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Existence of beam-equation solutions with strong damping and $p(x)$-biharmonic operator

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Abstract. In this paper, we consider a nonlinear beam equation with a strong damping and the $p(x)$-biharmonic operator. The exponent $p(\cdot)$ of nonlinearity is a given function satisfying some condition to be specified. Using Faedo-Galerkin method, the local and global existence of weak solutions is established with mild assumptions on the variable exponent $p(\cdot)$. This work improves and extends many other results in the literature.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$. We consider the following problem

$$
(1) \qquad \begin{cases} u_{tt} + \Delta^2_{p(x)} u - \Delta u_t + f(x,t,u_t) = g(x,t), & \text{in } Q_T, \\ u = 0, \;\; \Delta u = 0, & \text{on } \partial Q_T, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}
$$

where $\Delta^2_{p(x)}$ is the fourth-order operator called the $p(x)$-biharmonic operator and is defined by $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$. We introduce, for $0 < T < \infty$, $Q_T = \Omega \times (0,T)$, $\partial Q_T = \partial\Omega \times (0,T)$ and the functions $p(\cdot)$, $f(\cdot)$, $g(\cdot)$, $u_0(\cdot)$ and $u_1(\cdot)$ which satisfy the following conditions.

The function $p : \overline{\Omega} \to (1,\infty)$ is log-Hölder continuous, i.e., there are constants $c > 0$ and $0 < \delta < 1$ such that

$$
(2) \qquad |p(x) - p(y)| \log|x-y| \leq -c, \quad \forall x, y \in \overline{\Omega}, \;\; |x-y| < \delta.
$$

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The function $f \in C(\Omega \times [0,\infty) \times \mathbb{R})$ and satisfies for three positive constants c_1, c_2 and c_3, and for all $(x,t,s) \in \Omega \times [0,\infty) \times \mathbb{R}$

$$\begin{cases} f\,(x,t,s)\, s \geq c_1 |s|^{q(x)} - c_2, \\ |f\,(x,t,s)\,| \leq c_3 \left(|s|^{q(x)-1} + 1\right), \end{cases} \tag{3}$$

where $q : \overline{\Omega} \to (1,\infty)$ is log-Hölder continuous. For all $x \in \overline{\Omega}$, we have that $p(\cdot)$ and $q(\cdot)$ satisfy

$$1 < p^- \leq p(x) \leq p^+ < \frac{N}{2}, \tag{4}$$

$$1 < q^- \leq q(x) \leq q^+ < \frac{Np(x)}{N-2p(x)}, \tag{5}$$

where

$$p^- = ess \inf_{x\in\overline{\Omega}} p(x), \quad p^+ = ess \sup_{x\in\overline{\Omega}} p(x),$$
$$q^- = ess \inf_{x\in\overline{\Omega}} q(x), \quad q^+ = ess \sup_{x\in\overline{\Omega}} q(x).$$

Furthermore, we consider that

$$u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{1,2}(\Omega), \tag{6}$$

$$u_1 \in L^2(\Omega), \tag{7}$$

$$g \in L^{q'(x)}(Q_T), \tag{8}$$

where $q(\cdot)$ and $q'(\cdot)$ are conjugated exponents satisfying $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, for all $x \in \overline{\Omega}$.

Partial differential equations with variable exponents have many applications in mathematical physics, for example, problems of filtration processes in non-homogeneous porous media [1], wave equations [2, 3, 18], nonlinear beam equations [10], restoration and image processing [11–13], flow of electro-rheological or thermo-rheological fluids [14–17], plate equations with viscoelasticity, elasticity term or viscoelasticity term [19, 20]. The $p(x)$-biharmonic problems are at the intersection of these fields of study.

For the $p(x)$-biharmonic elliptical problems, Ge, Zhou and Wu [21] studied the problem

$$\begin{cases} \Delta^2_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $f(x,u) = \lambda V(x)|u|^{q(x)-2}u$, λ is a positive real number, V is a weight function and $p, q : \overline{\Omega} \to (1,\infty)$ are continuous functions. Considering different situations concerning the growth rates involved in Problem (9), they proved the existence of a continuous family of eigenvalues using the mountain pass theorem and Ekeland's variational principle.

Li and Tang [22] studied Problem (9) with Navier boundary condition and for $f(x,u)=\lambda|u|^{p(x)-2}u+g(x,u)$, where $\lambda\le 0$ and $g(x,u)$ is a Carathéodory function. Using the mountain pass theorem and Fountain theorem, they established the existence of at least one solution.

Kong, in [23], considered Problem (9), where $f(x,u)=\lambda b(x)|u|^{\gamma(x)-2}u-\lambda c(x)|u|^{\beta(x)-2}u-a(x)|u|^{p(x)-2}u$, with $\lambda>0$ is a parameter and $a,b,c,\beta,\gamma\in C(\overline{\Omega})$ are nonnegative functions. He proved the existence of weak solutions to the problem associated with Navier boundary conditions.

For the $p(x)$-biharmonic parabolic problem, recently Liu [24] studied the problem

$$\begin{cases} u_t+\Delta^2_{p(x)}u=|u|^{q(x)-2}u, & \text{in } \Omega\times(0,T],\\ u=0, \Delta u=0, & \text{on } \partial\Omega\times[0,T],\\ u(x,0)=u_0(x), & \text{in } \Omega. \end{cases}$$

The author established the local existence of weak solutions and determined the finite-time blowup of solutions with nonpositive initial energy. Regarding the equations with variable exponent nonlinearities, we also refer to [4–9].

To the best of our knowledge, the present paper is the first to study the $p(x)$-biharmonic hyperbolic problem related to the local and global existence of beam equation solutions with strong damping.

The paper is organized as follows. In Section 2, we present some known results concerning Lebesgue and Sobolev spaces with variable exponents that shall be required. In Section 3, we prove the local and global existence of weak solutions for Problem (1). Finally, in Section 4 we present the conclusions of the paper.

2. Preliminaries

In this section, we present some results about Lebesgue and Sobolev spaces with variable exponents, $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, respectively (see [3,16] for more details). Let $p:\Omega\to[1,\infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^N$. We define the variable-exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\to\mathbb{R},\ \text{mensurable in } \Omega;\ \rho(u)=\int_\Omega |u(x)|^{p(x)}dx<\infty\right\}.$$

Equipped with the following Luxemburg-type norm

$$\|u\|_{p(\cdot)}=\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0;\ \rho\left(\frac{u}{\lambda}\right)\le 1\right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space (see [16]).

If p^+ is finite, then $p(x)$ is bounded and

$$\begin{aligned}\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)},\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}&\le\rho(u)\\ &\le\max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)},\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}.\end{aligned}\tag{10}$$

The inequality (10) can be represented by

$$\min\left\{\rho(u)^{\frac{1}{p^-}},\rho(u)^{\frac{1}{p^+}}\right\}\le\|u\|_{L^{p(\cdot)}(\Omega)}\le\max\left\{\rho(u)^{\frac{1}{p^-}},\rho(u)^{\frac{1}{p^+}}\right\}. \tag{11}$$

Theorem 1 ([3, 16]). *If $p(x)$ and $q(x)$ are variable exponents with $p(x)\ge q(x)$ for a.e. x in Ω, then $L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.*

Theorem 2 (Hölder's inequality, see [3]). *Let $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$ with $1<p(x)<\infty$ and $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. Then*

$$\begin{aligned}\int_\Omega |uv|dx&\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}\\&\le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.\end{aligned} \tag{12}$$

The variable-exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ is defined by

$$W^{m,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega);\ D^\alpha u\in L^{p(\cdot)}(\Omega),\ \forall\alpha,|\alpha|\le m\right\},$$

where m is a non-negative integer and D^α is the derivative in the sense of distributions. The variable-exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{m,p(\cdot)}=\|u\|_{W^{m,p(\cdot)}(\Omega)}=\sum_{|\alpha|\le m}\|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with a compact support contained in Ω. Throughout this paper, we denote by c_i various positive constants which may be different at different occurrences.

If X is a Banach space, then we denote by $L^p(0,T;X)$, with $1\le p\le\infty$, the Banach space of measurable vector valued functions $u:(0,T)\to X$, such that $\|u(t)\|_X\in L^p(0,T)$, together with the norms:

$$\begin{aligned}\|u\|_{L^p(0,T;X)}&=\left(\int_0^T\|u(t)\|_X^p\,dt\right)^{\frac{1}{p}}, && 1\le p<\infty,\\ \|u\|_{L^p(0,T;X)}&=ess\sup_{0\le t<T}\|u(t)\|_X, && p=\infty.\end{aligned}$$

In addition, by $C^1(0,T;X)$ we denote the space of continuously differentiable functions on $[0,T]$ with values in X.

Theorem 3 ([16]). *Let $p:\Omega\to(1,\infty)$ be a bounded function and log-Hölder continuous. If $q:\Omega\to(1,\infty)$, with $q^+<N$, is a bounded and measurable function with*

$$q(x)\le p^*=\frac{Np(x)}{N-2p(x)},\quad \forall x\in\overline{\Omega},$$

then, there is a continuous embedding $W^{2,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.

Theorem 4 ([25]). *Let $p : \Omega \to (1, \infty)$ be a bounded function and log-Hölder continuous. Then, there is a constant c such that for each $u \in W_0^{2,p(\cdot)}(\Omega)$,*

$$\|u\|_{W_0^{2,p(\cdot)}(\Omega)} \leq c \|\Delta u\|_{L^{p(\cdot)}(\Omega)} .$$

Theorem 5 ([3]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\{\omega_i(x)\}_{i=1}^{\infty}$ an orthonormal base in $L^2(\Omega)$, then for any $\varepsilon > 0$, there is a constant $N_\varepsilon > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq \left(\sum_{i=1}^{N_\varepsilon} \left(\int_\Omega u\omega_i(x)dx \right)^2 \right)^{\frac{1}{2}} + \varepsilon \|u\|_{W_0^{1,p(\cdot)}(\Omega)} ,$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where $2 \leq p < \infty$.

Theorem 6 ([26]). *Let $p : \overline{\Omega} \to \mathbb{R}$ be a bounded log-Hölder continuous function with $p^- > 1$. If $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^{p(\cdot)}(Q_T)$ and $u_n \to u$ a.e. in Q_T as $n \to \infty$, then there exist a subsequence of u_n, still denoted by u_n, such that $u_n \rightharpoonup u$ in $L^{p(\cdot)}(Q_T)$ as $n \to \infty$.*

Theorem 7 (Peano, see [27]). *Let $I = [a, b]$ be a real interval, $D \subseteq \mathbb{R}^N$ and a continuous function $f : I \times D \to \mathbb{R}^N$. If $(t_0, x_0) \in I \times D$, $C > 0$ and $T > 0$ are such that $[t_0 - T, t_0 + T] \times \mathcal{B}(x_0, C) \subseteq I \times D$, where $\mathcal{B}(x_0, C)$ is the ball with center x_0 and radius C, then the problem*

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{13}$$

with $t \in [t_0 - \gamma, t_0 + \gamma]$, has at least one solution x, where $\gamma \leq \min\left\{T, \frac{C}{M}\right\}$ and $M = \max_{(t,x)\in[t_0-T,t_0+T]\times\mathcal{B}(x_0,C)} |f(t, x)|$.

3. Weak solutions

In this section, we establish the existence of weak solutions to Problem (1), where the functions p, f, g, u_0 and u_1 satisfy the conditions given by (2)-(8).

Definition 1. The scalar function $u : Q_T \to \mathbb{R}$ is a weak solution to Problem (1), if u satisfies simultaneously:

$$u \in L^\infty \left(0, T; W_0^{2,p(\cdot)}(\Omega)\right) \cap C\left(0, T; W_0^{1,2}(\Omega)\right),$$
$$\frac{\partial u}{\partial t} \in L^\infty \left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; W_0^{1,2}(\Omega)\right) \cap L^{q(\cdot)}(Q_T)$$

and

$$-\int_{\Omega} \frac{\partial u(x,0)}{\partial t}\varphi(x,0)dx - \int_{Q_T} \frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}dxdt + \int_{Q_T} |\Delta u|^{p(x)-2}\Delta u \Delta \varphi dxdt$$

$$+\int_{Q_T} \nabla\left(\frac{\partial u}{\partial t}\right)\nabla\varphi dxdt + \int_{Q_T} f\left(x,t,\frac{\partial u}{\partial t}\right)\varphi dxdt = e\int_{Q_T} g\left(x,t\right)\varphi dxdt,$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$ with $\varphi(x,T)=0$.

We apply the Faedo-Galerkin method to Problem (1) and show the existence of weak solutions. For that, as stated in [28, 29], we choose a sequence $\omega_j(x)\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ such that $C_0^\infty(\Omega) \subset \overline{\bigcup_{n=1}^\infty V_n}^{C^2(\overline{\Omega})}$ and $\{\omega_j(x)\}_{j=1}^\infty$ is a Hilbertian base in $L^2(\Omega)$, where $V_n = \langle \omega_1(x), \omega_2(x), \ldots, \omega_n(x)\rangle$. Due to the fact that $\bigcup_{n=1}^\infty V_n$ is dense in $C^2(\overline{\Omega})$, it is well known that, if $u_0 \in W^{2,p(\cdot)}(\Omega)\cap W_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega)$, then there are $\psi_n, \phi_n \in V_n$ such that, when $n \to \infty$,

$$\begin{cases} \psi_n \to u_0 & \text{in } W^{2,p(\cdot)}(\Omega)\cap W_0^{1,2}(\Omega), \\ \phi_n \to u_1 & \text{in } L^2(\Omega). \end{cases} \tag{14}$$

Multiplying the equation (1) by an arbitrary function $v \in V_n$, integrating over Ω and using Green's formula, we get

$$\int_{\Omega} \frac{\partial^2 u}{\partial t^2} v dx + \int_{\Omega} |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_{\Omega} \nabla\left(\frac{\partial u}{\partial t}\right)\nabla v dx$$
$$+\int_{\Omega} f\left(x,t,\frac{\partial u}{\partial t}\right) v dx = \int_{\Omega} g(x,t) v dx.$$

The Faedo-Galerkin method consists of finding a sequence of solutions

$$u_n(x,t) = \sum_{j=1}^n \eta_{nj}(t)\omega_j(x) \in V_n \tag{15}$$

to the approximate problem

$$\begin{aligned} &\int_{\Omega} \frac{\partial^2 u_n}{\partial t^2} v dx + \int_{\Omega} |\Delta u_n|^{p(x)-2}\Delta u_n \Delta v dx + \int_{\Omega} \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla v dx \\ &\qquad + \int_{\Omega} f\left(x,t,\frac{\partial u_n}{\partial t}\right) v dx = \int_{\Omega} g_n\left(x,t\right) v dx, \end{aligned} \tag{16}$$

for all $v \in V_n$ with $g_n \in C_0^\infty(Q_T)$ and $g_n \to g$ in $L^{q'(x)}(Q_T)$.

Substituting (15) in (16) and taking $v = \omega_i$ with $1 \le i \le n$, we obtain

$$\begin{aligned}
&\int_\Omega \sum_{j=1}^{n} \eta_{nj}''(t)\omega_j(x)\omega_i(x)dx \\
&\quad + \int_\Omega \left|\sum_{j=1}^{n} \eta_{nj}(t)\Delta\omega_j(x)\right|^{p(x)-2} \left(\sum_{j=1}^{n} \eta_{nj}(t)\Delta\omega_j(x)\right) \Delta\omega_i(x)dx \\
&\quad + \int_\Omega \left(\sum_{j=1}^{n} \eta_{nj}'(t)\nabla\omega_j(x)\right) \nabla\omega_i(x)dx \\
&\quad + \int_\Omega f\left(x,t,\sum_{j=1}^{n} \eta_{nj}'(t)\omega_j(x)\right) \omega_i(x)dx \\
&= \int_\Omega g_n(x,t)\,\omega_i(x)dx,
\end{aligned} \tag{17}$$

where $\eta_{nj}'(t) = \frac{\partial \eta_{nj}(t)}{\partial t}$ and $\eta_{nj}''(t) = \frac{\partial^2 \eta_{nj}(t)}{\partial t^2}$.

Defining the projection $P_{ni}(t,\mu,\nu) : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as being

$$\begin{aligned}
P_{ni}(t,\mu,\nu) = &\int_\Omega \left|\sum_{j=1}^{n} \mu_{nj}(t)\Delta\omega_j(x)\right|^{p(x)-2} \left(\sum_{j=1}^{n} \mu_{nj}(t)\Delta\omega_j(x)\right) \Delta\omega_i(x)dx \\
&+ \int_\Omega \left(\sum_{j=1}^{n} \nu_{nj}(t)\nabla\omega_j(x)\right) \nabla\omega_i(x)dx \\
&+ \int_\Omega f\left(x,t,\sum_{j=1}^{n} \nu_{nj}(t)\omega_j(x)\right) \omega_i(x)dx,
\end{aligned} \tag{18}$$

where $1 \le i \le n$, and using the fact that V_n is a Hilbertian base in $L^2(\Omega)$, then we get

$$\begin{cases}
\eta_{n1}''(t) + P_{n1}\left(t,\eta_{n1}(t),\eta_{n1}'(t)\right) = G_{n1}(t), \\
\eta_{n2}''(t) + P_{n2}\left(t,\eta_{n2}(t),\eta_{n2}'(t)\right) = G_{n2}(t), \\
\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\eta_{nn}''(t) + P_{nn}\left(t,\eta_{nn}(t),\eta_{nn}'(t)\right) = G_{nn}(t),
\end{cases} \tag{19}$$

where

$$G_{ni}(t) = \int_\Omega g_n(x,t)\,\omega_i(x)dx.$$

Problem (19) can be rewritten as

$$\begin{cases}
\eta''(t) + P_n\left(t,\eta(t),\eta'(t)\right) = G_n(t), \\
\eta(0) = U_{0n},\ \eta'(0) = U_{1n},
\end{cases} \tag{20}$$

with

$$\eta''(t) = \begin{bmatrix} \eta''_{n1}(t) \\ \eta''_{n2}(t) \\ \vdots \\ \eta''_{nn}(t) \end{bmatrix},$$

$$P_n\left(t, \eta(t), \eta'(t)\right) = \begin{bmatrix} P_{n1}\left(t, \eta_{n1}(t), \eta'_{n1}(t)\right) \\ P_{n2}\left(t, \eta_{n2}(t), \eta'_{n2}(t)\right) \\ \vdots \\ P_{nn}\left(t, \eta_{nn}(t), \eta'_{nn}(t)\right) \end{bmatrix},$$

$$G_n(t) = \begin{bmatrix} G_{n1}(t) \\ G_{n2}(t) \\ \vdots \\ G_{nn}(t) \end{bmatrix}.$$

We define

$$X(t) = \eta'(t), \tag{21}$$

$$Y(t) = (\eta(t), X(t)), \tag{22}$$

$$Z_n(t) = \left(X(t), G_n(t) - P_n(t, \eta(t))\right), \tag{23}$$

thus, Problem (20) becomes

$$\begin{cases} Y'(t) = Z_n(t, Y(t)), \\ Y(0) = (U_{0n}, U_{1n}). \end{cases} \tag{24}$$

Before we prove the solution to Problem (24), we will make some remarks.

Remark 1.

$$\int_\Omega f\left(x, t, \frac{\partial u_n}{\partial t}\right) \frac{\partial u_n}{\partial t} dx \geq c_1 \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx - c_4, \tag{25}$$

where $c_4 = c_2|\Omega| \geq 0$.

Proof. In fact, using the equation ((3))

$$\int_\Omega f\left(x, t, \frac{\partial u_n}{\partial t}\right) \frac{\partial u_n}{\partial t} dx \geq \int_\Omega \left(c_1 \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} - c_2\right) dx$$
$$= c_1 \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx - c_4. \qquad \square$$

Remark 2.

$$\int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n \Delta\left(\frac{\partial u_n}{\partial t}\right) dx = \frac{d}{dt} \frac{|\Delta u_n|^{p(x)}}{p(x)} dx. \tag{26}$$

Proof. Note that,

$$\frac{d}{dt}\left|\Delta u_n\right|^{p(x)} = p(x)\left|\Delta u_n\right|^{p(x)-2}\Delta u_n \Delta\left(\frac{\partial u_n}{\partial t}\right).$$

Thus, we obtain that

$$\begin{aligned}&\int_\Omega \left|\Delta u_n\right|^{p(x)-2}\Delta u_n \Delta\left(\frac{\partial u_n}{\partial t}\right)dx\\ &= \int_\Omega \frac{1}{p(x)}\frac{d}{dt}\left|\Delta u_n\right|^{p(x)}dx \;=\; \frac{d}{dt}\int_\Omega \frac{\left|\Delta u_n\right|^{p(x)}}{p(x)}dx.\end{aligned}$$

□

Remark 3.

$$\begin{aligned}P_n(t,\eta,\eta')\eta' \geq \frac{d}{dt}\int_\Omega \frac{\left|\Delta u_n\right|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx\\ + c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx - c_4.\end{aligned} \tag{27}$$

Proof. Using the projection (18), we get

$$\begin{aligned}P_n(t,\eta,\eta')\eta' = &\int_\Omega \left|\Delta u_n\right|^{p(x)-2}\Delta u_n \sum_{i=1}^n \Delta\omega_i(x)\eta'_{ni}(t)dx\\ &+\int_\Omega \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\sum_{i=1}^n \omega_i(x)\eta'_{ni}(t)dx\\ &+\int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)\sum_{i=1}^n \omega_i(x)\eta'_{ni}(t)dx.\end{aligned}$$

So, it follows that

$$\begin{aligned}P_n(t,\eta,\eta')\eta' = &\int_\Omega \left|\Delta u_n\right|^{p(x)-2}\Delta u_n \Delta\left(\frac{\partial u_n}{\partial t}\right)dx\\ &+\int_\Omega \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\left(\frac{\partial u_n}{\partial t}\right)dx\\ &+\int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)\frac{\partial u_n}{\partial t}dx.\end{aligned} \tag{28}$$

Replacing (25) and (26) in (28), we conclude (27). □

Now, returning to Problem (24) and in the seek of simplicity, we are omitting the arguments. Composing Problem ((24)) with Y, applying the inner product and using (21), (22) and (23), we obtain that

$$Y'Y - \eta'\eta - G_n\eta' = -P_n\eta'. \tag{29}$$

Applying the inequality (27) in (29),

$$
\begin{aligned}
Y'Y + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx \\
+c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \quad &\leq \quad \eta'\eta + G_n\eta' + c_4.
\end{aligned} \tag{30}
$$

We know that $\eta'\eta \leq |\eta'||\eta|$ and applying Young's inequality,

$$
\eta'\eta \leq |\eta'|\,|\eta| \leq \frac{1}{2}|\eta'|^2 + \frac{1}{2}|\eta|^2 \tag{31}
$$

and

$$
G_n\eta' \leq \frac{1}{2}|G_n|^2 + \frac{1}{2}|\eta'|^2. \tag{32}
$$

Replacing (3) and (32) in (30) results in

$$
\begin{aligned}
&Y'Y + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \\
&\leq \frac{1}{2}|\eta'|^2 + \frac{1}{2}|\eta|^2 + \frac{1}{2}|G_n|^2 + \frac{1}{2}|\eta'|^2 + c_4.
\end{aligned}
$$

By (22), we have

$$
\begin{aligned}
&Y'Y + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \\
&\leq |Y|^2 + \frac{1}{2}|G_n|^2 + c_4.
\end{aligned}
$$

Since $\{\omega_j\}_{j=1}^n$ is a Hilbertian basis, then $|G_n|^2 = \int_\Omega |g_n|^2 dx$. Thus,

$$
\begin{aligned}
&Y'Y + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \\
&\leq |Y|^2 + \frac{1}{2}\int_\Omega |g_n|^2 dx + c_4.
\end{aligned}
$$

We know that $\frac{1}{2}\frac{d}{dt}|Y|^2 = Y'Y$, then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|Y|^2 + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \\
&\quad + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_1\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \\
&\leq |Y|^2 + \frac{1}{2}\int_\Omega |g_n|^2 dx + c_4.
\end{aligned} \tag{33}
$$

Integrating (33) from 0 to t and since $t \leq T$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx - \int_\Omega \frac{|\Delta u_n(x,0)|^{p(x)}}{p(x)}dx \\
&+ \int_0^T\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dxdt + c_1\int_0^T\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dxdt \\
&\le \int_0^t |Y(s)|^2\,ds + \frac{1}{2}\int_0^t\int_\Omega |g_n|^2\,dxds + c_4T.
\end{aligned} \tag{34}
$$

By (14), $u_n(x,0)$ converges strongly in $W^{2,p(x)}(\Omega)\cap W_0^{1,2}(\Omega)$, then $|\Delta u_n(x,0)|$ is bounded by a constant, that is,

$$
|\Delta u_n(x,0)| \le c_5. \tag{35}
$$

In addition, we have

$$
\int_0^T\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dxdt = \left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|^2_{L^2(Q_T)} \le c_6, \tag{36}
$$

$$
\int_0^T\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dxdt = \left\|\frac{\partial u_n}{\partial t}\right\|^{q(x)}_{L^{q(\cdot)}(Q_T)} \le c_7, \tag{37}
$$

$$
\frac{1}{2}\int_0^t\int_\Omega |g_n|^2\,dxds \le c_8. \tag{38}
$$

Replacing (35), (36), (37) and (38) in (34), and defining $c_9(T) = c_4T + c_5 + c_8 - c_6 - c_1c_7$, then

$$
\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \le \int_0^t |Y(s)|^2\,ds + c_9(T).
$$

Applying Gronwall's lemma,

$$
\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \le c_{10}(T).
$$

Since $p:\overline{\Omega}\to(1,\infty)$, it follows that

$$
\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \le \int_\Omega |\Delta u_n|^{p(x)}\,dx. \tag{39}
$$

Thus,

$$
\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega |\Delta u_n|^{p(x)}\,dx \le c_{10}(T). \tag{40}
$$

We know that $\int_\Omega |\Delta u_n|^{p(x)}dx \ge 0$, then

$$
|Y(t) - Y(0)| \le \sqrt{c(T)},
$$

where $c(T) = 2c_{10}(T)$.

We denote

$$M_n = \max_{(t,Y) \in [0,T]\times\mathcal{B}(Y(0),\sqrt{c(T)})} |Z_n(t,Y)| \text{ and } \gamma_n \leq \min\left\{T, \frac{\sqrt{c(T)}}{M_n}\right\},$$

where $\mathcal{B}(Y(0), \sqrt{c(T)})$ is the ball with center $Y(0) \in \mathbb{R}^{2N}$ and radius $\sqrt{c(T)}$. By definition $Z_n(t,Y)$ is continuous with respect to (t,Y), then applying Peano's Theorem 7 it follows that Problem (24) has a solution C^1 over the interval $[0,\gamma_n]$, it implies that over the same interval Problem (20) has a solution C^2 denoted by $\eta_n^1(t)$.

Considering that $\eta(\gamma_n)$ and $\frac{\partial\eta(\gamma_n)}{\partial t}$ are the initial values of Problem (20), then we can repeat the previous process and over the interval $[\gamma_n, 2\gamma_n]$ we obtain a solution C^2 denoted by $\eta_n^2(t)$.

We define

$$T = \left[\frac{T}{\gamma_n}\right]\gamma_n + \left(\frac{T}{\gamma_n}\right)\gamma_n, \text{ with } 0 < \left(\frac{T}{\gamma_n}\right) < 1,$$

where $\left[\frac{T}{\gamma_n}\right]$ and $\left(\frac{T}{\gamma_n}\right)$ are, respectively, the integer part and the decimal part of $\frac{T}{\gamma_n}$. If we divide the interval $[0,T]$ in $[(i-1)\gamma_n, i\gamma_n]$, $i = 1,2,\ldots,L$ and $[L\gamma_n, T]$, where $L = \left[\frac{T}{\gamma_n}\right]$, then there is a solution C^2 over the interval $[(i-1)\gamma_n, i\gamma_n]$ denoted by $\eta_n^i(t)$ and there is $\eta_n^{L+1}(t)$ over $[L\gamma_n, T]$. Therefore, we obtain a solution $\eta_n(t) \in C^2([0,T])$ as follows

$$\eta_n(t) = \begin{cases} \eta_n^1(t), & \text{if } t \in [0,\gamma_n], \\ \eta_n^2(t), & \text{if } t \in (\gamma_n, 2\gamma_n], \\ \quad\vdots & \qquad\vdots \\ \eta_n^L(t), & \text{if } t \in ((L-1)\gamma_n, L\gamma_n], \\ \eta_n^{L+1}(t), & \text{if } t \in (L\gamma_n, T]. \end{cases}$$

Therefore, we conclude that Problem (1) has local solutions. Our next objective will be to prove that these solutions are global, before that we show Lemmas 1 and 2 that assist in the development of this paper.

Lemma 1. *The following estimates are uniform with respect to n for all $t \in [0,T]$*

$$\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dx + \int_\Omega |\Delta u_n|^{p(x)}\, dx + \int_\Omega |\nabla u_n|^2\, dx \leq C_1, \tag{41}$$

$$\int_{Q_T} \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dxdt + \int_{Q_T} |\Delta u_n|^{p(x)}\, dxdt + \int_{Q_T} \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dxdt \leq C_2. \tag{42}$$

Proof. Through equation (40) and since $|Y(0)|^2$ is bounded, then we have

$$|Y(t)|^2 + \int_\Omega |\Delta u_n|^{p(x)}\, dx \leq c_{11}.$$

Using (15)), (22), the fact that ω_j, $1 \leq j \leq n$, is a Hilbertian base and Poincaré's inequality, we obtain (41). On the other hand, in (33) since $|Y|^2$ and $\int_\Omega |g_n|^2 dx$ are bounded, then

$$\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_1 \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \leq c_{12}.$$

Integrating from 0 to T and using $Q_T = \Omega \times [0,T]$, it follows that

$$\int_{Q_T} \frac{|\Delta u_n|^{p(x)}}{p(x)} dxdt + \int_{Q_T} \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dxdt + c_1 \int_{Q_T} \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dxdt \leq c_{13}.$$

From (39) and by hypothesis $c_1 \geq 0$, we conclude (42). $\square$

Lemma 2. *The following estimate is uniform with respect to n for all $t \in [0,T]$,*

$$\| |\Delta u_n|^{p(x)-2} \Delta u_n \|_{L^{p'(\cdot)}(Q_T)} + \left\| f\left(x,t,\frac{\partial u_n}{\partial t}\right) \right\|_{L^{q'(\cdot)}(Q_T)} \leq C_3. \tag{43}$$

Proof. By Lemma 1, we obtain

$$\int_{Q_T} \left| |\Delta u_n|^{p(x)-2} \nabla u_n \right|^{p'(x)} dxdt \leq \int_{Q_T} |\nabla u_n|^{p(x)} dxdt \leq c_{14}.$$

Thus,

$$\begin{aligned} &\left\| |\Delta u_n|^{p(x)-2} \nabla u_n \right\|_{L^{p'(\cdot)}(Q_T)} \\ &\quad \leq \max\left\{ \left(\int_\Omega |\Delta u_n|^{p(x)} dx\right)^{\frac{p^- -1}{p^-}}, \left(\int_\Omega |\Delta u_n|^{p(x)} dx\right)^{\frac{p^+ -1}{p^+}} \right\}, \end{aligned}$$

that is,

$$\left\| |\Delta u_n|^{p(x)-2} \nabla u_n \right\|_{L^{p'(\cdot)}(Q_T)} \leq c_{15}. \tag{44}$$

Using (3) and (37), it follows that

$$\left\| f\left(x,t,\frac{\partial u_n}{\partial t}\right) \right\|_{L^{q'(\cdot)}(Q_T)} \leq c_{16}. \tag{45}$$

From inequalities (44) and (45), we conclude (43). $\square$

Next, we prove our main result of the paper in the form of Theorem 8, which guarantees the existence of weak global solutions to Problem (1).

Theorem 8 (Existence of weak global solutions)**.** *Under the conditions* (2)–(8)*, Problem* (1) *has a weak solution in the sense of Definition 1.*

Proof. Using Lemmas 1 and 2 there is a subsequence of u_n (still denoted by u_n) and u such that

$$\frac{\partial u_n}{\partial t} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial t} \text{ in } L^\infty(0,T;L^2(\Omega)),$$

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;W_0^{2,p(x)}(\Omega)) \cap L^\infty(0,T;W_0^{1,2}(\Omega)),$$

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^{q(x)}(Q_T) \cap L^2(0,T;W_0^{1,2}(\Omega)),$$

$$|\Delta u_n|^{p(x)-2}\Delta u_n \rightharpoonup \xi \text{ in } L^{p'(x)}(Q_T),$$

$$f\left(x,t,\frac{\partial u_n}{\partial t}\right) \rightharpoonup f\left(x,t,\frac{\partial u}{\partial t}\right) \text{ in } L^{q'(x)}(Q_T).$$

Our next objective will be to prove that there is a subsequence of u_n, such that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(\Omega) \quad\text{and}\quad u_n \to u \text{ in } L^{q(x)}(Q_T).$$

Note that, from (15) and since $\omega_j(x)\}_{j=1}^n$ is a Hilbertian base, then

$$\int_\Omega \frac{\partial u_n}{\partial t}\omega_j(x)dx = \eta'_{nj}(t) \text{ and } \int_\Omega \frac{\partial^2 u_n}{\partial t^2}\omega_j(x)dx = \eta''_{nj}(t). \tag{46}$$

Through Lemma 1, it follows that $\eta'_{nj}(t)$ is uniformly bounded in $[0,T]$. Consider that $0 \le t_1 < t_2 \le T$, integrating (20) from t_1 to t_2, using (46) and defining $Q_{t_1}^{t_2} = \Omega \times [t_1,t_2]$, we get

$$\begin{aligned}
&\int_\Omega \frac{\partial u_n(x,t_2)}{\partial t}\omega_j(x)dx - \int_\Omega \frac{\partial u_n(x,t_1)}{\partial t}\omega_j(x)dx \\
&+ \int_{Q_{t_1}^{t_2}} |\Delta u_n|^{p(x)-2}\Delta u_n \Delta\omega_j(x)dxdt + \int_{Q_{t_1}^{t_2}} \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\omega_j(x)dxdt \\
&+ \int_{Q_{t_1}^{t_2}} f\left(x,t,\frac{\partial u_n}{\partial t}\right)\omega_j(x)dxdt \;=\; \int_{Q_{t_1}^{t_2}} g_n\omega_j(x)dxdt.
\end{aligned} \tag{47}$$

By (46), we obtain

$$\int_\Omega \frac{\partial u_n(x,t_2)}{\partial t}\omega_j(x)dx - \int_\Omega \frac{\partial u_n(x,t_1)}{\partial t}\omega_j(x)dx \le |\eta'_{nj}(t_2) - \eta'_{nj}(t_1)|. \tag{48}$$

Replacing (48) in (47) and using Hölder's inequality (12),

$$
\begin{aligned}
&|\eta_{nj}'(t_2)-\eta_{nj}'(t_1)|\\
&\leq 2\left\||\Delta u_n|^{p(x)-2}\Delta u_n\right\|_{L^{p'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}\\
&\quad+2\left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|_{L^2\left(Q_{t_1}^{t_2}\right)}\|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}-2\|g_n\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}\\
&\quad+2\left\|f\left(x,t,\frac{\partial u_n}{\partial t}\right)\right\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}.
\end{aligned}
$$

As $\|g_n\|_{L^{q'(\cdot)}(Q_{t_1}^{t_2})}\|\omega_j\|_{L^{q(\cdot)}(Q_{t_1}^{t_2})}\geq 0$ and by Poincaré's inequality, it follows that

$$
\begin{aligned}
&|\eta_{nj}(t_2)-\eta_{nj}(t_1)|\\
&\leq c_{17}\left\||\Delta u_n|^{p(x)-2}\Delta u_n\right\|_{L^{p'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}\\
&\quad+c_{18}\left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|_{L^2\left(Q_{t_1}^{t_2}\right)}\|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}\\
&\quad+c_{19}\left\|f\left(x,t,\frac{\partial u_n}{\partial t}\right)\right\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}.
\end{aligned}
$$

By Lemmas 1 and 2, $\||\Delta u_n|^{p(x)-2}\Delta u_n\|_{L^{p'(\cdot)}(Q_{t_1}^{t_2})}$, $\|\nabla(\frac{\partial u_n}{\partial t})\|_{L^2(Q_{t_1}^{t_2})}$ and $\|f(x,t,\frac{\partial u_n}{\partial t})\|_{L^{q'(\cdot)}(Q_{t_1}^{t_2})}$ are bounded, then

$$
\begin{aligned}
&|\eta_{nj}(t_2)-\eta_{nj}(t_1)|\\
&\quad\leq c_{20}\left(\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}+\|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}+\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}\right).
\end{aligned}
$$

Through Theorems 1, 3 and 4, it follows that

$$
\begin{aligned}
&|\eta_{nj}(t_2)-\eta_{nj}(t_1)|\\
&\quad\leq c_{21}\left(\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}+\|\Delta\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}+\|\Delta\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}\right).
\end{aligned}
$$

Using (11), we get

$$
\begin{aligned}
&|\eta_{nj}(t_2)-\eta_{nj}(t_1)|\\
&\leq c_{21}\left[\max\left\{|t_2-t_1|^{\frac{1}{p^-}}\left(\int_\Omega|\Delta\omega_j(x)|^{p(x)}\,dx\right)^{\frac{1}{p^-}},|t_2-t_1|^{\frac{1}{p^+}}\left(\int_\Omega|\Delta\omega_j(x)|^{p(x)}\,dx\right)^{\frac{1}{p^+}}\right\}\right.\\
&\quad+\max\left\{|t_2-t_1|^{\frac{1}{q^-}}\left(\int_\Omega|\Delta\omega_j(x)|^{q(x)}\,dx\right)^{\frac{1}{q^-}},|t_2-t_1|^{\frac{1}{q^+}}\left(\int_\Omega|\Delta\omega_j(x)|^{q(x)}\,dx\right)^{\frac{1}{q^+}}\right\}\\
&\quad\left.+\max\left\{|t_2-t_1|^{\frac{1}{2}}\left(\int_\Omega|\Delta\omega_j(x)|^2\,dx\right)^{\frac{1}{2}}\right\}\right].
\end{aligned}
$$

Thus, the sequence $\eta_{nj}(t)$, with $1 \leq n < \infty$, is uniformly bounded and equicontinuous for fixed j and $n \geq j$ in $[0,T]$. Using Arzelà-Ascoli's theorem (see [30]), there is a subsequence such that $\eta_{nj}(t)$ converges uniformly in $[0,T]$ for some continuous function $\eta_j(t)$ for each fixed $j = 1, 2, \ldots$.

We define

$$\bar{u}(x,t) = \sum_{j=1}^{\infty} \eta_j(t)\omega_j(x),$$

then for each $j \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial t}\omega_j(x)dx = \int_\Omega \bar{u}\omega_j(x)dx$$

uniformly in $[0,T]$. With the completeness of $\omega_j(x)$ we obtain that

$$\frac{\partial u_n}{\partial t} \rightharpoonup \bar{u} \text{ in } L^2(\Omega)$$

and uniformly in $[0,T]$ when $n \to \infty$. Furthermore, it turns out that $\bar{u} = \frac{\partial u}{\partial t}$. Using Lemma 1 and the Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_j(x)dx \right)^2 dt = 0.$$

Through Theorem 5, there is a positive number N_ε independent of n such that

$$\begin{aligned} \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \leq {} & 2\sum_{j=1}^{N_\varepsilon} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_j(x)dx \right)^2 dt \\ & + 2\varepsilon^2 \int_0^T \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|^2_{W_0^{1,2}(\Omega)} dt. \end{aligned}$$

Furthermore, by Lemma 1

$$\limsup_{n\to\infty} \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \leq c_{22}\varepsilon^2.$$

The arbitrariness of ε implies that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{in} \quad L^2\left(Q_T\right).$$

Consequently, there is a subsequence of u_n such that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{a.e. in} \quad Q_T.$$

For the continuity of f, we get

$$f\left(x,t,\frac{\partial u_n}{\partial t}\right) \to f\left(x,t,\frac{\partial u}{\partial t}\right) \quad \text{a.e. in} \quad Q_T.$$

Our next objective will be to prove that $u_n \to u$ in $L^{q(x)}(Q_T)$. We know that $u_n \in W^{1,2}(Q_T)$ and by Theorem 3 we can obtain a subsequence such that $u_n \to u$ in $L^2(Q_T)$ and a.e. in Q_T. By (4), Lemma 1 and Theorem 3, we have

$$\int_\Omega |u_n|^{\frac{Np(x)}{N-2p(x)}}\,dx \le c_{23}, \quad \forall t \in [0,T].$$

This implies that,

$$\int_0^T \int_\Omega |u_n|^{\frac{Np(x)}{N-2p(x)}}\,dxdt \le c_{24}.$$

For any measurable subset $V \in Q_T$, if we use Hölder's inequality (12) and that $q(x) < p^* = \frac{Np(x)}{N-2p(x)}$, then

$$\int_V |u_n|^{q(x)}dxdt \le 2\,\||u_n|\|_{L^{\frac{p^*(\cdot)}{q(\cdot)}}(Q_T)} \|1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-q(\cdot)}}(V)} \le \|1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-q(\cdot)}}(V)}.$$

Thus, the sequence $|u_n|^{q(x)}$, with $1 \le n < \infty$, is equi-integrable in $L^1(Q_T)$. By Vitali's Convergence theorem (see [31])

$$\lim_{n\to\infty} \int_{Q_T} |u_n - u|^{q(x)}dxdt = 0.$$

Therefore, $u_n \to u$ in $L^{q(x)}(Q_T)$.

Finally, our next objective will be to prove that $\xi = |\Delta u|^{p(x)-2}\Delta u$. We know that for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$, we can choose a sequence $\varphi_k \in C^1(0,T;V_k)$ such that $\varphi_k \to \varphi$ in $C^{1,2}(Q_T)$, where for any $u \in C^{1,2}(Q_T)$ its norm is given by

$$\|u\| = \sup_{|\alpha|\le 2,\ (x,t)\ \in\ \overline{Q_T}} \left\{ |D^\alpha u|, \left|\frac{\partial u}{\partial t}\right| \right\}.$$

For all $\tau \in [0,T]$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2}\varphi_k dxdt \\
&= \lim_{k\to\infty}\lim_{n\to\infty}\left(\int_\Omega \frac{\partial u_n(x,\tau)}{\partial t}\varphi_k(x,\tau)dx - \int_\Omega \frac{\partial u_n(x,0)}{\partial t}\varphi_k(x,0)dx\right) \\
&\quad - \lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\tau} \frac{\partial u_n}{\partial t}\frac{\partial \varphi_k}{\partial t}dxdt \\
&= \lim_{k\to\infty}\left(\int_\Omega \bar{u}(x,\tau)\varphi_k(x,\tau)dx - \int_\Omega u_1\varphi_k(x,0)dx - \int_{Q_\tau}\frac{\partial u}{\partial t}\frac{\partial \varphi_k}{\partial t}dxdt\right). \\
&= \int_\Omega \bar{u}(x,\tau)\varphi(x,\tau)dx - \int_\Omega u_1\varphi(x,0)dx - \int_{Q_\tau}\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}dxdt. \\
&= \lim_{n\to\infty}\int_{Q_\tau}\frac{\partial^2 u_n}{\partial t^2}\varphi dxdt,
\end{aligned}$$

where $Q_\tau = \Omega \times (0,\tau)$. By (16), it follows that

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2}\varphi_k dxdt + \int_{Q_\tau} |\Delta u_n|^{p(x)-2}\Delta u_n \Delta\varphi_k + \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\varphi_k dxdt \\
&+ \int_{Q_\tau} f\left(x,t,\frac{\partial u_n}{\partial t}\right)\varphi_k dxdt \;=\; \int_{Q_\tau} g_n\varphi_k dxdt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2}\varphi dxdt \\
&\quad = \int_{Q_\tau} g\varphi - \xi\Delta\varphi - \nabla\left(\frac{\partial u}{\partial t}\right)\nabla\varphi - f\left(x,t,\frac{\partial u}{\partial t}\right)\varphi dxdt.
\end{aligned} \tag{49}$$

In addition, for any $\psi(x) \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (\bar{u}(x,\tau) - u_1)\varphi dx &= \lim_{n\to\infty}\int_\Omega \left(\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial u_n(x,0)}{\partial t}\right)\psi(x)dx \\
&= \lim_{n\to\infty}\int_0^\tau\int_\Omega \frac{\partial^2 u_n}{\partial t^2}\psi(x)dxdt \\
&= \int_{Q_\tau} g\varphi - \xi\Delta\varphi - \nabla\left(\frac{\partial u}{\partial t}\right)\nabla\varphi - f\left(x,t,\frac{\partial u}{\partial t}\right)\varphi dxdt, \\
&\to 0, \quad (\tau \to 0).
\end{aligned}$$

Consequently, $\bar{u}(x,t)$ is weakly continuous in $L^2(\Omega)$, that is, we have $\bar{u}(x,t) \in C_w(0,T;L^2(\Omega))$. For all $\eta \in C^1([0,T])$ with $\eta(T) = 0$ and $\eta(0) = 1$, we get

$$\begin{aligned}
\int_{Q_T} \frac{\partial u_n}{\partial t}\eta(t)\omega_i(x)dxdt = &-\int_\Omega u_n(x,0)\eta(0)\omega_i(x)dx \\
&- \int_{Q_T} u_n(x,t)\eta'(t)\omega_i(x)dxdt.
\end{aligned}$$

If $n \to \infty$, then

$$\int_\Omega (u(x,0) - u_0)\,\omega_i(x)dx = 0, \text{ with } i = 1,2,\dots$$

By the completeness of basis ω_i in $L^2(\Omega)$, we conclude that $u(x,0) = u_0$. Due to $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(0,T;L^2(\Omega))$ and $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^2(0,T;W_0^{1,2}(\Omega))$, just as was done by Lions [32] we assume that $u \in C(0,T;W_0^{1,2}(\Omega))$ and that there is a subsequence of u_n such that $\nabla u_n(x,T) \rightharpoonup \nabla u(x,T)$ in $(L^2(\Omega))^N$. Thus,

$$\int_\Omega |\nabla u(x,T)|^2 dx \le \liminf_{n\to\infty}\int_\Omega |\nabla u_n(x,T)|^2 dx.$$

We take $\varphi = u_k$ in equation (49) and if $k \to \infty$, then

$$\begin{aligned}
&\int_\Omega \bar{u}(x,T)u(x,T)dx - \int_\Omega u_1 u_0 dx + \int_{Q_T} \left(\xi \Delta u + \nabla \left(\frac{\partial u}{\partial t}\right) \nabla u\right) dxdt \\
&- \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_{Q_T} f\left(x,t,\frac{\partial u}{\partial t}\right) udxdt = \int_{Q_T} gudxdt.
\end{aligned} \tag{50}$$

Multiplying (20) by η_{nj}, adding j from 1 to n and integrating from 0 to T, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial^2 u_n}{\partial t^2} u_n dxdt + \int_0^T \int_\Omega |\Delta u_n|^{p(x)} + \nabla \left(\frac{\partial u_n}{\partial t}\right) \nabla u_n dxdt \\
&+ \int_0^T \int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right) u_n dxdt = \int_0^T \int_\Omega g_n(x,t) u_n dxdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\le \int_0^T \int_\Omega \left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\, dxdt. \\
&= \int_0^T \int_\Omega g_n u_n - f\left(x,t,\frac{\partial u_n}{\partial t}\right) u_n - \nabla \left(\frac{\partial u_n}{\partial t}\right) \nabla u_n dxdt \\
&\quad - \int_\Omega \frac{\partial u_n(x,T)}{\partial t} u_n(x,T) dx + \int_\Omega \frac{\partial u_n(x,0)}{\partial t} u_n(x,0) dx + \int_0^T \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dxdt \\
&\quad - \int_0^T \int_\Omega \left(|\Delta u_n|^{p(x)-2}\Delta u_n \Delta u + |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\, dxdt.
\end{aligned}$$

By equation (50), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^T \int_\Omega \left(|\Delta u_n|^{p(x)-2}\Delta u_n \Delta u + |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\, dxdt \\
&\le \int_0^T \int_\Omega gu - f\left(x,t,\frac{\partial u}{\partial t}\right) u - \xi \Delta u dxdt - \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |\nabla u(x,0)|^2 dx - \int_\Omega \bar{u}(x,T)u(x,T)dx + \int_\Omega u_1 u_0 dx + \int_0^T \int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dxdt. \\
&= 0.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\, dxdt = 0.$$

We define

$$
\begin{aligned}
Q_1 &= \big\{(x,t) \in Q_T;\ p(x) \geq 2\big\},\\
Q_2 &= \big\{(x,t) \in Q_T;\ 1 < p(x) < 2\big\},
\end{aligned}
$$

then, when $n \to \infty$,

$$
\begin{aligned}
&\int_{Q_1} |\Delta u_n - \Delta u|^{p(x)} dxdt\\
&\leq c_{25} \int_{Q_1} \left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\, dxdt \to 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_{Q_2} |\Delta u_n - \Delta u|^{p(x)} dxdt\\
&\leq c_{26} \Bigg(\left\| \left[\left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\right]^{\frac{p(x)}{2}} \right\|_{L^{\frac{2}{p(\cdot)}}(Q_2)}\\
&\qquad \left\| \left(|\Delta u_n|^{p(x)} + |\Delta u|^{p(x)}\right)^{\frac{2-p(x)}{2}} \right\|_{L^{\frac{2}{2-p(\cdot)}}(Q_2)} \Bigg) \to 0.
\end{aligned}
$$

Consequently, we obtain $\Delta u_n \to \Delta u$ in $L^{p(x)}(Q_T)$, then there is a subsequence of u_n such that $\Delta u_n \to \Delta u$ a.e. in Q_T. Besides that,

$$
|\Delta u_n|^{p(x)-2}\Delta u_n \to |\Delta u|^{p(x)-2}\Delta u \quad \text{a.e. for} \quad (x,t) \in Q_T.
$$

Using Theorem 6, we obtain that $\xi = |\Delta u|^{p(x)-2}\Delta u$.

Therefore, we conclude the proof of the theorem of the existence of weak global solutions to Problem (1). □

4. CONCLUSION

We studied a nonlinear fourth-order beam equation with a strong dissipation and a lower order perturbation with the $p(x)$-biharmonic operator considering $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$, a bounded domain. Using Faedo-Galerkin method, we proved the local and global existence of weak solutions with mild assumptions on the variable exponent $p(\cdot)$.

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