d$-frame.

**Theorem 6.** *Let $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ be an alternate dual d-frame for a d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$. Then, for every $N \times M \subseteq \mathbb{N}\times\mathbb{N}$ and $x \in \mathcal{H}$,*

$$\sum_{(i,j)\in N\times M} \langle x, \tilde{x}_{ij}\rangle \overline{\langle x, x_{ij}\rangle} - \left\| \sum_{(i,j)\in N\times M} \langle x, \tilde{x}_{ij}\rangle x_{ij} \right\|^2 \tag{8}$$

$$= \overline{\sum_{(i,j)\in N^c\times M^c} \langle x, \tilde{x}_{ij}\rangle \overline{\langle x, x_{ij}\rangle}} - \left\| \sum_{(i,j)\in N^c\times M^c} \langle x, \tilde{x}_{ij}\rangle x_{ij} \right\|^2.$$

*Proof.* For $x \in \mathcal{H}$ and $N \times M \subseteq \mathbb{N} \times \mathbb{N}$, define the operator $\mathcal{T}_{N\times M}$ as

$$\mathcal{T}_{N\times M}(x) = \sum_{(i,j)\in N\times M} \langle x, \tilde{x}_{ij}\rangle x_{ij}.$$

It is obvious that the operator $\mathcal{T}_{N\times M}(x)$ is well defined, linear and bounded. And by the definition of dual $d$-frame, we have

$$\mathcal{T}_{N\times M} + \mathcal{T}_{N^c\times M^c} = I.$$

Therefore,

$$\begin{aligned} \mathcal{T}_{N\times M} - \mathcal{T}^*_{N\times M}\mathcal{T}_{N\times M} = (I - \mathcal{T}^*_{N\times M})\mathcal{T}_{N\times M} &= \mathcal{T}^*_{N^c\times M^c}(I - \mathcal{T}_{N^c\times M^c}) \\ &= \mathcal{T}^*_{N^c\times M^c} - \mathcal{T}^*_{N^c\times M^c}\mathcal{T}_{N^c\times M^c}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{(i,j)\in N\times M} \langle x, \tilde{x}_{ij}\rangle \overline{\langle x, x_{ij}\rangle} - \left\| \sum_{(i,j)\in N\times M} \langle x, \tilde{x}_{ij}\rangle x_{ij} \right\|^2 \\ &\quad = \langle \mathcal{T}_{N\times M}(x), x\rangle - \langle \mathcal{T}^*_{N\times M}\mathcal{T}_{N\times M}(x), x\rangle \\ &\quad = \langle \mathcal{T}^*_{N^c\times M^c}(x), x\rangle - \langle \mathcal{T}^*_{N^c\times M^c}\mathcal{T}_{N^c\times M^c}(x), x\rangle \\ &\quad = \langle x, \mathcal{T}_{N^c\times M^c}(x)\rangle - \langle \mathcal{T}_{N^c\times M^c}(x), \mathcal{T}_{N^c\times M^c}(x)\rangle \\ &\quad = \overline{\sum_{(i,j)\in N^c\times M^c} \langle x, \tilde{x}_{ij}\rangle \overline{\langle x, x_{ij}\rangle}} - \left\| \sum_{(i,j)\in N^c\times M^c} \langle x, \tilde{x}_{ij}\rangle x_{ij} \right\|^2. \qquad \square \end{aligned}$$

**Remark 6.** Every Parseval $d$-frame is dual $d$-frame of itself. Hence identity (8) becomes

$$\begin{aligned} &\sum_{(i,j)\in N\times M} |\langle x, x_{ij}\rangle|^2 - \left\| \sum_{(i,j)\in N\times M} \langle x, x_{ij}\rangle x_{ij} \right\|^2 \\ &\qquad\qquad = \sum_{(i,j)\in N^c\times M^c} |\langle x, x_{ij}\rangle|^2 - \left\| \sum_{(i,j)\in N^c\times M^c} \langle x, x_{ij}\rangle x_{ij} \right\|^2. \end{aligned}$$

Which is called Parseval $d$-frame identity.

## 4. Stability of $d$-frames

In this section, we study the stability of $d$-frames, which is similar version of Paley-Wiener Theorem for frames [2]. Also, we prove similar results for the stability of canonical dual $d$-frame.

**Theorem 7.** *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a $d$-frame with lower and upper $d$-frame bounds $\lambda_1, \lambda_2$ respectively, and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in $\mathcal{H}$ such* *that* $\exists\ \lambda, \mu \geq 0$ *with* $\left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right) < 1$ *and*

$$\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}(x_{ij}-y_{ij})\right\| \leq \lambda \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij}\right\| + \mu\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|, \tag{9}$$

*for all* $\{\alpha_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$.

*Then,* $\{y_{ij}\}_{i,j\in\mathbb{N}}$ *is also a d-frame for* $\mathcal{H}$ *with lower and upper d-frame bounds* $\lambda_1\left(1-\left(\lambda+\frac{\mu}{\sqrt{\lambda_1}}\right)\right)^2$ *and* $\lambda_2\left(1+\lambda+\frac{\mu}{\sqrt{\lambda_2}}\right)^2$ *respectively.*

*Proof.* Given $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $d$-frame for $\mathcal{H}$. Let $\mathcal{T}$ be the pre $d$-frame operator. From Theorem 1, we have

$$\|\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\| = \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij}\right\| \leq \sqrt{\lambda}_2\,\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|, \tag{10}$$

for $\{\alpha_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$.

For the given double sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$, we have

$$\begin{aligned}\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij}\right\| &\leq \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}(x_{ij}-y_{ij})\right\| + \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij}\right\| \\ &\leq (1+\lambda)\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij}\right\| + \mu\,\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|.\end{aligned}$$

By equation (10), we get

$$\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij}\right\| \leq ((1+\lambda)\sqrt{\lambda}_2+\mu)\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|. \tag{11}$$

Now, define an another operator $\mathcal{U} : \ell^2(\mathbb{N}\times\mathbb{N}) \to \mathcal{H}$ as

$$\mathcal{U}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij}.$$

So, for $m > m'$ and $n > n'$, where $m, n, m', n' \in \mathbb{N}$, we have

$$\left\|\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij} - \sum_{i,j=1}^{m',n'}\alpha_{ij}y_{ij}\right\| \leq \left\|\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij}\right\| + \left\|\sum_{i,j=1}^{m',n'}\alpha_{ij}y_{ij}\right\|.$$

Using (10), we get

$$\left\| \sum_{i,j=1}^{m,n} \alpha_{ij} y_{ij} - \sum_{i,j=1}^{m',n'} \alpha_{ij} y_{ij} \right\| \leq 2((1+\lambda)\sqrt{\lambda_2} + \mu)\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|.$$

Since $\{\alpha_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$, hence sequence of partial sums of $\lim\limits_{m,n\to\infty} \sum\limits_{i,j=1}^{m,n} \alpha_{ij} y_{ij}$ is Cauchy, i.e. $\lim\limits_{m,n\to\infty} \sum\limits_{i,j=1}^{m,n} \alpha_{ij} y_{ij}$ exists. Which implies

$$\|\mathcal{U}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\| \leq \Big((1+\lambda)\sqrt{\lambda_2} + \mu\Big) \|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\|. \tag{12}$$

Therefore, operator $\mathcal{U}$ is linear, well defined and bounded. Thus, by Theorem 1, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a Bessel sequence for $\mathcal{H}$ with bound $\lambda_2\Big(1+\lambda+\frac{\mu}{\sqrt{\lambda_2}}\Big)^2$ i.e.,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big|\langle y_{ij}, x\rangle\big|^2 \leq \lambda_2\Big(1+\lambda+\frac{\mu}{\sqrt{\lambda_2}}\Big)^2. \tag{13}$$

Now, using $\mathcal{T}$ and $\mathcal{U}$ in equation (9)

$$\begin{aligned} &\big\|\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) - \mathcal{U}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\big\| \leq \\ &\lambda\big\|\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}})\big\| + \mu\big\|\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\big\|. \end{aligned} \tag{14}$$

By Theorem 5, we know that $S = \mathcal{T}\mathcal{T}^*$ is a $d$-frame operator for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ with upper bound $\lambda_1^{-1}$.

Again, consider $\mathcal{T}^\dagger : \mathcal{H} \to \ell^2(\mathbb{N}\times\mathbb{N})$ as

$$\begin{aligned} \mathcal{T}^\dagger(x) &= \mathcal{T}^*(\mathcal{T}\mathcal{T}^*)^{-1}(x) \\ &= \big\{\big\langle x, (\mathcal{T}\mathcal{T}^*)^{-1}(x_{ij})\big\rangle\big\}_{i,j\in\mathbb{N}}, \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

So,

$$\begin{aligned} \big\|\mathcal{T}^\dagger(x)\big\|^2 &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big|\big\langle x, (\mathcal{T}\mathcal{T}^*)^{-1}(x_{ij})\big\rangle\big|^2 \\ &\leq {\lambda_1}^{-1}\big\|x\big\|^2, \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

Put $\{\alpha_{ij}\}_{i,j\in\mathbb{N}} = \mathcal{T}^\dagger(x)$ in equation (14), we get

$$\|x - \mathcal{U}\mathcal{T}^\dagger(x)\| \leq \Big(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\Big)\|x\|, \quad \text{for all } x \in \mathcal{H}.$$

Given that, $\left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right) < 1$. Therefore, $\mathcal{U}\mathcal{T}^\dagger$ is invertible and

$$\|\mathcal{U}\mathcal{T}^\dagger\| \leq 1 + \lambda + \frac{\mu}{\sqrt{\lambda_1}}, \qquad \|(\mathcal{U}\mathcal{T}^\dagger)^{-1}\| \leq \frac{1}{1 - \left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right)}.$$

For $x \in \mathcal{H}$,

$$\begin{aligned} x &= (\mathcal{U}\mathcal{T}^\dagger)(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x) \\ &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big\langle (\mathcal{U}\mathcal{T}^\dagger)^{-1}(x), (\mathcal{T}\mathcal{T}^*)^{-1}(x_{ij}) \big\rangle y_{ij}. \end{aligned}$$

This implies

$$\|x\|^2 = \langle x, x\rangle = \left| \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big\langle (\mathcal{U}\mathcal{T}^\dagger)^{-1}(x), (\mathcal{T}\mathcal{T}^*)^{-1}(x_{ij}) \big\rangle \langle y_{ij}, x\rangle \right|.$$

Squaring both sides, we get

$$\begin{aligned} \|x\|^4 &= \left| \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big\langle (\mathcal{U}\mathcal{T}^\dagger)^{-1}(x), (\mathcal{T}\mathcal{T}^*)^{-1}(x_{ij}) \big\rangle \big\langle y_{ij}, x\big\rangle \right|^2 \\ &\leq \lim_{m,n\to\infty} \frac{1}{\lambda_1} \big\|(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x)\big\|^2 \sum_{i,j=1}^{m,n} \big|\big\langle y_{ij}, x\big\rangle\big|^2 \\ &\leq \frac{1}{\lambda_1} \frac{1}{\left(1 - \left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right)\right)^2} \|x\|^2 \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big|\big\langle y_{ij}, x\big\rangle\big|^2, \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

Therefore,

$$\lambda_1 \left(1 - \left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right)\right)^2 \|x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \big|\big\langle y_{ij}, x\big\rangle\big|^2. \tag{15}$$

From (13) and (15), we have $\{y_{ij}\}_{i,j\in\mathbb{N}}$ a $d$-frame for $\mathcal{H}$. $\square$

Following lemma is used to study stability theorem for canonical dual $d$-frame.

**Lemma 1.** *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-Bessel sequence in $\mathcal{H}$ with d-Bessel bound $\beta$. Then for any $\{c_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$,*

$$\Big\| \sum_{i,j\in\mathbb{N}} c_{ij} x_{ij} \Big\|^2 \leq \beta \sum_{i,j\in\mathbb{N}} |c_{ij}|^2.$$

*Proof.* For any $\{c_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N})$, we have

$$\begin{aligned}
\Big\| \sum_{i,j\in\mathbb{N}} c_{ij}x_{ij} \Big\|^2 &= \sup_{\|x\|=1} \Big| \Big\langle \sum_{i,j\in\mathbb{N}} c_{ij}x_{ij}, x \Big\rangle \Big|^2 \\
&= \sup_{\|x\|=1} \Big| \sum_{i,j\in\mathbb{N}} c_{ij}\langle x_{ij}, x\rangle \Big|^2 \\
&\le \sup_{\|x\|=1} \sum_{i,j\in\mathbb{N}} |c_{ij}|^2 \sum_{i,j\in\mathbb{N}} |\langle x_{ij}, x\rangle|^2 \\
&\le \beta \sum_{i,j\in\mathbb{N}} |c_{ij}|^2.
\end{aligned}$$

□

**Theorem 8.** *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be d-frames for $\mathcal{H}$ with $(\lambda_1,\mu_1)$ and $(\lambda_2,\mu_2)$ as respective lower and upper bounds. And let $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ and $\{\tilde{y}_{ij}\}_{i,j\in\mathbb{N}}$ are their respective canonical dual d-frames.*

*(i) If $\{x_{ij} - y_{ij} : i,j \in \mathbb{N}\}$ is a d-Bessel sequence with bound $\beta$, then $\{\tilde{x}_{ij} - \tilde{y}_{ij} : i,j\in\mathbb{N}\}$ is also d-Bessel sequence with bound*

$$\beta\Big(\frac{\lambda_1+\lambda_2+\lambda_2^{\frac12}\mu_2^{\frac12}}{\lambda_1\mu_1}\Big)^2.$$

*(ii) If*

$$\Big| \sum_{i,j\in\mathbb{N}} |\langle x_{ij}, x\rangle|^2 - \sum_{i,j\in\mathbb{N}} |\langle y_{ij}, x\rangle|^2 \Big| \le \gamma\|x\|^2, \quad \text{for all } x\in\mathcal{H},$$

*then*

$$\Big| \sum_{i,j\in\mathbb{N}} |\langle \tilde{x}_{ij}, x\rangle|^2 - \sum_{i,j\in\mathbb{N}} |\langle \tilde{y}_{ij}, x\rangle|^2 \Big| \le \frac{\gamma}{\lambda_1\mu_1}\|x\|^2, \quad \text{for all } x\in\mathcal{H},$$

*where $\gamma$ is a real positive number.*

*Proof.* Let $\mathcal{S}$ and $\mathcal{T}$ are $d$-frame operator for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ respectively. Then, for $x\in\mathcal{H}$

$$\mathcal{S}(x) = \sum_{i,j\in\mathbb{N}} \langle x, x_{ij}\rangle x_{ij} \quad \text{and} \quad \mathcal{T}(x) = \sum_{i,j\in\mathbb{N}} \langle x, y_{ij}\rangle y_{ij}.$$

Since $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ and $\{\tilde{y}_{ij}\}_{i,j\in\mathbb{N}}$ are canonical dual $d$-frames, so $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}} = \{S^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ and $\{\tilde{y}_{ij}\}_{i,j\in\mathbb{N}} = \{T^{-1}(y_{ij})\}_{i,j\in\mathbb{N}}$.

(i) Given that $\{x_{ij} - y_{ij} : i,j\in\mathbb{N}\}$ is a $d$-Bessel sequence with bound $\beta$. So,

$$\|\mathcal{S}(x) - \mathcal{T}(x)\| = \Big\| \sum_{i,j\in\mathbb{N}} \Big( \langle x, x_{ij}\rangle x_{ij} - \langle x, y_{ij}\rangle y_{ij} \Big) \Big\|$$

$$\leq \quad \Big\| \sum_{i,j\in\mathbb{N}} \langle x, x_{ij}\rangle (x_{ij} - y_{ij}) \Big\| + \Big\| \sum_{i,j\in\mathbb{N}} \langle x, x_{ij} - y_{ij}\rangle y_{ij} \Big\|.$$

Using Lemma 1 we get

$$\begin{aligned}\|\mathcal{S}(x) - \mathcal{T}(x)\| &\leq \beta^{1/2}\Big(\sum_{i,j\in\mathbb{N}} |\langle x, x_{ij}\rangle|^2\Big)^{1/2} + \mu_2^{1/2}\Big(\sum_{i,j\in\mathbb{N}} |\langle x, x_{ij} - y_{ij}\rangle|^2\Big)^{1/2}\\ &\leq \beta^{1/2}(\lambda_2^{1/2} + \mu_2^{1/2})\|x\|.\end{aligned}$$

Hence,

$$\|\mathcal{S} - \mathcal{T}\| \leq \beta^{1/2}(\lambda_2^{1/2} + \mu_2^{1/2}),$$

and

$$\begin{aligned}\|\mathcal{S}^{-1} - \mathcal{T}^{-1}\| &= \|\mathcal{T}^{-1}(\mathcal{T} - \mathcal{S})\mathcal{S}^{-1}\|\\ &\leq \|\mathcal{T}^{-1}\|\|\mathcal{T} - \mathcal{S}\|\|\mathcal{S}^{-1}\|\\ &\leq \frac{1}{\lambda_1\mu_1}\beta^{1/2}(\lambda_2^{1/2} + \mu_2^{1/2}).\end{aligned}$$

Now, we prove that $\{\tilde{x}_{ij} - \tilde{y}_{ij} : i, j \in \mathbb{N}\}$ is a $d$-Bessel sequence.

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} |\langle \tilde{x}_{ij} - \tilde{y}_{ij}, x\rangle^2 &= \sum_{i,j\in\mathbb{N}} |\langle \mathcal{S}^{-1}(x_{ij}) - \mathcal{T}^{-1}(y_{ij}), x\rangle|^2\\ &= \sum_{i,j\in\mathbb{N}} |\langle x_{ij}, \mathcal{S}^{-1}(x)\rangle - \langle x_{ij}, \mathcal{T}^{-1}(x)\rangle\\ &\qquad + \langle x_{ij}, \mathcal{T}^{-1}(x)\rangle - \langle y_{ij}, \mathcal{T}^{-1}(x)\rangle|^2 \qquad (16)\\ &= \sum_{i,j\in\mathbb{N}} |\langle x_{ij}, (\mathcal{S}^{-1} - \mathcal{T}^{-1})(x)\rangle + \langle x_{ij} - y_{ij}, \mathcal{T}^{-1}(x)\rangle|^2.\end{aligned}$$

For right hand side calculation, we have

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} |\langle x_{ij}, (\mathcal{S}^{-1} - \mathcal{T}^{-1})(x)\rangle|^2 &\leq \lambda_2\|(\mathcal{S}^{-1} - \mathcal{T}^{-1})(x)\|^2\\ &\leq \frac{\lambda_2}{\lambda_1^2\mu_1^2}\beta(\lambda_2^{1/2} + \mu_2^{1/2})^2\|x\|^2,\end{aligned} \qquad (17)$$

and

$$\sum_{i,j\in\mathbb{N}} |\langle x_{ij} - y_{ij}, \mathcal{T}^{-1}(x)\rangle|^2 \leq \beta\|\mathcal{T}^{-1}(x)\|^2 \leq \frac{\beta}{\mu_1^2}\|x\|^2. \qquad (18)$$

Using equation (17) and (18) in equation (16), we get

$$\sum_{i,j\in\mathbb{N}} |\langle \tilde{x}_{ij} - \tilde{y}_{ij}, x\rangle|^2 \leq \beta\Big(\frac{\lambda_1 + \lambda_2 + \lambda_2^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\lambda_1\mu_1}\Big)^2\|x\|^2.$$

(*ii*) Since both $\mathcal{S}$ and $\mathcal{T}$ are self adjoint, we have

$$\begin{aligned}\|\mathcal{S}-\mathcal{T}\| &= \sup_{\|x\|=1} |\langle(\mathcal{S}-\mathcal{T})(x), x\rangle| \\ &= \sup_{\|x\|=1} |\langle\mathcal{S}(x), x\rangle - \langle\mathcal{T}(x), x\rangle| \\ &= \sup_{\|x\|=1} \left|\sum_{i,j\in\mathbb{N}} |\langle x_{ij}, x\rangle|^2 - \sum_{i,j\in\mathbb{N}} |\langle y_{ij}, x\rangle|^2\right| \\ &\leq \gamma\end{aligned}$$

and

$$\|\mathcal{S}^{-1}-\mathcal{T}^{-1}\| \leq \frac{1}{\lambda_1\mu_1}\gamma.$$

We have $\tilde{x}_{ij} = \mathcal{S}^{-1}(x_{ij})$, so

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} |\langle\tilde{x}_{ij}, x\rangle|^2 &= \sum_{i,j\in\mathbb{N}} |\langle\mathcal{S}^{-1}(x_{ij}), x\rangle|^2 = \sum_{i,j\in\mathbb{N}} |\langle x_{ij}, \mathcal{S}^{-1}x\rangle|^2 \\ &= \langle\mathcal{S}\mathcal{S}^{-1}(x), \mathcal{S}^{-1}(x)\rangle = \langle x, \mathcal{S}^{-1}(x)\rangle.\end{aligned}$$

Similarly,

$$\sum_{i,j\in\mathbb{N}} |\langle\tilde{y}_{ij}, x\rangle|^2 = \langle x, \mathcal{T}^{-1}(x)\rangle.$$

Hence,

$$\begin{aligned}\left|\sum_{i,j\in\mathbb{N}} |\langle\tilde{x}_{ij}, x\rangle|^2 - \sum_{i,j\in\mathbb{N}} |\langle\tilde{y}_{ij}, x\rangle|^2\right| &= |\langle x, \mathcal{S}^{-1}(x)\rangle - \langle x, \mathcal{T}^{-1}(x)\rangle| \\ &\leq \|\mathcal{S}^{-1}-\mathcal{T}^{-1}\|\|x\|^2 \\ &\leq \frac{\gamma}{\lambda_1\mu_1}\|x\|^2.\end{aligned}$$

□

## 5. Conclusion

The paper introduces frames generated by double sequences in Hilbert spaces and named as $d$-frames. Some of the properties of $d$-frames, frame operators, alternate dual $d$-frames and stability for $d$-frames are also discussed in details. Applications of $d$-frames in other areas of study, specially in signal processing, can be considered as future scope of the work.

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**NARENDRA BISWAS**
DEPARTMENT OF MATHEMATICS
D.S.B. CAMPUS, KUMAUN UNIVERSITY
NAINITAL-263002
INDIA
*E-mail address*: narendrabiswas@yahoo.com

**CHETNA MEHRA**
DEPARTMENT OF MATHEMATICS
D.S.B. CAMPUS, KUMAUN UNIVERSITY
NAINITAL-263002
INDIA
*E-mail address*: chetnamehra@gmail.com

**MAHESH C. JOSHI**
DEPARTMENT OF MATHEMATICS
D.S.B. CAMPUS, KUMAUN UNIVERSITY
NAINITAL-263002
INDIA
*E-mail address*: mcjoshi69@gmail.com