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Anti-periodic boundary value problems for Caputo-Fabrizio fractional impulsive differential equations

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Abstract. In this paper, we shall discuss the existence and uniqueness of solutions for a nonlinear anti-periodic boundary value problem for fractional impulsive differential equations involving a Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$. Our results are based on some fixed point theorem, nonlinear alternative of Leray-Schauder type and coupled lower and upper solutions.

1. Introduction

Fractional calculus and impulsive fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics, control theory, for more details on this topics see, for instance,[1, 7, 9, 12, 15, 21, 26, 34] and the references therein.

In recent times, a new fractional differential equations having a kernel with exponential decay has been introduced by Caputo and Fabrizio [17].

Anti-periodic boundary value problems have been studied extensively in the last ten years, for differential equations, a Massera's type criterion is presented in [18]. Also anti-periodic boundary value problems accur in mathematical modeling of a variety of physical processes and have recently received considerable attention for details, see [3, 4, 6, 8, 16]. The monotone iterative method is based on coupled lower and upper solutions is an effective and flexible mechanism for details, see [33].

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Bekada et al. [14] discussed the following boundary value problem for Caputo-Fabrizio random fractional equations:

$$\begin{cases} \left({}^{CF}D_0^{\alpha}u\right)(t,w) = f(t;u(t,w),w),\ t \in I := [0,T],\ w \in \Omega, \\ au(0,w) + bu(T,w) = c(w),\ w \in \Omega, \end{cases}$$

where $T > 0$, $f : I \times E \times \Omega \to E$ is a given function, $a, b \in \mathbb{R}$, $c : \Omega \to E$ is the Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ and Ω is the sample space in a probability space (Ω, F) and E is a real (or complex) Banach space. The anti-periodic boundary value problem for Caputo-Fabrizio random fractional differential equations is a partial case with $a = b = 1$ and $c(w) = 0$. In this work we consider the following nonlinear anti-periodic problem (short BVP) with non-monotone term:

$$(1) \qquad {}^{CF}D_0^r u(t) = f(t, u(t)), \quad \text{a.e. } t \in I := [0,T],$$

$$(2) \qquad \Delta u(t_k) = I_k(u_k(t_k), \quad k = 1, 2, \ldots, m,$$

$$(3) \qquad u(0) = -u(T),$$

where $T > 0$, ${}^{CF}D_0^r$ is the standard Caputo-Fabrizio fractional derivative of order $r \in (0,1)$, $I_k \in C(I,\mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $f : I \times \mathbb{R} \to \mathbb{R}$ map satisfying some assumptions. In this paper, we shall discuss existence results of the solution of the anti-periodic problem (1)–(3) by using monotone iterative method.

This paper is organized in the following way. In the next section we recall briefly some notion of fractional calculus and theory of operators. The existence results are given in Section 3. In Section 4, using coupled upper and lower solutions. Finally, a conlusion is given in Section 5.

2. Preliminaries

In this section, we introduce the notations, definitions, and preliminary facts that will be used in the rest of this paper.

Let $C(I)$ be the Banach space of all continuous functions from I with the supremum norm defined by

$$\|u\| = \sup\{|u(t)|, t \in [0,T]\}.$$

By $L^1(I)$, we denote the space of Lebesgue integrable functions, $v : I \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

By $AC(I) = C([0,T],\mathbb{R})$ is the space of coninuous absolutely functions.

Definition 1. Let f is said to be L^1-Carathéodory

(i) $t \to f(t,u)$ is Lebesgue measurable for each $u \in \mathbb{R}$,

(ii) $u \to f(t,u)$ is continuous for almost all $t \in [0,T]$,

(iii) for each $R > 0$, there exists $\varphi \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq \varphi_R(t),$$

for all $\|u\|_\infty \leq R$ and for a.e.$t \in [0,T]$.

Definition 2 ([17, 27])**.** The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(I)$ is defined by

$$(4) \qquad {}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(\alpha)(2-r)}h(\tau) + \frac{2r}{M(r)(2-r)}\int_0^\tau h(s)ds, \quad \tau \geq 0,$$

where $M(r)$ is normalization constant depending on r.

Definition 3 ([17])**.** The Caputo-Fabrizio fractional derivative for a function $h \in AC(I)$ of order $0 < r < 1$, is defined by for $\tau \in I$,

$$(5) \qquad {}^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)}\int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - s)\right) h'(s)ds.$$

Note that $({}^{CF}D^r)(h) = 0$ if only if h is a constant function.

Lemma 1. *Let $u \in AC(I)$, and let $\lambda > 0$, the solution of the Cauchy problem ${}^{CF}D^r u(t) + \lambda u(t) = h(t)$ with initial boundary condition defined by $u(0) = u_0$, is given by*

$$(6) \qquad \begin{aligned} u(t) = {}& \frac{u_0}{1+\lambda(1-r)}\exp\left(\frac{-\lambda r t}{1+\lambda(1-r)}\right) \\ &+ \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right) h(s)ds. \end{aligned}$$

Proof. We first apply the Laplace transform, thus it follows that

$$(7) \qquad \mathcal{L}({}^{CF}D^r u(t)) = -\lambda\mathcal{L}(u(t)) + H(s),$$

such that $H(s) = \mathcal{L}(h)(s)$, on other hand

$$\begin{aligned} \frac{s\mathcal{L}\{u(t)\}(s) - u_0}{s + r(1-s)} &= -\lambda\mathcal{L}(u(t)), \\ s\mathcal{L}\{u(t)\}(s) - u_0 &= -s\lambda\mathcal{L}(u(t)) - r(1-s)\lambda\mathcal{L}(u(t)), \end{aligned}$$

so

$$\begin{aligned} [s(1+\lambda(1-r)) + \lambda r]\,\mathcal{L}\{u(t)\}(s) &= u_0, \\ \left[s + \lambda r(\lambda(1-r))^{-1}\right]\mathcal{L}\{u(t)\}(s) &= u_0(1+\lambda(1-r))^{-1}. \end{aligned}$$

Hence

$$\mathcal{L}_p(u(t)) = \frac{u_0(1+\lambda(1-r))^{-1}}{s+\lambda r(\lambda(1-r))^{-1}}.$$

Then by (7) we have

$$\mathcal{L}_p = \frac{u_0(1+\lambda(1-r))^{-1}}{s+\lambda r(\lambda(1-r))^{-1}} + \frac{H(s)}{s+\lambda r(\lambda(1-r))^{-1}}.$$

Applying the inverse of Laplace transform on the previous equation, we obtain (6). ☐

Lemma 2. *Let $h \in L^1(I)$ consider the boundary value problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) + \lambda u(t) = h(t), a.e.\ t \in I, \\ u(0) = -u(T). \end{cases} \tag{8}$$

Then the solution of (8) *given by*

$$u(t) = \int_0^T g(t,s)h(s)ds, \tag{9}$$

where g is the Green's function

$$g(t,s) = \begin{cases} \dfrac{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(T-t+s)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}, & 0 \le s \le t \le T, \\ \dfrac{-\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(s-t)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}, & 0 \le t < s \le T. \end{cases}$$

Definition 4 ([29])**.** The set $S \subset AC(I,\mathbb{R}^n)$ is said to be quasiequicontinuous in I if for any $\varepsilon > 0$ there exists a $\delta > 0$, such that if $u \in S$, $k \in \mathbb{Z}$, $\tau_1, \tau_2 \in (t_{k-1}, t_k) \cap I$ and $|\tau_1 - \tau_2| < \delta$, then $\|u(\tau_1) - u(\tau_2)\| < \varepsilon$.

Lemma 3. *The set $S \subset AC(I,\mathbb{R}^n)$ is relatively compact if and only if*

(i) *S is bounded that is $\|u\| \le c$ for each $u \in S$ and some $c > 0$,*

(ii) *S is quasiequicontinuous in I.*

Theorem 1. *Let V be a complete convex subset of a locally convex Hausdorff linear topological space E and U an open subset of V with $p \in U$. In addition let $N : \overline{U} \to V$ be a continuous, compact map. Then either:*

(A_1) *N has a fixed point in $\overline{U}$, or*

(A_2) *there is a $u \in \partial U$ and $\mu \in (0,1)$, with$u = \mu N(u) + (1-\mu)p$.*

3. Existence Results

Lemma 4. *Let $h : L^1(I)$ consider the problem*

$$(10)\qquad \begin{cases} ({}^{CF}D_0^r u)(t) + \lambda u(t) = h(t), a.e.\ t \in I, \\ \Delta u(t_k) = I_k(u(t_k)),\ \ k = 1, 2, \dots, m, \\ u(0) = -u(T). \end{cases}$$

Then the solution of (10) *given by*

$$(11)\qquad u(t) = \int_0^T g(t,s)h(s) + \sum_{k=1}^{m} g(t,t_k)I_k(u(t_k)).$$

Proof. Rewriting (9), if $t \in [0, t_1)$ then

$$u(t) = \int_0^T g(t,s)h(s)ds,$$

then if $t \in [t_1, t_2)$, we obtain

$$\begin{aligned} u(t) &= u(t_1^+) + \int_{t_1}^T g(t,s)h(s)ds \\ &= g(t,t_1)I_1(u(t_1) + g(t,t_1)u(t_1) + \int_{t_1}^T g(t,s)h(s)ds \\ &= g(t,t_1)I_1(u(t_1) + \int_0^{t_1} g(t,t_1)g(t_1,s)h(s)ds + \int_{t_1}^T g(t,s)h(s)ds \\ &= g(t,t_1)I_1(u(t_1) + \int_0^T g(t,s)h(s)ds. \end{aligned}$$

If $t \in [t_2, t_3)$, then from (10) we get

$$\begin{aligned} u(t) &= u(t_2^+) + \int_0^T g(t,s)h(s)ds \\ &= g(t,t_2)I_2(u(t_2)) + g(t,t_2)u(t_2) + \int_{t_2}^T g(t,s)h(s)ds \\ &= g(t,t_2)I_2(u(t_2) + g(t,t_2)g(t_2,t_1)I_1(u(t_1) + \int_0^T g(t,s)h(s)ds \\ &= g(t,t_1)I_1(u(t_1)) + g(t,t_2)I_2(u(t_2)) + \int_0^T g(t,s)h(s)ds. \end{aligned}$$

If $t \in [t_k, t_{k+})$, then again from (9), we get

$$u(t) = \int_0^T g(t,s)h(s) + \sum_{k=1}^{m} g(t,t_k)I_k(u(t_k)). \qquad \square$$

Let $\lambda > 0$, $F : I \times \mathbb{R} \to \mathbb{R}$ a L^1-Carathéodory function and consider the problem

$$(12) \qquad \begin{cases} ({}^{CF}D_0^r u)(t) + \lambda u(t) = F(t, u(t)), \text{a.e. } t \in I, \\ \Delta u(t_k) = I_k(u(t_k)), \ k = 1, 2, \cdots, m, \\ u(0) = -u(T). \end{cases}$$

Evident if $F(t, u) = f(t, u) + \lambda u$ and u, is a solution to (12) then u is a solution to (1)–(3). Furthermore, it is easy to show that solving (12) is equivalent to finding a $u \in AC(I)$ that satisfies $u = Nu$. Here $N : AC(I) \to AC(I)$ is given by

$$(13) \qquad (Nu)(t) = \int_0^T g(t, s)F(s, u(s))ds + \sum_{k=1}^{m} g(t, t_k)I_k(u(t_k)).$$

Note that if $F(t, u) = \delta(t)$ problem (12) is linear and solavable for each $\lambda \in \mathbb{R}$ and the solution is given by expression (13). Using a nonlinear alternative of Leray-Schauder type we now establish existence principles for (13).

Theorem 2. *Suppose that there exist a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $q \in L^1(I)$*

- (H_1) $|F(t, u)| \leq q(t)\psi(|s|)$, *for a.e. $t \in I$ and all $s \in I$.*
- (H_2) I_k $(k = 1, 2, \ldots, m)$ *is continuous.*
- (H_3) *There exist $b_k \geq 0$, $k = 1, 2, \cdots, m$, such that*

$$|I_k(u)| \leq b_k|u| \ \textit{and} \ \sum_{k=1}^{m} b_k < 1 + \exp\left(\frac{-\lambda r}{1 + \lambda(1 - r)}T\right).$$

In addition, suppose that

$$(14) \qquad \sup_{c \geq 0} \frac{c}{\psi(c)} > K = \frac{\|q\|_{L^1}}{1 + \exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right) - \sum\limits_{k=1}^{m} b_k}.$$

Then (12) *has at least one solution.*

Proof. Consider the problem

$$(15) \qquad \begin{cases} ({}^{CF}D_0^\alpha u)(t) + \lambda u(t) = \mu F(t, u(t)), \text{a.e. } t \in I, \\ \Delta u(t_k) = \mu I_k(u(t_k)), \ k = 1, 2, \ldots, m, \\ u(0) = -u(T). \end{cases}$$

It is easy to see that a function u is a solution to (15) if and only if

$$u = \mu N u.$$

We show that N satisfies the assumptions of Theorem 2 by three steps.

Step 1. N is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $AC(I)$. Then, for each $t \in I$ we have

$$\begin{aligned}
|(Nu_n)(t)-(Nu)(t)| &\le \int_0^T |g(t,s)||F(s,u_n(s))-F(s,u(s))|ds \\
&\quad + \sum_{k=1}^{m} |g(t,t_k)||I_k(u_n(t_k))-I_k(u(t_k))| \\
&\le \frac{1}{1+\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)}\|F(s,u_n(s))-F(s,u(s))\|_{L^1} \\
&\quad + \frac{1}{1+\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)}\sum_{k=1}^{m}|I_k(u_n(t_k))-I_k(u(t_k))|.
\end{aligned}$$

Since $u_n \to u$ is $n \to \infty$ and F is of L^1-Carathéodory type, then we get

$$|N(u_n)-N(u)| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence, the operator N is continuous.

Step 2. N bounded sets into bounded sets.
Indeed, for given $R>0$, if $u \in B_R = \{u \in AC(I) : \|u\| \le R\}$ thus, we have

$$\begin{aligned}
|(Nu_n)(t)-(Nu)(t)| &\le \int_0^T |g(t,s)||F(s,u(s))|ds + \sum_{k=1}^{m}|g(t,t_k)||I_k(u(t_k))| \\
&\le \frac{1}{1+\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)}\left[\|q\|_{L^1}\psi(\|u\|)+\sum_{k=1}^{m}b_kR\right] := l.
\end{aligned}$$

Step 3. N bounded sets into quasiequicontinuous sets.
Let $\tau_1,\tau_2 \in (t_{k-1},t_k]\cap I$ $k=1,2,\dots,m+1$, $\tau_1<\tau_2$, $u \in B_r$, then, we have

$$\begin{aligned}
|(Nu)(\tau_2)-(Nu)(\tau_1)| &\le \int_0^{\tau_1}|g(\tau_2,s)-g(\tau_1,s)||F(s,u(s))|ds \\
&\quad + \int_{\tau_2}^{\tau_1}|g(\tau_2,s)-g(\tau_1,s)||F(s,u(s))|ds \\
&\quad + \int_{\tau_1}^{\tau_2}(|g(\tau_2,s)+g(\tau_1,s)|)|F(s,u(s))|ds \\
&\quad + \sum_{k=1}^{m}|g(\tau_2,t_k)-g(\tau_1,t_k)||I_k(u(t_k)|.
\end{aligned}$$

From the definition of the function $g(t,s)$ and that F is L^1-Carathéodory it follows that $N(B_R)$ is quasiequicontinuous.
By (14) there exists $M > 0$ independent of μ, such that $\|u\| \neq M$ with

$$\frac{M}{\psi(M)} > K. \tag{16}$$

For $\mu \in (0,1)$ let $u \in AC(I)$ be any solution of (15). Then, for $t \in I$ we have

$$u(t) = \mu \int_0^T g(t,s)F(s,u(s))ds + \mu \sum_{k=1}^{m} g(t,t_k)I_k(u(t_k)),$$

and so

$$\begin{aligned}(Nu)(t) &\leq \int_0^T |g(t,s)|q(s)\psi(|u(s)|)ds + \sum_{k=1}^{m} |g(t,t_k)|b_k|u(t_k))| \\ &\leq \frac{1}{1+\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)} \left[\|q\|_{L^1}\psi(\|u\|) + \sum_{k=1}^{m} b_k\|u\|\right].\end{aligned}$$

Thus, $\|u\| \leq K\psi(\|u\|)$ and so, $\|u\| \neq M$ from, now the proof is complete from Theorem 2, deduce that (12) has a solution in $AC(I)$. □

4. COUPLED UPPER AND LOWER SOLUTIONS

In the space $AC(I)$ we also consider the usual pointwise parcial ordering. In such a case we define the interval

$$[\beta,\alpha] = \{u \in AC(I) : \beta \leq u \leq \alpha\}.$$

Definition of lower and upper solution is presented.

Definition 5. A function $\alpha(t) \in AC(I)$ is called a lower solution of (1)–(3) if it satisfies

$$\begin{cases} ({}^{CF}D_0^r\alpha)(t) \leq f(t,\alpha(t)) \text{ a.e. } t \in I\ t \neq t_k, \\ \Delta\alpha(t) \leq I_k(\alpha(t_k)),\ k = 1,2,\ldots,m, \\ \qquad \beta(0) \leq -\alpha(T), \end{cases} \tag{17}$$

Definition 6. A function $\beta(t) \in AC(I)$ is called a upper solution of (1)–(3) if it satisfies

$$\begin{cases} ({}^{CF}D_0^r\beta)(t) \geq f(t,\beta(t)) \text{ a.e. } t \in I\ t \neq t_k, \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)),\ k = 1,2,\ldots,m, \\ \qquad \alpha(0) \geq -\beta(T). \end{cases} \tag{18}$$

Note that g is not of constant sign on $I \times I$. Let $g = g^+ - g^-$ with

$$g^+(t,s) = \max\{g(t,s),0\} \text{ and } g^- = \max\{-g(t,s),0\}$$

and we can write the operator given in (10) as

$$(19)\qquad (Nu)(t)=\int_0^T g^+(t,s)F(s,u(s))ds-\int_0^T g^-(t,s)dsF(s,u(s))ds,$$

or equivalently as

$$\begin{aligned}(Nu)(t)=&\int_0^t \frac{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(T-t+s)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}F(s,u(s))ds\\ &-\int_t^T \frac{-\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(s-t)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}F(s,u(s))ds.\end{aligned}$$

For $\eta\in C(I)$, $t\in I$, we define

$$(N^+\eta)(t)=\int_0^T g^+(t,s)F(s,\eta(s))ds,$$
$$(N^-\eta)(t)=\int_0^T g^-(t,s)F(s,\eta(s))ds.$$

Note that $N^+:AC(I)\to AC(I)$ and $N^-:AC(I)\to AC(I)$ are continuous and completely continuous.

Definition 7. We say that functions $\alpha,\beta\in AC(I)$ are coupled lower and upper solutions for the anti-periodic problem (1)–(3) if $\beta(t)\le\alpha(t)$ and

$$(20)\qquad \beta\le N^+\beta-N^-\alpha$$

and

$$(21)\qquad \alpha\ge N^+\alpha-N^-\beta.$$

Theorem 3. *Let $\alpha,\beta\in AC(I)$ be a coupled lower and upper for (1)–(3). Suppose that f satisfies for a.e. $t\in I$*

$$(22)\qquad f(t,u)-f(t,v)+\lambda(u-v)\ge 0,\quad \beta(t)\le v\le u\le\alpha(t),$$

and that I_k, $k=1,2,\cdots,m$, are nondecreasing. Then, there exist monotone sequences $\{\beta_n\}$ and $\{\alpha_n\}$ such that $\{\beta_n\}\nearrow\phi$ and $\{\alpha_n\}\searrow\xi$ uniformly on I and any solution to (1)–(3) such that $u\in[\beta,\alpha]$ satisfies $u\in[\phi,\xi]$.

Proof. Clearly, if functions α,β are coupled lower and upper solutions for (1)–(3), then there are (20) and (21). In fact, by the definition of the lower and upper solutions there exist function $p(t)\ge 0$ such that

$${}^{CF}D_0^r\alpha(t)=f(t,\alpha(t))-p(t),\ \text{a.e. } t\in I$$

and

$${}^{CF}D_0^r\beta(t)=f(t,\beta(t))+p(t),\ \text{a.e. } t\in I.$$

Then

$$\begin{aligned}\beta(t) = &\int_0^t \frac{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(T-t+s)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}\big[f(s,\beta(s))+\lambda\beta(s)+p(s)\big]ds \\ &+ g^+(t,t_k)I_k(\beta(t_k) \\ &- \int_t^T \frac{-\exp\left(\frac{\lambda r}{1+\lambda(1-r)}(s-t)\right)}{\exp\left(\frac{\lambda r}{1+\lambda(1-r)}T\right)+1}\big[f(s,\alpha(s))+\lambda\alpha(s)-p(s)\big]ds \\ &- g^-(t,t_k)I_k(\alpha(t_k) \\ \leq\ & N^+\beta(t) - N^-\alpha(t).\end{aligned}$$

Similarly, there is

$$\alpha \geq N^+\alpha - N^-\beta.$$

We define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ by $\alpha_0 = \alpha$, $\beta_0 = \beta$ and for each $n \geq 1$

$$\begin{aligned}\beta_n &= N^+\beta_{n-1} - N^-\alpha_{n-1}, \\ \alpha_n &= N^+\alpha_{n-1} - N^-\beta_{n-1}.\end{aligned} \tag{23}$$

In view that N^+, N^- are completely continuous and $\beta \leq \beta_n \leq \alpha_n \leq \alpha$ for all $n \geq 0$, we can deduce that $\{\beta_n\}$ converges to ϕ uniformly on I, and $\{\alpha_n\}$ converges to ξ uniformly on I. Now suppose that u is solution (1)–(3) and $u \in [\beta, \alpha]$, we get for each $n \geq 0$

$$\beta_n \leq u \leq \alpha_n.$$

Thus, passing to the limit when $n \to \infty$ we obtain $\phi \leq u \leq \xi$. □

Theorem 4. *Let $\alpha, \beta \in AC(I)$ be a coupled lower and upper solutions for* (1)–(3)*, with following assumptions:*

(H_4) *There exists constant $M > 0$ such that $\lambda > 0$ for a.e. $t \in I$*

$$f(t,u) - f(t,v) + \lambda(u-v) \leq M(u-v), \quad \beta(t) \leq v \leq u \leq \alpha(t). \tag{24}$$

(H_5) *There exists constant $L > 0$ such that*

$$I_k(u) - I_k(v) \leq L(u-v), \quad \beta(t) \leq v \leq u \leq \alpha(t). \tag{25}$$

If

$$\frac{M(1+\lambda(1-r))\left[1-\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)\right] + m\lambda r L}{\lambda r\left[1+\exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)\right]} < 1, \tag{26}$$

then (1)–(3) *has a unique solution $u \in [\beta, \alpha]$.*

Proof. We pass to the limit in expression (23) to obtain that ϕ and ξ satisfy

$$\phi = N^{+}\phi - N^{-}\xi, \quad \xi = N^{+}\xi - N^{-}\phi.$$

We show that $\xi = \phi$ on consider

$$\xi(t) - \phi(t) = (N^{+}\xi)(t) - (N^{+}\phi)(t) - (N^{-}\xi)(t) + (N^{-}\phi)(t).$$

Using conditions (24) and (25) we obtain

$$\begin{aligned}\xi(t) - \phi(t) \leq & \int_0^T M[g^{+}(t,s) + g^{-}(t,s)](\xi(s) - \phi(s))ds \\ & + L\sum_{k=1}^{m}[g^{+}(t,t_k) + g^{-}(t,t_k)](\xi(s) - \phi(s)).\end{aligned}$$

Note that

$$\int_0^T [g^{+}(t,s) + g^{-}(t,s)]ds = \frac{(1+\lambda(1-r))\left(1 - \exp\left(-\frac{\lambda r}{1+\lambda(1-r)}T\right)\right)}{\lambda r\left(1 + \exp\left(-\frac{\lambda r}{1+\lambda(1-r)}T\right)\right)}$$

and

$$\sum_{k=1}^{m}[g^{+}(t,t_k) + g^{-}(t,t_k)] \leq \frac{mL}{1 + \exp\left(-\frac{\lambda r}{1+\lambda(1-r)}T\right)}.$$

We obtain

$$\|\xi - \phi\| \leq \frac{M(1+\lambda(1-r))\left[1 - \exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)\right] + m\lambda r L}{\lambda r\left[1 + \exp\left(\frac{-\lambda r}{1+\lambda(1-r)}T\right)\right]}\|\xi - \phi\|.$$

Therefore, $\xi = \phi$. □

5. Conclusion

This manuscript presents the existence and uniqueness of the solution of considered anti-periodic boundary value problems for Caputo-Fabrizio fractional impulsive differential equations. Further, we use the monotone iterative method of coupled lower and upper solutions. It yields monotone sequences of coupled lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solution or upper and lower solutions. For future, we intended to search the existence results for Caputo-Fabrizio fractional impulsive differential inclusions with anti-periodic boundary conditions by using differential inequalities method.

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