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The Tribonacci-type balancing numbers and their applications

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Abstract. In this paper, we define the Tribonacci-type balancing numbers via a Diophantine equation with a complex variable and then give their miscellaneous properties. Also, we study the Tribonacci-type balancing sequence modulo m and then obtain some interesting results concerning the periods of the Tribonacci-type balancing sequences for any m. Furthermore, we produce the cyclic groups using the multiplicative orders of the generating matrices of the Tribonacci-type balancing numbers when read modulo m. Then give the connections between the periods of the Tribonacci-type balancing sequences modulo m and the orders of the cyclic groups produced. Finally, we expand the Tribonacci-type balancing sequences to groups and give the definition of the Tribonacci-type balancing sequences in the 3-generator groups and also, investigate these sequences in the non-abelian finite groups in detail. In addition, we obtain the periods of the Tribonacci-type balancing sequences in the polyhedral groups $(2,2,n)$, $(2,n,2)$, $(n,2,2)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$.

1. Introduction

Behera and Panda [2] introduced the balancing numbers n and balancers r as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r). \tag{1}$$

First few balancing numbers $1, 6, 35, 204$ are and 1189 with balancers 0, 2, 14, 84 and 492, respectively. For $n \geq 1$, the n^{th} balancing number B_n is described [2] by

$$B_{n+1} = 6B_n - B_{n-1}$$

with initial conditions $B_0 = 1$ and $B_1 = 6$.

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Ray [35] showed that the balancing numbers are also generated by a matrix

$$Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}, \quad Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

It is well-known that the tribonacci (3-step Fibonacci) sequence $\{T_n\}$ is defined by the following homogeneous linear recurrence relation:

$$T_{n+2} = T_{n+1} + T_n + T_{n-1}$$

for $n \geq 1$, with initial conditions $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$.

Komatsu [24] defined Tribonacci-type numbers by the following recurrence relation:

$$T_n^{(T_0,T_1,T_2)} = T_{n-1}^{(T_0,T_1,T_2)} + T_{n-2}^{(T_0,T_1,T_2)} + T_{n-3}^{(T_0,T_1,T_2)}$$

for $(n \geq 3)$, where $T_0^{(T_0,T_1,T_2)} = T_0$, $T_1^{(T_0,T_1,T_2)} = T_1$ and $T_2^{(T_0,T_1,T_2)} = T_2$. It is important to note that $T_n = T_n^{(0,1,1)}$ are ordinary Tribonacci numbers.

For a finitely generated group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_{i+1}$, $0 \leq i \leq n-1$, $x_{n+i} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$ (cf. [4,5]).

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, \ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

The polyhedral (triangle) group (l, m, n) for $l, m, n > 1$, is defined by the presentation

$$(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = e \rangle.$$

The polyhedral group (l, m, n) is finite if and only if the number $k = mn + nl + lm - lmn$ is positive, that is in the case $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Its order is $\frac{2lmn}{k}$. By Tietze transformations, we can easily prove that $(l, m, n) \cong (m, n, l) \cong (n, l, m)$ (cf. [6,7]).

Behera and Panda [2] defined the sequence of balancing numbers by the aid of the equation (1) and then gave its miscellaneous properties. Since then obtaining a recurrence sequence by using a certain Diophantine equation have been a topic of current. In literature, one can find any interesting properties and applications of the balancing-like sequences which are obtained from a certain Diophantine equation; see for example, [3,8,20,25–27,31,32]. We derive here a new recurrence sequence by using a Diophantine equation with a complex variable and called the Tribonacci-type balancing sequence.

In the first part of the paper, we give number theoretic properties of the Tribonacci-type balancing sequence.

The study of the behavior of the linear recurrence sequences under a modulus began with the earlier work of Wall [37], where the periods of the ordinary Fibonacci sequences modulo m were investigated. It is important to note that the period of a recurrence sequence modulo m with the period of this sequence in the cyclic group C_m are the same. Lu and Wang contributed to the study of Wall numbers for k-step Fibonacci sequence [28]. Recently, the theory extended to some special linear recurrence sequences by several authors; see, for example, [9–12, 16, 17, 19, 34, 36]. Patel and Ray [33] studied the period, rank and order of the sequence of the balancing number modulo m. In the second part of the paper, we consider the Tribonacci-type balancing sequence modulo m and then we derive some interesting results concerning the periods of the Tribonacci-type balancing sequences for any m. Also, we produce the cyclic groups using the multiplicative orders of the generating matrices of the Tribonacci-type balancing numbers when read modulo m. Then we give the connections between the periods of the Tribonacci-type balancing sequences modulo m and the orders of the cyclic groups produced.

In the mid-eighties, Wilcox applied the idea which was firstly introduced by Wall to the abelian groups [38]. The theory was expanded to some finite simple groups by Campbell et al. [5], where the Fibonacci sequence in a non-abelian group generated by two generators were introduced. The concept of the Fibonacci sequence for more two generators had also been considered by several authors; see, for example, [1, 4, 18, 22, 23, 29, 30]. In [9, 11, 12, 16, 17, 21, 29], the authors studied some special linear recurrence sequences defined by the aid of the elements of a group. In the next process, the theory was extended to the quaternions and the complex numbers, see [13–15]. In the third part of the paper, we give the definition of the Tribonacci-type balancing sequences in the 3-generator groups and then we investigate these sequences in the non-abelian finite groups in detail. Finally, we obtain the periods of the Tribonacci-type balancing sequences in the polyhedral groups $(2, 2, n)$, $(2, n, 2)$, $(n, 2, 2)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ as applications of the results produced.

2. Results

A positive integer n is called a Tribonacci-type balancing number if

$$i + i^2 + i^3 + \cdots + i^{n-1} = i^{n+1} + i^{n+2} + \cdots + i^{n+k}$$

for some positive integer k, where $i = \sqrt{-1}$. The positive integer k is called as the Tribonacci-type balancer of corresponding to the Tribonacci-type balancing number n.

First few Tribonacci-type balancing numbers are $4, 5, 8, 9$ and 12 with balancer $3, 4, 7, 8$ and 9, respectively. For $n \geq 1$, the n^{th} Tribonacci-type balancing number $B_{i,n}$ is defined recursively by

$$B_{i,n+2} = B_{i,n+1} + B_{i,n} - B_{i,n-1}, \tag{2}$$

with initial conditions $B_{i,0} = 4$, $B_{i,1} = 5$ and $B_{i,2} = 8$.

Using an inductive argument, we derive the following relations via the equation in the definition of the Tribonacci-type balancing numbers:

$$i + 2i^2 + 3i^3 + \cdots + (n-1)i^{n-1} =$$
$$= (-i)\begin{cases} (n+1)i^{n+1} + (n+2)i^{n+2} + \cdots + 2ni^{2n}, & \text{if } n \equiv 0 \pmod 4, \\ (n+1)i^{n+1} + (n+2)i^{n+2} + \cdots + (2n-1)i^{2n-1}, & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

and

$$i + 2i^2 + 3i^3 + \cdots + (n-1)i^{n-1} = \begin{cases} \frac{n}{3}(-2-2i), & \text{if } n \equiv 0 \pmod 4, \\ \frac{n}{3}(2-2i), & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

It is clear that the auxiliary equation of the Tribonacci-type balancing sequence $\{B_{i,n}\}$ is

$$x^3 = x^2 + x - 1. \tag{3}$$

Using the equation (3), we can give a Binet formula for the Tribonacci-type balancing numbers by

$$B_{i,n} = 2n + \frac{7}{2} + (-1)^n \frac{1}{2}.$$

By a simple calculation, we obtain the generating function of the Tribonacci-type balancing numbers as shown:

$$g(x) = \frac{-x^2 + x + 4}{x^3 - x^2 - x + 1}$$

for $0 \le -x^3 + x^2 + x < 1$.

Now we give an exponential representation for the Tribonacci-type balancing numbers by the aid of the generating function $g(x)$ with the following Proposition.

Proposition 1. *The Tribonacci-type balancing sequence $\{B_{i,n}\}$ have the following exponential representation:*

$$g(x) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(-x^3 + x^2 + x\right)^n - \left(x^2 - x - 3\right)^n \right).$$

Problem 1. By a simple calculation, we may write

$$\begin{aligned} \ln(g(x)) &= \ln\left(1 - \left(x^2 - x - 3\right)\right) - \ln\left(1 - \left(-x^3 + x^2 + x\right)\right) \\ &= -\left(x^2 - x - 3 + \frac{1}{2}\left(x^2 - x - 3\right)^2 + \cdots\right) \\ &\quad + \left(-x^3 + x^2 + x + \frac{1}{2}\left(-x^3 + x^2 + x\right)^2 + \cdots\right) \end{aligned}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}\left(x^2-x-3\right)^n+\sum_{n=1}^{\infty}\frac{1}{n}\left(-x^3+x^2+x\right)^n.$$

So we have the conclusion.

If we reduce the Tribonacci-type balancing sequence $\{B_{i,n}\}$ by a modulus m, taking least nonnegative residues, then we get the following recurrence sequence:

$$\{B_{i,n}(m)\} = \{B_{i,0}(m), B_{i,1}(m), B_{i,2}(m), \ldots, B_{i,j}(m), \ldots\}$$

where $B_{i,j}(m)$ is used to mean the jth element of the Tribonacci-type balancing sequence when read modulo m. We note here that the recurrence relations in the sequences $\{B_{i,n}(m)\}$ and $\{B_{i,n}\}$ are the same.

Theorem 1. *$\{B_{i,n}(m)\}$ forms a simply periodic sequence for any $m \geq 2$.*

Proof. Consider the set

$$S = \{(s_1, s_2, s_3) \mid s_i\text{'s are integers such that } 0 \leq s_i \leq m-1\}.$$

Since $|S| = m^3$, there are m^3 distinct 3-tuples of the Tribonacci-type balancing sequence modulo m. Thus, it is clear that at least one of these 3-tuples appears twice in the sequence $\{B_{i,n}(m)\}$. Therefore, the subsequence following this 3-tuple repeats; that is, $\{B_{i,n}(m)\}$ is a periodic sequence. Let $B_{i,u}(m) \equiv B_{i,v}(m)$, $B_{i,u+1}(m) \equiv B_{i,v+1}(m)$ and $B_{i,u+2}(m) \equiv B_{i,v+2}(m)$ such that $v > u$, then we get $v \equiv u \pmod 3$. From the equation (2), we may write the following relations:

$$B_{i,u}(m) = -B_{i,u+3}(m) + B_{i,u+2}(m) + B_{i,u+3}(m)$$

and

$$B_{i,v}(m) = -B_{i,v+3}(m) + B_{i,v+2}(m) + B_{i,v+3}(m).$$

Thus, we obtain

$$\begin{aligned} B_{i,u-1}(m) &\equiv B_{i,v-1}(m), \\ B_{i,u-2}(m) &\equiv B_{i,v-2}(m), \\ \vdots \quad & \quad \vdots \\ B_{i,0}(m) &\equiv B_{i,v-u}(m), \end{aligned}$$

which implies that the Tribonacci-type balancing sequence modulo m is simply periodic. □

Let the notation $PB_i(m)$ denote the smallest period of the sequence $\{B_{i,n}(m)\}$.

From the equation (2), we may write the following companion matrix:

$$C_i = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The matrix C^i is said to be the Tribonacci-type balancing matrix. Then we can write the following matrix relation:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} B_{i,n+2} \\ B_{i,n+1} \\ B_{i,n} \end{bmatrix} = \begin{bmatrix} B_{i,n+3} \\ B_{i,n+2} \\ B_{i,n+1} \end{bmatrix}.$$

By mathematical induction on n, it is easy to see that the n^{th} powers of the matrix C_i are

$$(4) \qquad (C_i)^n = \begin{bmatrix} \frac{n}{2}+1 & 0 & -\frac{n}{2} \\ \frac{n}{2} & 1 & -\frac{n}{2} \\ \frac{n}{2} & 0 & -\frac{n}{2}+1 \end{bmatrix}, \quad \text{if } n \text{ is even;}$$

$$(5) \qquad (C_i)^n = \begin{bmatrix} \frac{n+1}{2} & 1 & \frac{-n-1}{2} \\ \frac{n+1}{2} & 0 & \frac{-n+1}{2} \\ \frac{n-1}{2} & 1 & \frac{-n+1}{2} \end{bmatrix} \quad \text{if } n \text{ is odd.}$$

Given an integer matrix $A = [a_{ij}]$, $A \pmod m$ means that all entries of A are modulo m, that is, $A \pmod m = (a_{ij} \pmod m)$. Let us consider the set $\langle A\rangle_m = \{(A)^n \pmod m \mid n \geq 0\}$. If $(\det A, m) = 1$, then the set $\langle A\rangle_m$ is a cyclic group; if $(\det A, m) \neq 1$, then the set $\langle A\rangle_m$ is a semigroup. Since $\det C_i = -1$, the set $\langle C_i\rangle_m$ is a cyclic group for every positive integer $m \geq 2$. From (5), it is easy to see that the cardinality of the set $\langle C_i\rangle_m$ cannot be odd. Thus, for $m \geq 2$, we obtain

$$(C_i)^{2m} = \begin{bmatrix} m+1 & 0 & -m \\ m & 1 & -m \\ m & 0 & -m+1 \end{bmatrix},$$

which yields that $|\langle C_i\rangle_m| = 2m$. Now we give the connections between the periods of the Tribonacci-type balancing sequences modulo m and the orders of the cyclic groups produced with the following Theorem.

Theorem 2. *For any $m \geq 2$,*

$$PB_i(m) = \begin{cases} \frac{|\langle C_i\rangle_m|}{4}, & \text{if } m \equiv 0 \pmod 4, \\ \frac{|\langle C_i\rangle_m|}{2}, & \text{if } m \equiv 2 \pmod 4, \\ |\langle C_i\rangle_m|, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. In fact it is easy to see that the Tribonacci-type balancing sequence $\{B_{i,n}\}$ conforms to the following pattern:

$$\begin{aligned} B_{i,4k} &= 4+8k, \\ B_{i,4k+1} &= 5+8k, \\ B_{i,4k+2} &= 8+8k, \\ B_{i,4k+3} &= 9+8k, \end{aligned}$$

where $k \in \mathbb{N}$. So we need to find the smallest natural number k to determine the period of the sequence $\{B_{i,n}(m)\}$. If $m \equiv 0 \pmod 4$, then the smallest positive value k is $\frac{m}{8}$ providing conditions $B_{i,4k} \equiv 4$, $B_{i,4k+1} \equiv 5$ and $B_{i,4k+2} \equiv 8$ and hence $PB_i(m) = \frac{m}{2} = \frac{|\langle C_i \rangle_m|}{4}$. If $m \equiv 2 \pmod 4$, then $k = \frac{m}{4}$. So we get $PB_i(m) = m = \frac{|\langle C_i \rangle_m|}{2}$. Similarly, we obtain $k = \frac{m}{2}$ when m is odd. Thus it is veried that $PB_i(m) = 2m = |\langle C_i \rangle_m|$. $\square$

Let G be a finite k-generator group and let

$$X = \left\{ (x_1, x_2, \ldots, x_k) \in \underbrace{G \times G \times \cdots \times G}_{k} \mid \langle \{x_1, x_2, \ldots, x_k\} \rangle = G \right\}.$$

We call $(x_1, x_2, \ldots, x_k)$ a generating k-tuple for G.

Now we redefine the Tribonacci-type balancing sequence by means of the elements of a group which have three generators.

Definition 1. Let G be a 3-generator group and let (x_1, x_2, x_3) be a generating 3-tuple of G. For generating 3-tuple (x_1, x_2, x_3), we define the Tribonacci-type balancing orbits of the first and second kind of the group G, respectively by:

$$b_0^{(1)} = x_1, b_1^{(1)} = x_2 x_3, b_2^{(1)} = (x_3)^4, b_{n+3}^{(1)} = \left(b_n^{(1)}\right)^{-1} b_{n+1}^{(1)} b_{n+2}^{(1)}, \quad (n \geq 0)$$

and

$$b_0^{(2)} = x_1, b_1^{(2)} = x_3 x_2, \; b_2^{(2)} = (x_3)^4, b_{n+3}^{(2)} = \left(b_n^{(2)}\right)^{-1} b_{n+1}^{(2)} b_{n+2}^{(2)}, \quad (n \geq 0).$$

For generating 3-tuple (x_1, x_2, x_3), we denote the Tribonacci-type balancing orbits of the first and second kind of G by the notations $B_{(x_1,x_2,x_3)}^{(1)}(G)$ and $B_{(x_1,x_2,x_3)}^{(2)}(G)$, respectively.

Theorem 3. *Let G be a 3-generator group and let (x_1, x_2, x_3) be a generating 3-tuple for G. If G is finite, then the sequences $B_{(x_1,x_2,x_3)}^{(1)}(G)$ and $B_{(x_1,x_2,x_3)}^{(2)}(G)$ are simply periodic.*

Proof. Let us consider the sequence $B_{(x_1,x_2,x_3)}^{(1)}(G)$. Suppose that n is the order of G. Since there are n^3 distinct 3-tuples of elements of G, at least one of the 3-tuples appears twice in the sequence $B_{(x_1,x_2,x_3)}^{(1)}(G)$. Therefore, the subsequence following this 3-tuple repeats. Because of the repetition, the sequence is periodic. Then we have the natural numbers i and j, with $i > j$, such that

$$b_{i+1}^{(1)} = b_{j+1}^{(1)}, \quad b_{i+2}^{(1)} = b_{j+2}^{(1)}, \quad b_{i+3}^{(1)} = b_{j+3}^{(1)}.$$

From the defining recurrence relation of the Tribonacci-type balancing orbit of G, it is easy to see that

$$b_k^{(1)} = b_{k+1}^{(1)} \left(b_{k+2}^{(1)}\right)^{-1} \left(b_{k+2}^{(1)}\right)^{-1}$$

for $k = i, j$. Thus we obtain $b_i^{(1)} = b_j^{(1)}$ and so

$$b_{i-1}^{(1)} = b_{j-1}^{(1)}, \quad b_{i-2}^{(1)} = b_{j-2}^{(1)}, \quad \ldots, \quad b_{i-j}^{(1)} = b_{j-j}^{(1)} = b_0^{(1)},$$

which implies that the sequence $B_{(x_1,x_2,x_3)}^{(1)}(G)$ is simply periodic.

The proof for the Tribonacci-type balancing orbit of the second kind of G is similar to the above and is omitted. □

Let the notations $LB_{(x_1,x_2,x_3)}^{(1)}(G)$ and $LB_{(x_1,x_2,x_3)}^{(2)}(G)$ denote the smallest periods of the sequences $B_{(x_1,x_2,x_3)}^{(1)}(G)$ and $B_{(x_1,x_2,x_3)}^{(2)}(G)$, respectively. From the definitions, it is clear that the lengths of the periods of the Tribonacci-type balancing orbits of the first and second kinds of a finite non-abelian 3-generator group depend on the chosen generating set and the order in which the assignments of x_1, x_2, x_3 are made.

We shall now address the lengths of the periods of the Tribonacci-type balancing orbits of the first and second kinds of the polyhedral groups $(2, 2, n)$, $(2, n, 2)$, $(n, 2, 2)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ with respect to the generating 3-tuple (x, y, z).

Theorem 4. *Let $G_n = \langle x, y, z \mid x^2 = y^2 = z^n = xyz = e\rangle$, where $n \geq 2$. Then*

(i) $LB_{(x,y,z)}^{(1)}(G_2) = LB_{(x,y,z)}^{(1)}(G_4) = 4$ *and* $LB_{(x,y,z)}^{(1)}(G_n) = 8$ *for* $n \neq 2, 4$.

(ii) $LB_{(x,y,z)}^{(2)}(G_2) = LB_{(x,y,z)}^{(2)}(G_4) = 4$ *and*

$$LB_{(x,y,z)}^{(2)}(G_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 8, \\ 2n, & \text{if } n \equiv 4 \pmod 8, \\ 4n, & \text{if } n \equiv 2, 6 \pmod 8, \\ 8n, & \text{if } n \text{ is odd}, \end{cases}$$

for $n \neq 2, 4$.

Proof. We prove this by direct calculation. We first note that $x = yz$, $y = zx$ and $z = yx$.

(i) The sequence $LB_{(x,y,z)}^{(1)}(G_n)$ is

$$x,\ x,\ z^4,\ z^4,\ xz^8,\ xz^8,\ z^{-4},\ z^{-4},\ x,\ x,\ z^4,\ \ldots$$

Thus we have the conclusion.

(ii) Now we consider the start of the Tribonacci-type balancing orbit of the second kind of the polyhedral group $(2, 2, n)$

$$\begin{aligned}&x, zy, z^4, z^2, yz^5, yz^7, e, z^2,\\&yz^9, yz^7, z^{-4}, z^{-6}, z^3y, zy, z^8, z^{10},\\&yz^{17}, yz^{15}, z^{-12}, z^{-14}, z^{11}y, z^9y, z^{16}, z^{18},\\&yz^{25}, yz^{23}, z^{-20}, z^{-22}, z^{19}y, z^{17}y, z^{24}, z^{26},\\&yz^{33}, yz^{31}, z^{-28}, z^{-30}, z^{27}y, z^{25}y, z^{32}, z^{34}, \ldots,\end{aligned}$$

which is verifed that $LB^{(2)}_{(x,y,z)}((2,2,2)) = LB^{(2)}_{(x,y,z)}((2,2,4)) = 4$. Using the above, the sequence $B^{(2)}_{(x,y,z)}(G_n)$ becomes:

$$\begin{array}{lll}b^{(2)}_0 = x, & b^{(2)}_1 = zy, & b^{(2)}_2 = z^4, \ \ldots,\\ b^{(2)}_8 = xz^8, & b^{(2)}_9 = z^{-7}y, & b^{(2)}_{10} = z^{-4}, \ \ldots,\\ b^{(2)}_{16} = xz^{16}, & b^{(2)}_{17} = z^{-15}y, & b^{(2)}_{18} = z^{-12}, \ \ldots,\\ b^{(2)}_{8i} = xz^{8i}, & b^{(2)}_{8i+1} = z^{1-8i}y, & b^{(2)}_{8i+2} = z^{4-8i}, \ \ldots\end{array}$$

So we need the smallest $i \in \mathbb{N}$ such that $8i = nk$ $(n \neq 2, 4)$ for $k \in \mathbb{N}$. If $n \equiv 0 \pmod 8$, then $i = \frac{n}{8}$. Thus, we obtain $8i = n$ and so $LB^{(2)}_{(x,y,z)}(G_n) = n$. If $n \equiv 4 \pmod 8$, then the smallest positive value for i is $\frac{n}{4}$, giving a period $2n$. If $n \equiv 2 \pmod 8$ or $n \equiv 6 \pmod 8$, then $i = \frac{n}{2}$ and hence the period is $4n$. Similarly, we obtain $i = n$ when n is odd. Then, we get $LB^{(2)}_{(x,y,z)}(G_n) = 8n$. $\square$

Consider the sequences

$$u_0 = 1,\ u_1 = -1\ u_2 = 0\ u_{n+3} = u_{n+2} + u_{n+1} - u_n, \quad (n \geq 0)$$

and

$$v_0 = 1\ v_1 = 1\ v_2 = 0\ v_{n+3} = v_{n+2} + v_{n+1} - v_n, \quad (n \geq 0).$$

It is easy to prove that the sequences $\{u_n\}$ and $\{v_n\}$ modulo m are periodic. Reducing the sequences $\{u_n\}$ and $\{v_n\}$ by a modulus m, then we get the repeating sequences, respectively denoted by

$$\{u_n(m)\} = \{u_0(m),\ u_1(m),\ \ldots,\ u_\tau(m),\ \ldots\}$$

and

$$\{v_n(m)\} = \{v_0(m),\ v_1(m),\ \ldots\ v_\tau(m),\ \ldots\}.$$

They have the same recurrence relation as in the definitions of the sequences $\{u_n\}$ and $\{v_n\}$. We denote the lengths of the periods of the sequences $\{u_n(m)\}$ and $\{v_n(m)\}$ by $h_{u_n}(m)$ and $h_{v_n}(m)$. By mathematical induction on n, we find the relationships between the Tribonacci-type balancing matrix C_i and the elements of the sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$(C_i)^n = \begin{bmatrix} -u_{n+1} & 0 & u_{n-1} \\ -u_{n+1}-1 & 1 & u_{n-1} \\ -u_{n+1}-1 & 0 & u_{n-1}+1 \end{bmatrix} = \\ = \begin{bmatrix} -v_{n+1} & 0 & v_{n-1} \\ -v_{n+1}-1 & 1 & v_{n-1} \\ -v_{n+1}-1 & 0 & v_{n-1}+1 \end{bmatrix}, \quad \text{if } n \text{ is even;}$$

$$(C_i)^n = \begin{bmatrix} -u_{n} & 1 & u_{n} \\ -u_{n} & 0 & u_{n}+1 \\ -u_{n}-1 & 1 & u_{n}+1 \end{bmatrix} = \\ = \begin{bmatrix} -v_{n} & 1 & v_{n} \\ -v_{n} & 0 & v_{n}+1 \\ -v_{n}-1 & 1 & v_{n}+1 \end{bmatrix}, \quad \text{if } n \text{ is odd.}$$

From the above matrix relations, it is clear that $h_{u_n}(m) = h_{v_n}(m) = |\langle C_i \rangle_m| = 2m$.

Now we give the lengths of the periods of the sequences $B^{(1)}_{(x,y,z)}((n,2,2))$, $B^{(2)}_{(x,y,z)}((n,2,2))$ and $B^{(1)}_{(x,y,z)}((2,n,2))$ via $|\langle C_i \rangle_m|$.

Theorem 5. *For $n \geq 2$, $LB^{(1)}_{(x,y,z)}((n,2,2)) = h_{u_n}(n)$ and $LB^{(2)}_{(x,y,z)}((n,2,2)) = h_{v_n}(n)$.*

Proof. We first note that in the group defined by

$$\langle x, y, z \mid x^n = y^2 = z^2 = xyz = e \rangle,$$

$x = zy$, $y = zx$ and $z = xy$. Clearly, the Tribonacci-type balancing orbits of the first and second kind of the polyhedral group $(n,2,2)$ are as follows, respectively:

$$x\ , yz\ , z^4,\ x^{-1}yz^5,\ z^{-1}y^{-1}x^{-1}yz^9,\ x^{-2}y^3z^9,\ \ldots$$

and

$$x,\ zy,\ z^4,\ x^{-1}z^5y,\ y^{-1}z^{-1}x^{-1}z^9y,\ x^{-2}z^9y,\ \ldots$$

By direct calculation it is easy to see that the sequences $B^{(1)}_{(x,y,z)}((n,2,2))$ and $B^{(2)}_{(x,y,z)}((n,2,2))$ conform to the following patterns:

$$\begin{aligned} &b^{(1)}_0 = x = x^{u_0}, && b^{(1)}_1 = x^{-1} = x^{u_1}, && b^{(1)}_2 = e = x^{u_2}, \\ &b^{(1)}_3 = x^{-2} = x^{u_3}, && b^{(1)}_4 = x^{-1} = x^{u_4}, && b^{(1)}_5 = x^{-3} = x^{u_5}, \ldots \end{aligned}$$

and

$$\begin{aligned} &b^{(2)}_0 = x = x^{v_0}, && b^{(2)}_1 = x = x^{v_1}, && b^{(2)}_2 = e = x^{v_2}, \\ &b^{(2)}_3 = e = x^{v_3}, && b^{(2)}_4 = x^{-1} = x^{v_4}, && b^{(1)}_5 = x^{-1} = x^{v_5}, \ldots \end{aligned}$$

Since the order of the element x is n, we get

$$LB^{(1)}_{(x,y,z)}((n,2,2)) = LB^{(2)}_{(x,y,z)}((n,2,2)) = |\langle C_i \rangle_n| = 2n. \qquad \square$$

Theorem 6. *Let G_n, $n \geq 2$, be the group defined by the presentation $\langle x,y,z \mid x^2 = y^n = z^2 = xyz = e \rangle$. Then*

i. *For $n \geq 2$, $LB^{(1)}_{(x,y,z)}(G_n) = h_{v_n}(n) = |\langle C_i \rangle_n| = 2n$.*

ii. *$LB^{(2)}_{(x,y,z)}(G_2) = LB^{(2)}_{(x,y,z)}(G_4) = 4$ and*

$$LB^{(2)}_{(x,y,z)}(G_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 8, \\ 2n, & \text{if } n \equiv 4 \pmod 8, \\ 4n, & \text{if } n \equiv 2,6 \pmod 8, \\ 8n, & \text{if } n \text{ is odd}, \end{cases}$$

for $n \neq 2, 4$.

Proof. The proof is similar to the above and is omitted. $\square$

Now we concentrate on finding the lengths of the periods of the Tribonacci-type balancing orbits of the first and second kind of the polyhedral groups $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. The results are summarized in the following table:

G_n	$LB^{(1)}_{(x,y,z)}(G_n)$	$LB^{(2)}_{(x,y,z)}(G_n)$
$(2,3,3)$	12	24
$(2,3,4)$	4	24
$(2,3,5)$	12	60

3. CONCLUSION

In this paper, the Tribonacci-type balancing numbers were defined and their miscellaneous properties were given. Also, taking into account the Tribonacci-type balancing sequence modulo m, some interesting results concerning the periods of the Tribonacci-type balancing sequence for any m were obtained. In addition, the cyclic groups from the generating matrices of the Tribonacci-type balancing numbers when read modulo m were produced. Finally, the Tribonacci-type balancing sequence to groups were expanded and then the periods of these sequences in the finite polyhedral groups were examined.

4. Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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