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Ideal convergence and ideal Cauchy sequences in intuitionistic fuzzy metric spaces

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Abstract. The present study introduces the concepts of ideal convergence (I–convergence), ideal Cauchy (I–Cauchy) sequences, I^*–convergence, and I^*–Cauchy sequences in intuitionistic fuzzy metric spaces. It defines I–limit and I–cluster points as a sequence in these spaces. Afterward, it examines some of their basic properties. Lastly, the paper discusses whether phenomena should be further investigated.

1. Introduction

Based on the concept of density of positive natural numbers, statistical convergence was independently defined by Fast [8] and Steinhaus [9] in 1951. Adopting an ideal I of some subsets of the set of positive integers, Kostryko et al. [18] have characterized ideal convergence (I–convergence) as a generalization of ordinary and statistical convergence and also conceptualized the I^*–convergence closely related to I–convergence. Besides, Dems [13] has extended the statistical Cauchy sequence [10] to ideals and introduced ideal Cauchy (I–Cauchy) sequences. Nabiyev et al. [3] have proposed I^*–Cauchy sequences and investigated the relationship between these sequences and I–Cauchy sequences.

Fuzzy sets, defined by Zadeh [15] in 1965, have been used in many fields, such as artificial intelligence, decision-making, image analysis, probability theory, and weather forecasting. In particular, Kramosil and Michalek [12] and Kaleva and Seikkala [17] have first examined the concept of fuzzy metric spaces (FMSs). Furthermore, George and Veeramani [2], using continuous t–norms, extensively revised the concept of fuzzy metric space originally proposed by Kramosil. As a result, they established a Hausdorff topology for fuzzy metric spaces and have introduced significant advancements in this field.

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Lately, Mihet [6] has studied the notion of point convergence (p–convergence), a weaker concept than ordinary convergence. Moreover, Gregori et al. [20] have suggested the s–convergence. Morillas and Sapena have defined the concept of standard convergence (std–convergence) [19]. Gregori and Miñana [21] have introduced the strong convergence (st–convergence), a stronger concept than ordinary convergence. Li et al. [5] have propounded the statistical convergence and statistical Cauchy sequence in FMSs and have examined some of their basic properties.

In 1986, Atanasov [14] generalized a fuzzy set introduced by Zadeh [15], accepting the membership as a fuzzy logic value rather than a single truth value, and introduced the Intuitionistic Fuzzy Set (IFS). Later, in 2004, Park [11] generalized the notion of fuzzy metric spaces to the intuitionistic fuzzy metric spaces (IFMSs) with the help of an intuitionistic set. Many studies, such as fixed point theory [16] and convergence types [1], have been studied and introduced in IFMSs. One of these studies, the statistical convergence in IFMSs, was dealt with by Varol in 2022 [4].

The current paper can be summarized in the following way. Section 2 presents some basic definitions and properties required in the following sections. Section 3 proposes the concepts of I and I^*–convergence, I and I^*–Cauchy sequence in IFMSs and suggests some of their basic properties. Section 4 defines the notions of I–limit points and I–cluster points of a sequence in IFMSs. The final section discusses the need for further research.

2. PRELIMINARIES

This section presents the exhaustive definitions, basic properties, and theorems for ideal convergence, ideal Cauchy sequences, IFMSs and statistical convergence in IFMSs.

Definition 1 ([7])**.** Let $\circ : [0,1]^2 \to [0,1]$ be a binary operation. We say that $\circ$ is a triangular norm (t–norm) if it satisfies the following conditions:

(1) $\circ$ is both associative and commutative;
(2) $t \circ 1 = t$ for all $t \in [0,1]$;
(3) Whenever $t_1 \leq t_3$ and $t_2 \leq t_4$ for each $t_1, t_2, t_3, t_4 \in [0,1]$, it holds that $t_1 \circ t_3 \leq t_2 \circ t_4$.

Definition 2 ([7])**.** Let $\bigtriangledown : [0,1]^2 \to [0,1]$ be a binary operation. We say that $\bigtriangledown$ is a triangular conorm (t–conorm) if it satisfies the following conditions:

(1) $\bigtriangledown$ is both associative and commutative;
(2) $t \bigtriangledown 0 = t$ for all $t \in [0,1]$;
(3) Whenever $t_1 \leq t_3$ and $t_2 \leq t_4$ for each $t_1, t_2, t_3, t_4 \in [0,1]$, it holds that $t_1 \bigtriangledown t_3 \leq t_2 \bigtriangledown t_4$.

Remark 1. We utilize the concepts of the triangular norm, often referred to as t–norm, and triangular conorm, commonly known as t–conorm, to define and characterize fuzzy intersections and fuzzy unions.

Example 1 ([7])**.** According to the previous two definitions, the following operators are basic examples of t–norm and t–conorms, respectively.

(1) $a \circ b = ab$;
(2) $a \circ b = \min\{a, b\}$;
(3) $a \bigtriangledown b = \max\{a, b\}$;
(4) $a \bigtriangledown b = \min\{a + b, 1\}$.

With the help of definition 1 and 2; Park [11] has recently introduced the IFMS as follows.

Definition 3 ([11])**.** Let $\mathbb{X}$ be an arbitrary set, $\circ$ be a continuous t–norm, $\bigtriangledown$ be a continuous t–conorm, and μ, ν be fuzzy sets on $\mathbb{X}^2 \times (0, \infty)$. If μ and ν satisfy the following conditions: for all $x_1, x_2, x_3 \in \mathbb{X}$ and $u, s > 0$,

(1) $\mu(x_1, x_2, u) + \nu(x_1, x_2, u) \leq 1$;
(2) $\mu(x_1, x_2, u) > 0$;
(3) $\mu(x_1, x_2, u) = 1 \Leftrightarrow x_1 = x_2$;
(4) $\mu(x_1, x_2, u) = \mu(x_2, x_1, u)$;
(5) $\mu(x_1, x_3, u + s) \geq \mu(x_1, x_2, u) \circ \mu(x_2, x_3, s)$;
(6) The function $(\mu)_{x_1 x_2} : (0, \infty) \to (0, 1]$ is continuous;
(7) $\nu(x_1, x_2, u) > 0$;
(8) $\nu(x_1, x_2, u) = 0 \Leftrightarrow x_1 = x_2$;
(9) $\nu(x_1, x_2, u) = \nu(x_2, x_1, u)$;
(10) $\nu(x_1, x_3, u + s) \leq \nu(x_1, x_2, u) \bigtriangledown \nu(x_2, x_3, s)$;
(11) The function $(\nu)_{x_1 x_2} : (0, \infty) \to (0, 1]$ is continuous;

then a 5-tuple $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ is said to be an intuitionistic fuzzy metric space.

The functions $\mu(x_1, x_2, u)$ and $\nu(x_1, x_2, u)$ denote the degree of nearness and the degree of non-nearness between x_1 and x_2 concerning u, respectively.

Example 2 ([11])**.** Let $(\mathbb{X}, d)$ be a metric space. Define $a \circ b = ab$ and $a \bigtriangledown b = \min\{a + b, 1\}$, for all $a, b \in [0, 1]$, and let μ and ν be fuzzy sets on $\mathbb{X}^2 \times (0, \infty)$ defined as

$$\mu(x_1, x_2, u) = \frac{u}{u + d(x_1, x_2)}, \quad \nu(x_1, x_2, u) = \frac{d(x_1, x_2)}{u + d(x_1, x_2)}$$

for $x_1, x_2 \in \mathbb{X}$ and $u > 0$. Then $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ is an IFMS.

Remark 2 ([4])**.** Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, $(\mathbb{X}, \mu, \circ)$ is a FMS. Conversely, if $(\mathbb{X}, \mu, \circ)$ is a FMS, then $(\mathbb{X}, \mu, 1 - \mu, \circ, \bigtriangledown)$ is an IFMS, where $a \bigtriangledown b = 1 - [(1 - a) \circ (1 - b)]$, for all $a, b \in [0, 1]$.

Park [11] introduced a comprehensive definition of convergence of sequence in IFMSs as below.

Definition 4 ([11]). Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{X}$ is said to be convergent to $x_0 \in \mathbb{X}$, if for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n \geq n_\varepsilon$ implies

$$\mu(x_n, x_0, u) > 1 - \varepsilon, \quad \nu(x_n, x_0, u) < \varepsilon$$

or equivalently

$$\lim_{n\to\infty} \mu(x_n, x_0, u) = 1, \quad \lim_{n\to\infty} \nu(x_n, x_0, u) = 0$$

and is denoted by $\frac{\mu}{\nu} - \lim_{n\to\infty} x_n = x_0$ or $x_n \xrightarrow{\frac{\mu}{\nu}} x_0$ as $n \to \infty$.

Definition 5 ([11]). Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, a sequence (x_n) is referred to as Cauchy sequence in $\mathbb{X}$, if for all $u > 0$ and $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n, N \geq n_\varepsilon$ implies

$$\mu(x_n, x_N, u) > 1 - \varepsilon, \quad \nu(x_n, x_N, u) < \varepsilon$$

or equivalently

$$\lim_{n,N\to\infty} \mu(x_n, x_N, u) = 1, \quad \lim_{n,N\to\infty} \nu(x_n, x_N, u) = 0.$$

Definition 6 ([4]). Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, a sequence (x_n) is called statistically convergent to $x_0 \in \mathbb{X}$, if for all $\varepsilon \in (0,1)$ and $u > 0$,

$$\delta(\{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{|\{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\}|}{n} = 0.$$

Example 3 ([4]). Let $\mathbb{X} = \mathbb{R}$, $a \circ b = ab$, and $a \bigtriangledown b = \min\{a + b, 1\}$ for all $a, b \in [0,1]$. Define μ and ν by

$$\mu(x_1, x_2, u) = \frac{u}{u + |x_1 - x_2|}, \quad \nu(x_1, x_2, u) = \frac{|x_1 - x_2|}{u + |x_1 - x_2|}$$

for all $x_1, x_2 \in \mathbb{X}$ and $u > 0$. Then, $(\mathbb{R}, \mu, \nu, \circ, \bigtriangledown)$ is an IFMS. Now define a sequence (x_n) by

$$x_n := \begin{cases} 1, & \forall k \in \mathbb{N}, n = k^2; \\ 0, & \exists k \in \mathbb{N} \ni n \neq k^2. \end{cases}$$

Then, (x_n) is statistically convergent to 0.

Definition 7 ([4]). Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, a sequence (x_n) is called statistically Cauchy sequence, if for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $N \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : \mu(x_n, x_N, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_N, u) \geq \varepsilon\}) = 0.$$

An interesting generalization of statistical convergence was introduced by Kostryko et al. [18] with the help of an admissible ideal I of subsets of $\mathbb{N}$, the set of positive integers. Next, we recall the basic terminology used by the authors to define this new type of convergence.

Definition 8 ([18])**.** Let $\mathbb{X}$ be a non-empty set. A family of subsets $I \subseteq P(\mathbb{X})$ is referred to as an ideal in $\mathbb{X}$, if

(1) $\emptyset \in I$;
(2) $T, S \in I \Rightarrow T \cup S \in I$;
(3) $(T \in I \wedge S \subseteq T) \Rightarrow S \in I$.

Definition 9 ([18])**.** Let I be an ideal in $\mathbb{X}$. Then, I is called non-trivial ideal such that $P(\mathbb{X}) \neq I$ and $I \neq \emptyset$. Additionally, I is defined admissible ideal, which is a non-trivial ideal $I \subseteq P(\mathbb{X})$, if $\{\{x\} : x \in \mathbb{X}\} \subseteq I$.

Example 4 ([18])**.** Let $\mathbb{N} = \bigcup\limits_{k=1}^{\infty} T_k$ be a decomposition of $\mathbb{N}$, assume that T_k $(k = 1, 2, \dots)$ are infinite sets. Express by $\mathcal{K}$ the family of all $A \subseteq \mathbb{N}$ such that A coincides only a finite number of T_k. Then, it is easy to see that $\mathcal{K}$ is an admissible ideal in $\mathbb{N}$.

Definition 10 ([18])**.** Let $I \subseteq P(\mathbb{N})$ be an admissible ideal, (P_i) be a sequence of mutually disjoint sets of I, and (R_i) be a subset of $\mathbb{N}$. Then, I satisfies the condition (AP), if for all (P_i), there is a sequence (R_i) such that for all $i \in \mathbb{N}$, $P_i \Delta R_i$ is finite and $R = \bigcup\limits_i R_i \in I$. Here, Δ denotes the symmetric difference. It must be noted that $R_i \in I$.

Definition 11 ([18])**.** Let $\mathbb{X}$ be a non-empty set. A family of subsets $\emptyset \neq F \subseteq P(\mathbb{X})$ is referred to as a filter in $\mathbb{X}$, if

(1) $\emptyset \notin F$;
(2) $T, S \in F \Rightarrow T \cap S \in F$;
(3) $(S \in F \wedge S \subseteq T) \Rightarrow T \in F$.

Remark 3 ([18])**.** The filter $F(I) = \{\mathbb{X} \setminus S : S \in I\}$ in $\mathbb{X}$ is called the associated filter with ideal I.

Proposition 1 ([3])**.** *Let $I \subseteq P(\mathbb{N})$ be an admissible ideal with the condition (AP), (P_i) be a countable collection of subsets of $\mathbb{N}$, and $(P_i) \in F(I)$. Then, there exists a set $P \subset \mathbb{N}$ such that $P \in F(I)$ and for all i, $P \setminus P_i$ is finite.*

Definition 12 ([18])**.** Let I be a non-trivial ideal in $\mathbb{N}$. A sequence (x_n) in $\mathbb{R}$ is called ideal convergent (I–convergent) to $x_0 \in \mathbb{R}$, if for all $\varepsilon > 0$,

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\} \in I$$

and is denoted by $I - \lim\limits_{n\to\infty} x_n = x_0$ or $x_n \xrightarrow{I} x_0$ as $n \to \infty$.

Here, if I is an admissible ideal, then convergence in the ordinary sense implies I–convergence.

Definition 13 ([18])**.** Let I be a non-trivial ideal in $\mathbb{N}$. A sequence (x_n) is referred to as I^*–convergent to $x_0 \in \mathbb{R}$, if there exists a set

$$H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \in F(I)$$

such that

$$\lim_{\substack{h_k \to \infty \\ h_k \in H}} x_{h_k} = x_0.$$

Definition 14 ([3])**.** Let I be an admissible ideal in $\mathbb{N}$. A sequence (x_n) is called an ideal Cauchy (I–Cauchy) sequence in $\mathbb{R}$, if for all $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in I.$$

Definition 15 ([3])**.** Let I be an admissible ideal in $\mathbb{N}$. A sequence (x_n) is referred to as an I^*–Cauchy sequence in $\mathbb{R}$, if there exists a set

$$H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \in F(I)$$

such that

$$\lim_{\substack{h_k, h_p \to \infty \\ h_k, h_p \in H}} |x_{h_k} - x_{h_p}| = 0.$$

3. ${}^{\mu}_{\nu}I$–CONVERGENCE AND ${}^{\mu}_{\nu}I$–CAUCHY SEQUENCES

This section defines the concepts of ideal convergence and ideal Cauchy sequences in IFMSs. In addition, it provides some of basic properties.

Definition 16. Let I non-trivial ideal in $\mathbb{N}$ and $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{X}$ is said to be ideal convergent to $x_0 \in \mathbb{X}$, if for all $u > 0$ and $\varepsilon \in (0, 1)$,

$$A(u, \varepsilon) = \{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon, \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\} \in I$$

and is denoted by ${}^{\mu}_{\nu}I - \lim_{n \to \infty} x_n = x_0$ or $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$ as $n \to \infty$. The number x_0 is called ${}^{\mu}_{\nu}I$-limit of the sequence (x_n).

Example 5. If we take

$$I = I_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$$

and

$$I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\},$$

then ${}^{\mu}_{\nu}I$–convergence is the same as ordinary convergence and statistical convergence in IFMS, respectively.

Remark 4. The ordinary convergence in IFMSs implies ${}^{\mu}_{\nu}I$–convergence, if I is an admissible ideal.

Proof. Let $x_n \overset{\mu}{\underset{\nu}{\to}} x_0$ and I is an admissible ideal. In this case, for all $u > 0$ and $\varepsilon \in (0,1)$, there exists a positive integer n_0 such that $n \geq n_0$ implies

$$\mu(x_n, x_0, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_n, x_0, u) < \varepsilon,$$

$$\begin{aligned} K &= \{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\} \\ &\subseteq \mathbb{N} \setminus \{n_0 + 1, n_0 + 2, \dots\}. \end{aligned}$$

Since the set of K is finite and I is an admissible ideal, $K \in I$. Hence, ${}^{\mu}_{\nu}I - \lim\limits_{n\to\infty} x_n = x_0$. □

Next, we shall explore the compatibility of ideal convergence with various convergence axioms. Presented below are the widely recognized axioms of classical convergence:

I A constant sequence $(x_0, x_0, \dots, x_0, \dots)$ converges to x_0;

II The limit of a convergent sequence is unique;

III Every subsequence of the converged sequence is convergent and has the same limit.

Theorem 1. *Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS and (x_n) be a sequence in $\mathbb{X}$.*

(1) *The ${}^{\mu}_{\nu}I$–convergence satisfies (I) and (II).*

(2) *Every subsequence of an ${}^{\mu}_{\nu}I$–convergent sequence is not ${}^{\mu}_{\nu}I$–convergent, if I contains an infinite set.*

Proof.

(1) It is obvious that ${}^{\mu}_{\nu}I$–convergence satisfies the proposition (I). We prove that it satisfies (II) as well. Suppose that $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$, $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_1$, and $x_0 \neq x_1$. Choose $u > 0$ and $\varepsilon = \frac{1}{n}$, $(n = 2, 3, \dots)$. Then, by assumption and Remark 3 the sets

$$\mathbb{N} \setminus A = \{n \in \mathbb{N} : \mu(x_n, x_1, u) > 1 - \varepsilon, \ \text{ and } \ \nu(x_n, x_1, u) < \varepsilon\} \in F(I),$$

$$\mathbb{N} \setminus B = \{n \in \mathbb{N} : \mu(x_n, x_2, u) > 1 - \varepsilon \ \text{ and } \ \nu(x_n, x_2, u) < \varepsilon\} \in F(I).$$

But then the set $(\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$ belongs to $F(I)$, too. Hence, there is an $m \in \mathbb{N}$ such that

$$\mu(x_m, x_1, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_m, x_1, u) < \varepsilon,$$

$$\mu(x_m, x_2, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_m, x_2, u) < \varepsilon.$$

From this $\mu(x_1, x_2, u) = 1$ and $\nu(x_1, x_2, u) = 0$ which is a contradiction to $x_1 \neq x_2$.

(2) Suppose that an infinite set $A = \{n_1 < n_2 < \cdots < n_k < \cdots\} \subseteq \mathbb{N}$ belongs to I. Put

$$B = \mathbb{N} \setminus A = \{m_1 < m_2 < \cdots < m_k < \cdots\}.$$

The set B is infinite because in the opposite case $\mathbb{N}$ would belong to I. Define the sequence (x_n) as follows $x_{n_k} = x_0$, $x_{m_k} = x_1$, $k \in \mathbb{N}$.

Obviously ${}^{\mu}_{\nu}I - \lim_{n\to\infty} x_n = x_0$. In addition, the sequence (x_{m_k}) of (x_n) is constant and thus ${}^{\mu}_{\nu}I - \lim_{m_k\to\infty} x_{m_k} = x_1$ (see proposition (I)). Hence, ${}^{\mu}_{\nu}I$–convergence does not satisfy the proposition (III). $\square$

Definition 17. Let I be an admissible ideal in $\mathbb{N}$ and $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{X}$ is said to be ${}^{\mu}_{\nu}I$–Cauchy sequence, if for all $u > 0$ and $\varepsilon \in (0,1)$, there exists an integer $N \in \mathbb{N}$ such that

$$A(u,\varepsilon) = \{n \in \mathbb{N} : \mu(x_n, x_N, u) \leq 1-\varepsilon \quad \text{or} \quad \nu(x_n, x_N, u) \geq \varepsilon\} \in I.$$

Theorem 2. *Let I be an admissible ideal in $\mathbb{N}$, $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS and (x_n) is a sequence in $\mathbb{X}$. If the sequence (x_n) is a ${}^{\mu}_{\nu}I$–convergent sequence in $\mathbb{X}$, then it is ${}^{\mu}_{\nu}I$–Cauchy sequence in $\mathbb{X}$.*

Proof. Let $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$. Then, for all $u > 0$ and $\varepsilon \in (0,1)$, we have

$$A(u,\varepsilon) = \{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1-\varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\} \in I.$$

Because of the definition of an admissible ideal, there exists an $N \notin A(u,\varepsilon)$. Assume that

$$B = \{n \in \mathbb{N} : \mu(x_n, x_N, u) \leq 1-\varepsilon \quad \text{or} \quad \nu(x_n, x_N, u) \geq \varepsilon\}.$$

Consider the following inequalities

$$\mu(x_n, x_N, u) \geq \mu\left(x_n, x_0, \frac{u}{2}\right) \circ \mu\left(x_N, x_0, \frac{u}{2}\right),$$

$$\nu(x_n, x_N, u) \leq \nu\left(x_n, x_0, \frac{u}{2}\right) \triangledown \nu\left(x_N, x_0, \frac{u}{2}\right).$$

Let $n \in B$. Then, $\mu(x_n, x_N, u) \leq 1-\varepsilon$ or $\nu(x_n, x_N, u) \geq \varepsilon$.
If $\mu(x_n, x_N, u) \leq 1-\varepsilon$, then

$$(1-\varepsilon) \circ (1-\varepsilon) \geq \mu\left(x_n, x_0, \frac{u}{2}\right) \circ \mu\left(x_N, x_0, \frac{u}{2}\right).$$

Moreover, we have $\mu(x_N, x_0, u) > 1-\varepsilon$ because $N \notin A(u,\varepsilon)$. Hence, $\mu(x_n, x_0, u) \leq 1-\varepsilon$, then $n \in A(u,\varepsilon)$. In this case, $B \subseteq A(u,\varepsilon) \in I$ for all $u > 0$ and $\varepsilon \in (0,1)$. Similarly, we observe that if $\nu(x_n, x_N, u) \geq \varepsilon$, then $B \subseteq A(u,\varepsilon) \in I$ for all $u > 0$ and $\varepsilon \in (0,1)$. Consequently, (x_n) is an ${}^{\mu}_{\nu}I$–Cauchy sequence in $\mathbb{X}$. $\square$

4. ${}^{\mu}_{\nu}I^*$–CONVERGENCE AND ${}^{\mu}_{\nu}I^*$–CAUCHY SEQUENCES

Varol [4] proved that a sequence (x_n) in an IFMSs $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ is statistically convergent to $x_0 \in \mathbb{X}$ if and only if there exists an increasing index sequence $K = \{k_1 < k_2 < \cdots\}$ of natural numbers such that $\delta(K) = 1$ and

$${}^{\mu}_{\nu} - \lim_{\substack{k_n\to\infty \\ k_n \in K}} x_{k_n} = x_0.$$

We use this result to introduce the concept of I^*–convergence in an IFMS as follows.

Definition 18. Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{X}$ is said to be I^*–convergent to $x_0 \in \mathbb{X}$, if there exists a subset $H = \{h_1 < h_2 < \cdots\} \in F(I)$ such that

$$\tag{1} {}^{\mu}_{\nu}\!-\lim_{\substack{h_k \to \infty \\ h_k \in H}} x_{h_k} = x_0.$$

The element x_0 is called the I^*–limit of the sequence (x_n) and we write ${}^{\mu}_{\nu}I^* - \lim\limits_{n\to\infty} x_n = x_0$.

Theorem 3. *Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS and (x_n) be a sequence in $\mathbb{X}$. If $x_n \xrightarrow{{}^{\mu}_{\nu}I^*} x_0$, then $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$.*

Proof. By hypothesis, there is a set $K \in I$ such that (1) holds, where

$$H = \mathbb{N} \setminus K = \{h_1 < h_2 < \cdots < h_k < \cdots\}.$$

Let $u > 0$ and $\varepsilon \in (0, 1)$. By (1), there is a $k_0 \in \mathbb{N}$, such that $\mu(x_n, x_0, u) > 1 - \varepsilon$ and $\nu(x_n, x_0, u) < \varepsilon$ for $n > k_0$. Put

$$A(u, \varepsilon) = \{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\}.$$

Then,

$$A(u, \varepsilon) \subseteq K \cup \{h_1, h_2, \ldots, h_{k_0}\}.$$

Since I is an admissible ideal and $K \in I$,

$$K \cup \{h_1, h_2, \ldots, h_{k_0}\} \in I$$

and therefore $A(u, \varepsilon) \in I$. □

The following Example 6 states that the converse of Theorem 3 does not always hold.

Example 6. Assume that $(\mathbb{R}, |.|)$ denotes the space of real numbers with the usual metric, and let $a \circ b = ab$, $a \bigtriangledown b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Define μ and ν by

$$\mu(x_1, x_2, u) = \frac{u}{u + |x_1 - x_2|} \quad \text{and} \quad \nu(x_1, x_2, u) = \frac{|x_1 - x_2|}{u + |x_1 - x_2|}$$

for all $x_1, x_2 \in \mathbb{R}$ and $u > 0$. Put $I = \mathcal{K}$ (see Example 4). Suppose that x_0 is accumulation point of $\mathbb{R}$. Hence, there exists a sequence (x_n) in $\mathbb{R}$ such that ${}^{\mu}_{\nu} - \lim\limits_{n\to\infty} x_n = x_0$. Define

$$y_n := \begin{cases} x_j, & \text{if } n \in T_j,\ j = 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

We choose $u > 0$ and $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$ for $\varepsilon \in (0, 1)$. Therefore,

$$A(u, \varepsilon) \subseteq \left\{ n \in \mathbb{N} : \mu(y_n, x_0, u) \leq 1 - \frac{1}{m} \right\} \subseteq \bigcup_{s=1}^{m} T_s,$$

where $A(u,\varepsilon) = \{n \in \mathbb{N} : \mu(y_n, x_0, u) \leq 1 - \varepsilon\}$. Hence, according to the notion of ideal $A(u,\varepsilon) \in \mathcal{I}$ and so ${}^{\mu}_{\nu}I - \lim_{n\to\infty} y_n = x_0$. Now, assume that ${}^{\mu}_{\nu}I^* - \lim_{n\to\infty} y_n = x_0$. Then, there exists a set $H = \{m_k : t > k, m_k < m_t\} \in \mathcal{I}$ such that

$$ {}^{\mu}_{\nu} - \lim_{\substack{m_k\to\infty \\ m_k\in H}} y_n = x_0. $$

From the notion of ideal, there exists a $s \in \mathbb{N}$ such that

$$ H \subseteq T_1 \cup T_2 \cup \cdots \cup T_s. $$

But by notation used in proof of Theorem 3 and $T_{s+1} \subseteq \mathbb{N} \setminus H$, we have $y_{m_k} = 0$ for infinitely many of m_k's. Consequently, ${}^{\mu}_{\nu} \lim_{m_k\to\infty} y_{m_k} = x_0$ can not be true.

Theorem 4. *Let* $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ *be an IFMS, I be an admissible ideal in* $\mathbb{N}$*, (x_n) be a sequence in* $\mathbb{X}$*, and* $x_0 \in \mathbb{X}$*.*

(1) *If I has the condition (AP), then* ${}^{\mu}_{\nu}I - \lim_{n\to\infty} x_n = x_0$ *implies* ${}^{\mu}_{\nu}I^* - \lim_{n\to\infty} x_n = x_0$.

(2) *If* $\mathbb{X}$ *has at least one accumulation point and* ${}^{\mu}_{\nu}I - \lim_{n\to\infty} x_n = x_0$ *implies* ${}^{\mu}_{\nu}I^* - \lim_{n\to\infty} x_n = x_0$*, then I has the property (AP).*

Proof.

(1) Let $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$ and I satisfies the condition (AP). Then, for all $u > 0$ and $\varepsilon \in (0,1)$ the set

$$ A(u,\varepsilon) = \{n : \mu(x_n, x_0, u) \leq 1 - \varepsilon \quad \text{or} \quad \nu(x_n, x_0, u) \geq \varepsilon\} \in I. $$

Consequently, each of the following sets $P_k \in I$ $(k = 1, 2, \dots)$

$$ P_1 = \left\{ n \in \mathbb{N} : \mu(x_n, x_0, u) \leq \frac{1}{2} \quad \text{or} \quad \nu(x_n, x_0, u) \geq \frac{1}{2} \right\} $$

$$ P_k = \left\{ n \in \mathbb{N} : \frac{k-1}{k} < \mu(x_n, x_0, u) \leq \frac{k}{k+1} \quad \text{or} \quad \frac{1}{k+1} \leq \nu(x_n, x_0, u) < \frac{1}{k} \right\} $$

for $k \geq 2$. Obviously $P_i \cap P_j = \emptyset$ for $i \neq j$. Since I satisfies (AP), there exist sets $R_j \subseteq \mathbb{N}$ such that $P_j \Delta R_j$ is a finite set $(j = 1, 2, \dots)$ and $R = \bigcup_{j=1}^{\infty} R_j \in I$.

It suffices to prove that

$$ {}^{\mu}_{\nu} - \lim_{\substack{n\to\infty \\ n\in H}} x_n = x_0, \tag{2} $$

where $H = \mathbb{N} \setminus R$.

Let $\lambda \in (0,1)$ and $u > 0$. Choose a $m \in \mathbb{N}$ such that $\frac{1}{m} < \lambda$. Then,

$$\{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \lambda \quad \text{or} \quad \nu(x_n, x_0, u) \geq \lambda\} \subseteq \bigcup_{j=1}^{m+1} P_j.$$

The set on right-hand side belongs to I by the additivity of I. Since $P_j \Delta R_j$ is finite ($j = 1, 2, \dots$), there is an $n_\varepsilon \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{m+1} R_j \cap (n_\varepsilon, \infty) = \bigcup_{j=1}^{m+1} P_j \cap (n_\varepsilon, \infty).$$

If we now $n \notin R$, $n > n_\varepsilon$, then $n \notin \bigcup_{j=1}^{m+1} R_j$ and thus $n \notin \bigcup_{j=1}^{m+1} P_j$. But then

$$n \in \{n \in \mathbb{N} : \mu(x_n, x_0, u) > 1 - \lambda \quad \text{and} \quad \nu(x_n, x_0, u) < \lambda\}.$$

Hence, (2) holds.

(2) Suppose $x_0 \in \mathbb{X}$ is an accumulation point of $\mathbb{X}$. Then, there exists a sequence (y_n) of distinct elements of $\mathbb{X}$ such that $y_n \neq x_0$ for any n, and ${}^{\mu}_{\nu} - \lim_{n\to\infty} y_n = x_0$. Let $\{P_1, P_2, \dots\}$ be a disjoint family of nonempty sets in I. Define a sequence (x_k) in the following way: $x_k = y_n$ if $k \in P_j$ and $x_k = x_0$ if $k \notin P_j$, for all j. Let $\eta \in (0,1)$ and $u > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$. Then,

$$A(u, \eta) \subseteq \bigcup_{j=1}^{n+1} P_j \tag{3}$$

where $A(u, \eta) = \{k \in \mathbb{N} : \mu(x_k, x_0, u) \leq 1 - \eta \text{ or } \nu(x_k, x_0, u) \geq \eta\}$. Hence, $A(u, \eta) \in I$ and ${}^{\mu}_{\nu}I - \lim_{k\to\infty} x_k = x_0$. By virtue of our assumption, we have ${}^{\mu}_{\nu}I^* - \lim_{k\to\infty} x_k = x_0$. Therefore, there exists a set $R \in I$ such that $H = \mathbb{N} \setminus R \in F(I)$ and

$${}^{\mu}_{\nu}\!-\lim_{\substack{k_n\to\infty \\ k_n \in H}} x_{k_n} = x_0 \tag{4}$$

Put $R_j = P_j \cap R$ for $j \in \mathbb{N}$. Then, $R_j \in I$ for all $j \in \mathbb{N}$. Moreover,

$$\bigcup_{j=1}^{\infty} R_j = R \cap \bigcup_{j=1}^{\infty} P_j \subset R$$

and thus $\bigcup_{j=1}^{\infty} R_j \in I$. Since (4), for all $\eta \in (0,1)$ and $u > 0$,

$$B = \{k_n \in \mathbb{N} : \mu(x_{k_n}, x_0, u) \leq 1 - \eta \quad \text{or} \quad \nu(x_{k_n}, x_0, u) \geq \eta\} \subset H$$

and B is finite. Since (3), $H \cap P_j$ is finite. In addition,

$$P_j \Delta R_j = P_j \setminus R_j = P_j \setminus R = P_j \cap H$$

and $P_j \Delta R_j$ is finite. This proves that ideal I has the property (AP). □

Theorem 5. *Let I be an admissible ideal in $\mathbb{N}$ and $\mathbb{X}$ be an IFMS. If $\mathbb{X}$ has no accumulation point, then ${}^{\mu}_{\nu}I$–convergence and ${}^{\mu}_{\nu}I^*$–convergence are the same.*

Proof. Let $x_0 \in \mathbb{X}$ and $x_n \xrightarrow{{}^{\mu}_{\nu}I} x_0$. Thanks to Theorem 3, it suffices to prove that $x_n \xrightarrow{{}^{\mu}_{\nu}I^*} x_0$ as $n \to \infty$. Since $\mathbb{X}$ has no accumulation points, there exists $u > 0$ and $\varepsilon \in (0,1)$ such that

$$B(x_0, \varepsilon, u) = \{x \in \mathbb{X} : \mu(x_n, x_0, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_n, x_0, u) < \varepsilon\} = \{x_0\}$$

From the assumption $\{n \in \mathbb{N} : \mu(x_n, x_0, u) \leq 1 - \varepsilon \text{ or } \nu(x_n, x_0, u) \geq \varepsilon\} \in I$. Hence,

$$\{n \in \mathbb{N} : \mu(x_n, x_0, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_n, x_0, u) < \varepsilon\} = \{n \in \mathbb{N} : x_n = x_0\} \in F(I)$$

and obviously $x_n \xrightarrow{{}^{\mu}_{\nu}I^*} x_0$. □

Definition 19. Let I be an admissible ideal in $\mathbb{N}$ and $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS. Then, a sequence (x_n) is referred to as ${}^{\mu}_{\nu}I^*$–Cauchy sequence in $\mathbb{X}$, if there exists a set

$$H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \in F(I)$$

such that

$$\lim_{\substack{h_k, h_p \to \infty \\ h_k, h_p \in H}} \mu(x_{h_k}, x_{h_p}, u) = 1 \quad \text{and} \quad \lim_{\substack{h_k, h_p \to \infty \\ h_k, h_p \in H}} \nu(x_{h_k}, x_{h_p}, u) = 0. \tag{5}$$

Theorem 6. *If a sequence (x_n) is an ${}^{\mu}_{\nu}I^*$–Cauchy sequence, then it is ${}^{\mu}_{\nu}I$–Cauchy, for all I is an admissible ideal in $\mathbb{N}$.*

Proof. Suppose that (x_n) be an ${}^{\mu}_{\nu}I^*$–Cauchy sequence. In that case, there exists a set

$$H = \mathbb{N} \setminus K = \{h_1 < h_2 < \cdots < h_k < \cdots\} \in F(I)$$

such that $\mu(x_{h_k}, x_{h_p}, u) > 1 - \varepsilon$ and $\nu(x_{h_k}, x_{h_p}, u) < \varepsilon$, for all $u > 0$, $\varepsilon \in (0,1)$ and $h_k, h_p > k_0$. We choose $N = h_{k_0+1}$. Then, for all $u > 0$ and $\varepsilon \in (0,1)$,

$$\mu(x_{h_k}, x_N, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_{h_k}, x_N, u) < \varepsilon, \quad h_k > k_0.$$

Hence, $K \in I$ and

$$A(u, \varepsilon) \subset K \cup \{h_1 < h_2 < \cdots < h_{k_0}\}, \tag{6}$$

where $A(u,\varepsilon) = \{h_k : \mu(x_{h_k}, x_N, u) \leq 1-\varepsilon \text{ or } \nu(x_{h_k}, x_N, u) \geq \varepsilon\}$. From here

$$K \cup \{h_1 < h_2 < \cdots < h_{k_0}\} \in I.$$

Consequently, the sequence (x_n) is an ${}^{\mu}_{\nu}I$–Cauchy sequence. □

Theorem 7. *Let I be an admissible ideal in $\mathbb{N}$ and $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS. ${}^{\mu}_{\nu}I$–Cauchy sequence in $\mathbb{X}$ implies that ${}^{\mu}_{\nu}I^*$–Cauchy sequence in $\mathbb{X}$ if and only if the I ideal has the condition (AP).*

Proof. Suppose that a sequence (x_n) be an ${}^{\mu}_{\nu}I$–Cauchy sequence in $\mathbb{X}$ and the I ideal has the condition (AP). Then, there exists an $N(\varepsilon)$ such that for all $u > 0$ and $\varepsilon \in (0,1)$

$$\{n \in \mathbb{N} : \mu(x_n, x_N, u) \leq 1-\varepsilon \quad \text{or} \quad \nu(x_n, x_N, u) \geq \varepsilon\} \in I.$$

We choose

$$S_i = \left\{ n \in \mathbb{N} : \mu(x_n, x_{m_i}, u) > \frac{i-1}{i} \quad \text{and} \quad \nu(x_n, x_{m_i}, u) < \frac{1}{i} \right\},$$

for $i = 1, 2, \ldots,$ where $m_i = N(\frac{1}{i})$. $S_i \in F(I)$ is obvious for $i = 1, 2, \ldots$. Since I has the condition (AP), then by Proposition 1 there exists a set $S \in F(I)$, and $S \setminus S_i$ is finite for all i. We prove that

$$\lim_{\substack{n,m\to\infty \\ n,m\in S}} \mu(x_n, x_m, u) = 1 \quad \text{and} \quad \lim_{\substack{n,m\to\infty \\ n,m\in S}} \nu(x_n, x_m, u) = 0.$$

Assume that $\varepsilon \in (0,1)$, $u > 0$ and $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon}$. If $n, m \in S$, then $S \setminus S_k$ is a finite set. Hence, there exists $j = j(k)$ such that $m \in S_k$ and $n \in S_k$ for all $m, n > j(k)$. Thus,

$$\mu(x_n, x_{m_k}, u) > \frac{k-1}{k} \quad \text{and} \quad \mu(x_m, x_{m_k}, u) > \frac{k-1}{k},$$

$$\nu(x_n, x_{m_k}, u) < \frac{1}{k} \quad \text{and} \quad \nu(x_n, x_{m_k}, u) < \frac{1}{k},$$

for all $n, m > j(k)$. In that case,

$$\mu(x_n, x_m, u) \geq \mu\left(x_n, x_{m_k}, \frac{u}{2}\right) \circ \mu\left(x_m, x_{m_k}, \frac{u}{2}\right) > (1-\varepsilon) \circ (1-\varepsilon) = \delta(\varepsilon),$$

$$\nu(x_n, x_m, u) \leq \nu\left(x_n, x_{m_k}, \frac{u}{2}\right) \triangledown \nu\left(x_m, x_{m_k}, \frac{u}{2}\right) < \varepsilon \triangledown \varepsilon = \delta(\varepsilon)$$

for $m, n > j(k)$. Consequently, the proof is complete. □

Theorem 8. *Let I be an admissible ideal in $\mathbb{N}$ and $(\mathbb{X}, \mu, \nu, \circ, \triangledown)$ be an IFMS. If a sequence in $\mathbb{X}$ is an ${}^{\mu}_{\nu}I^*$–convergent sequence, then it is an ${}^{\mu}_{\nu}I^*$–Cauchy sequence.*

Proof. Let $x_n \xrightarrow{{}^{\mu}_{\nu}I^*} x_0$. Then, we have

$$H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \in F(I)$$

such that

$$\tfrac{\mu}{\nu} - \lim_{\substack{h_k \to \infty \\ h_k \in H}} x_{h_k} = x_0.$$

Consider the following inequalities

$$\mu(x_{h_k}, x_{h_p}, u) \geq \mu\left(x_{h_k}, x_0, \frac{u}{2}\right) \circ \mu\left(x_{h_p}, x_0, \frac{u}{2}\right) > (1-\varepsilon) \circ (1-\varepsilon) = \delta(\varepsilon),$$

$$\nu(x_{h_k}, x_{h_p}, u) \leq \nu\left(x_{h_k}, x_0, \frac{u}{2}\right) \bigtriangledown \nu\left(x_{h_p}, x_0, \frac{u}{2}\right) < \varepsilon \bigtriangledown \varepsilon = \delta(\varepsilon),$$

we observe that

$$\lim_{\substack{h_k, h_p \to \infty \\ h_k, h_p \in H}} \mu(x_{h_k}, x_{h_p}, u) = 1 \quad \text{and} \quad \lim_{\substack{h_k, h_p \to \infty \\ h_k, h_p \in H}} \nu(x_{h_k}, x_{h_p}, u) = 0.$$

Consequently, the sequence (x_n) is an ${}^{\mu}_{\nu}I^*$–Cauchy sequence. □

5. ${}^{\mu}_{\nu}I$–LIMIT POINTS AND ${}^{\mu}_{\nu}I$–CLUSTER POINTS

This section defines the notions of ${}^{\mu}_{\nu}I$–limit points and ${}^{\mu}_{\nu}I$–cluster points in IFMS. Moreover, it analyses the connection between these concepts. Finally, it studies that set of ${}^{\mu}_{\nu}I$–cluster points is closed.

Definition 20. Let I be a non-trivial ideal in $\mathbb{N}$, $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS, and $x = (x_n)$ be a sequence in $\mathbb{X}$. Then, an element $x_0 \in \mathbb{X}$ is referred to as an ${}^{\mu}_{\nu}I$–limit point of x, if there is a set $H = \{h_1 < h_2 < \cdots\} \notin I$ and $\tfrac{\mu}{\nu} - \lim_{\substack{h_k \to \infty \\ h_k \in H}} x_{h_k} = x_0$.

Definition 21. Let I be a non-trivial ideal in $\mathbb{N}$, $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS, and $x = (x_n)$ be a sequence in $\mathbb{X}$. Then, an element $x_0 \in \mathbb{X}$ is called an ${}^{\mu}_{\nu}I$–cluster point of x, if for all $u > 0$ and $\varepsilon \in (0, 1)$

$$\{n \in \mathbb{N} : \mu(x_n, x_0, u) > 1 - \varepsilon \quad \text{and} \quad \nu(x_n, x_0, u) < \varepsilon\} \notin I.$$

The set of all ${}^{\mu}_{\nu}I$–limit points and ${}^{\mu}_{\nu}I$–cluster points of a sequence x are denoted by ${}^{\mu}_{\nu}I(\Lambda_x)$ and ${}^{\mu}_{\nu}I(\Gamma_x)$, respectively.

Proposition 2. *Let I be an admissible ideal in $\mathbb{N}$, $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS, and $x = (x_n)$ be a sequence in $\mathbb{X}$. Then, ${}^{\mu}_{\nu}I(\Lambda_x) \subset {}^{\mu}_{\nu}I(\Gamma_x)$.*

Proof. Let $x_0 \in {}^{\mu}_{\nu}I(\Lambda_x)$, then there exists a set $H = \{h_1 < h_2 < \cdots\} \notin I$ such that

$$\tfrac{\mu}{\nu} - \lim_{\substack{h_k \to \infty \\ h_k \in H}} x_{h_k} = x_0. \tag{7}$$

Take $u > 0$ and $\varepsilon \in (0,1)$. According to (7), there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$, $\mu(x_{h_k}, x_0, u) > 1-\varepsilon$ and $\nu(x_{h_k}, x_0, u) < \varepsilon$. Hence,

$$H \backslash \{h_1, h_2, \ldots, h_{k_0}\} \subset \{n \in \mathbb{N} : \mu(x_n, x_0, u) > 1-\varepsilon \quad \text{and} \quad \nu(x_n, x_0, u) < \varepsilon\}$$

and thus $\{n \in \mathbb{N} : \mu(x_n, x_0, u) > 1-\varepsilon$ and $\nu(x_n, x_0, u) < \varepsilon\} \notin I$ which means that $x_0 \in {}^{\mu}_{\nu}I(\Gamma_x)$. □

Theorem 9. *Let $(\mathbb{X}, \mu, \nu, \circ, \bigtriangledown)$ be an IFMS and $x = (x_n)$ be a sequence in $\mathbb{X}$. Then, the set ${}^{\mu}_{\nu}I(\Gamma_x)$ is closed in $\mathbb{X}$, if I is an admissible ideal in $\mathbb{N}$.*

Proof. Let $y \in \overline{{}^{\mu}_{\nu}I(\Gamma_x)}$ and $u > 0$, $\varepsilon \in (0,1)$. Then, $x_0 \in B(y, \varepsilon, u) \cap {}^{\mu}_{\nu}I(\Gamma_x)$. Suppose that $\delta \in (0,1)$ and $u > 0$ such that

$$B(x_0, \delta, u) \subset B(y, \varepsilon, u).$$

Hence, $T \subset K$, where

$$T = \{n \in \mathbb{N} : \mu(x_0, x_n, u) > 1-\delta, \quad \nu(x_0, x_n, u) < \delta\},$$
$$K = \{n \in \mathbb{N} : \mu(y, x_n, u) > 1-\varepsilon, \quad \nu(y, x_n, u) < \varepsilon\}.$$

Consequently,

$$\{n \in \mathbb{N} : \mu(y, x_n, u) > 1-\varepsilon, \quad \nu(y, x_n, u) < \varepsilon\} \notin I, \quad y \in^{\mu}_{\nu} I(\Gamma_x). \qquad \square$$

6. Conclusion

This paper studies the concept of ideal convergence, which is a generalization of ordinary convergence and statistical convergence in intuitionistic fuzzy metric spaces. In addition, it studies the concepts of ${}^{\mu}_{\nu}I^*$–convergent, ${}^{\mu}_{\nu}I$–Cauchy sequences, and ${}^{\mu}_{\nu}I^*$–Cauchy sequences and analyses the basic properties of these concepts. Finally, it defines the concepts of ${}^{\mu}_{\nu}I$–limit points and ${}^{\mu}_{\nu}I$–cluster points in intuitionistic fuzzy metric spaces and examines the connection between them.

In further research, it would be interesting to investigate similar results for double sequences.

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