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Convergence analysis for Suzuki's generalized nonexpansive mappings

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Abstract. In this paper, we study the Picard-Mann hybrid iteration process to approximate fixed points of Suzuki's generalized nonexpansive mappings. We establish some weak and strong convergence theorems for such mappings in uniformly convex Banach space.

1. Introduction

The iteration method for approximating fixed points of various classes of nonlinear mapping is available in the literature. The class of nonexpansive operators via iteration method is extensively studied by many authors [2, 4–6, 8, 14, 20]. In many research papers have appeared, some generalizations of nonexpansive mapping, namely quasi-nonexpansive mapping [3, 12], asymptotically nonexpansive mapping [4, 9, 15–17, 21].

In 2008, Suzuki [19] introducing an interesting extension of nonexpansive mappings. A self-mapping T on C is said to satisfy condition (C) if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|, \quad \forall x, y \in C,$$

where C is a nonempty subset of a Banach space X. He showed that there exist mappings that satisfy condition (C) but do not belong to the class of nonexpansive mappings. Suzuki also proved the existence of fixed point and convergence results for such mappings. In 2013, Khan [7] introduced an iterative process defined by

$$\begin{cases} x_{n+1} = Ty_n, \\ \quad y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \end{cases} \tag{1}$$

for all $n \geq 0$ and $\alpha_n \in (0, 1)$. This process is independent of all Picard and Mann iterative processes. He named it as Picard-Mann hybrid iterative

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process. Khan also proved the strong convergence and weak convergence theorems for the class of nonexpansive mappings in the Banach spaces.

Motived and inspired by these works, we study the Picard-Mann iterative process to approximate fixed points of mappings that satisfy condition (C). We also prove some weak and strong convergence results for such mappings in uniformly convex Banach space.

2. Preliminaries

In the following, we will name the mappings that satisfy condition (C), *Suzuki's generalized nonexpansive mappings.* Now, we recall some definitions and lemmas which will be used in our presentation.

Let C be a nonempty subset of a real Banach space X and $T\colon C \to C$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{p \in C : Tp = p\}$.

Definition 1 ([4])**.** The function $\delta_X\colon [0,2] \to [0,1]$ is said to be the modulus of convexity of X if

$$\delta_X(\varepsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\Big\}.$$

X is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition 2 ([11])**.** A Banach space X is said to satisfy Opial's condition if for each weakly convergent sequence $\{x_n\}$ to x,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds, for all $y \in X$, with $y \neq x$.

Lemma 1 ([17])**.** *Let X be a real uniformly convex Banach space and $0 < a \le t_n \le b < 1$, for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|(1-t_n)x_n + t_n y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 3. A mapping $T\colon C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall\, x, y \in C.$$

Definition 4. A mapping $T\colon C \to C$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\| \quad \forall\, x \in C,\ \forall\, p \in F(T).$$

Lemma 2 ([19])**.** *Let C be a nonempty subset of a Banach space X and $T\colon C \to C$. If T is a Suzuki's generalized nonexpansive mappings and $F(T) \neq \emptyset$ then T is quasi-nonexpansive.*

Lemma 3 ([19]). *Let C be a nonempty subset of a Banach space X and $T\colon C \to C$. If T is a Suzuki's generalized nonexpansive mappings, then for every two elements $x, y \in C$, we have*

$$\|x - Ty\| \leq 3\|x - Tx\| + \|x - y\|.$$

Lemma 4 ([19]). *Let C be a nonempty subset of a Banach space X having the Opial property and suppose $T\colon C \to C$ is a Suzuki's generalized nonexpansive mappings. If $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx = x$. That is, $I - T$ is demiclosed at zero.*

Theorem 1 ([19]). *Let C be a weakly compact convex subset of a uniformly convex Banach space X and suppose $T\colon C \to C$. If T is a Suzuki's generalized nonexpansive mappings, then T has a fixed point.*

Now we consider some well-known iteration schemes. Let C be a nonempty convex subset of normed space X and T a self-mapping on C.

(a) The Picard iteration process is defined by

$$x_{n+1} = Tx_n, \tag{2}$$

for all $n \geq 0$ (see, for example, [13]).

(b) The Mann iteration process (see, for example, [10]) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{3}$$

for all $n \geq 0$, $\alpha_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

3. Main results

In this section, we establish some weak and strong convergence theorems for Suzuki's generalized nonexpansive mappings in uniformly convex Banach space.

Theorem 2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and suppose $T\colon C \to C$ is a Suzuki's generalized nonexpansive mappings with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence given in* (1)*. Then the following hold:*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in F(T)$;*

(ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. (i) Let $p \in F(T)$, we have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \alpha_n)x_n + \alpha_n Tx_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \|Tx_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{4}$$

and by Lemma 2, we get

$$\|x_{n+1} - p\| = \|Ty_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|. \tag{5}$$

It follows from (5) that the sequence $\{\|x_n - p\|\}$ is non-increasing and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.

(ii) Now, if

$$\lim_{n\to\infty} \|x_n - p\| = r \tag{6}$$

then, by (4), we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{7}$$

Applying Lemma 2, we get

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r, \tag{8}$$

and by the proof of (i), we have

$$\|x_{n+1} - p\| \leq \|y_n - p\|, \tag{9}$$

taking limit infimum, we have

$$r \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{10}$$

Combining (7) and (10), we get

$$r = \lim_{n\to\infty} \|y_n - p\|. \tag{11}$$

From (11), we have

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\|. \end{aligned} \tag{12}$$

Therefore by Lemma 1, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{13}$$

□

Now, we prove our weak and strong convergent results.

Theorem 3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial condition, and $T\colon C \to C$ be a Suzuki's generalized nonexpansive mappings with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by (1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since every uniformly convex Banach space X is reflexive, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p \in C$. It follows by Lemma 4 that $p \in F(T)$. We suppose that p is not weak limit of $\{x_n\}$. Then, there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y$ and $p \neq q$. Obviously, $q \in F(T)$. Now, using Theorem 1 and Opial condition, we have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_{n_i} - p\| < \lim_{n\to\infty} \|x_{n_i} - q\| = \lim_{n\to\infty} \|x_n - q\|,$$

but

$$\lim_{n\to\infty}\|x_n-q\| = \lim_{n\to\infty}\left\|x_{n_j}-q\right\| < \lim_{n\to\infty}\left\|x_{n_j}-p\right\| = \lim_{n\to\infty}\|x_n-p\|,$$

which is a contradiction. Thus $\{x_n\}$ converges weakly to p. □

Theorem 4. *Let C be a nonempty compact convex subset of a uniformly convex Banach space X, and $T\colon C\to C$ be a Suzuki's generalized nonexpansive mappings. Suppose $\{x_n\}$ is a sequence given in (1). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. It follows from Theorem 1, that $F(T)\neq\emptyset$. By Theorem 2, $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F(T)$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Since C is compact, we can find a strongly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to p$ and $p\in C$. By Lemma 3, we have

$$\|x_{n_k}-Tp\| \le 3\|x_{n_k}-Tx_{n_k}\| + \|x_{n_k}-p\|. \tag{14}$$

It follows that $x_{n_k}\to Tp$ as $k\to\infty$ and from the uniqueness of limits we get $Tp=p$. Hence, $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $T\colon C\to C$ be a Suzuki's generalized nonexpansive mappings with $F(T)\neq\emptyset$. Suppose $\{x_n\}$ is a sequence given in (1). Then $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n,F(T))=0,$$

where $d(x,F(T))=\inf\{d(x,p):p\in F(T)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$. From Theorem 2 we know that $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F(T)$, so $\lim_{n\to\infty} d(x_n,F(T))$ exists for all $p\in F(T)$. Thus by hypothesis

$$\lim_{n\to\infty} d(x_n,F(T))=0. \tag{15}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n,F(T))=0$ for any $\varepsilon>0$ there exists a positive integer n_0 such that

$$d(x_n,F(T))<\frac{\varepsilon}{2},\quad \forall\, n\ge n_0.$$

Therefore, there exists $z\in F(T)$ such that

$$\|x_{n_0}-z\|<\frac{\varepsilon}{2}.$$

Thus, for all $m,n\ge n_0$, we get from the above inequality that

$$\begin{aligned}\|x_m-x_n\| &\le \|x_m-z\|+\|x_n-z\|\\ &\le \|x_{n_0}-z\|+\|x_{n_0}-z\|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus, it follows that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of Banach space X, the sequence $\{x_n\}$ converges strongly to some $p \in C$. Since $F(T)$ is a closed subset of C and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ we have $p \in F(T)$. Thus, the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Remark 1. The convergence of the Picard-Mann hybrid iterative process implies its consistency.

In 1974, Senter and Dotson [18] introduced the notion of mapping satisfying condition (I) which is defined as follows.

A mapping $T\colon C \to C$ is said to satisfy condition (I), if there exists a non-decreasing function $\phi\colon [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$ such that

$$d(x, Tx) \geq \phi(d(x, F(T)),$$

for all $x \in C$.

Theorem 6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $T\colon C \to C$ be a Suzuki's generalized nonexpansive mappings with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence given in* (1). *If T satisfies condition (I), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Theorem 2, that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from condition (I) that

$$0 \leq \lim_{n\to\infty} \phi(d(x_n, F(T)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

i.e.

$$\lim_{n\to\infty} \phi(d(x_n, F(T)) = 0.$$

Since $\phi\colon [0,\infty) \to [0,\infty)$ is a a non-decreasing function with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$, we obtain

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Consequently, $\{x_n\}$ converges strongly to a fixed point of T. □

4. Numerical examples

In this section, we provide some examples to illustrate our results.

Example 1. Let $X = \mathbb{R}$, $C = [0, 20]$, $x_0 = 10$, $\alpha_n = 0.85$ for all $n \geq 0$. Define the map $T\colon C \to C$ by the formula $Tx = \dfrac{x}{2} + 1$ for all $x \in C$.
It can be seen that T is a nonexpansive mapping, so T is a Suzuki mapping and $F(T) = \{2\}$. Table 1, shows that the Picard-Mann hybrid iteration (1) requires less number of iterations as compared to the Picard iteration (2) and the Mann iteration (3).

Table 1. Comparison between Picard, Mann and Picard-Mann iterations.

Step	Picard	Mann	Picard-Mann
0	10.00000000	10.00000000	10.00000000
1	6.000000000	6.600000000	4.300000000
2	4.000000000	4.645000000	2.661250000
3	3.000000000	3.520875000	2.190109375
4	2.500000000	2.874503125	2.054656445
5	2.250000000	2.502839296	2.015713728
6	2.125000000	2.289132595	2.004517696
7	2.062500000	2.166251242	2.001298837
8	2.031250000	2.095594464	2.000373415
9	2.015625000	2.054966817	2.000107357
10	2.007812562	2.031605919	2.000030865
11	2.003906250	2.018173403	
12	2.001953125	2.010449707	
13	2.000976562	2.006008581	
14	2.000488281	2.003454934	
15	2.000244140	2.001986587	
16	2.000122070	2.001142287	
17	2.000061035	2.000656815	
18		2.000377668	
19		2.000217159	
20		2.000124866	
21		2.000071798	

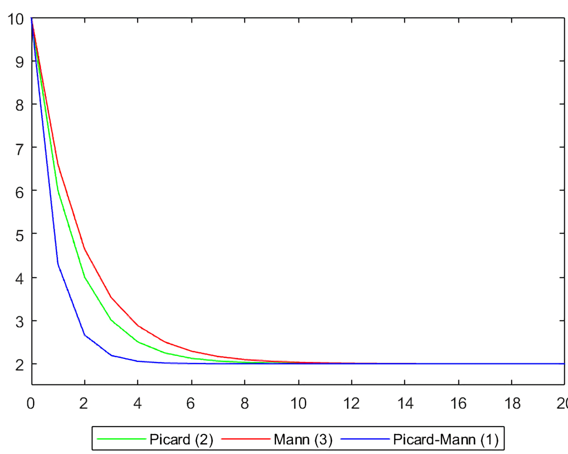

Figure 1. Convergence behavior of the iteration methods.

Example 2. Consider $C = [0,1]$ a closed convex subset of a Banach space $X = \mathbb{R}$ and define $T\colon C \to C$ by

$$T(x) = \begin{cases} 1 - x, & \text{if } x \le \frac{1}{8}, \\ \dfrac{x+4}{5}, & \text{if } x > \frac{1}{8}. \end{cases}$$

One can observe that T is Suzuki's generalized mapping with $F(T) = \{1\}$, but T is not nonexpansive mapping, see for example, [1]. In Table 2, we compare the number of iterations in which the fixed point is reached. We choose $x_0 = 0.1,\ \alpha_n = 0.85$ and we get:

TABLE 2. Comparison between Picard, Mann and Picard-Mann iterations.

Step	**Picard**	**Mann**	**Picard-Mann**
0	0.100000000	0.100000000	0.100000000
1	0.900000000	0.780000000	0.956000000
2	0.980000000	0.929600000	0.997184000
3	0.996000000	0.977472000	0.999819776
4	0.999200000	0.992791040	0.999988465
5	0.999840000	0.997693132	0.999999261
6	0.999968000	0.999261802	0.999999952
7	0.999993600	0.999763776	0.999999996
8	0.999998720	0.999924408	0.999999999
9	0.999999744	0.999975810	
10	0.999999948	0.999992259	
11	0.999999989	0.999997523	
12	0.999999997	0.999999207	
13		0.999999746	
14		0.999999918	
15		0.999999974	
16		0.999999991	
17			

5. CONCLUSION

We have proved strong and weak convergence results for Suzuki's generalized nonexpansive mappings. Moreover, we give illustrative numerical examples and approximate fixed points of these mappings using the Matlab program.

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