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Fixed point theorems for cyclic contractions in S-metric spaces involving C-class function

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Abstract. In this paper, we study the class of cyclic contractions in the setting of S-metric spaces involving C-class function and establish some fixed point theorems in the setting of complete S-metric spaces. We support our results with some examples. Our results extend and generalize several results from the existing literature (see, e.g., [3, 8, 9, 14, 15, 20] and many others) to the case of more general ambient space and contraction condition.

1. Introduction

The Banach contraction mappings principle is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. In fixed point theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle, which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. For the sake of completeness here we mention this celebrated theorem below.

Let (X, d) be a metric space. A mapping $S\colon X \to X$ is called contraction if for each $x, y \in X$, there exists a constant $k \in [0, 1)$ such that

$$d(S(x), S(y)) \leq k\, d(x, y). \tag{1}$$

If the metric space (X, d) is complete, then the mapping satisfying (1) has a unique fixed point (Banach contraction mapping principle). Inequality (1) also implies the continuity of S.

It is no surprise that there is a great number of generalizations of this fundamental result. They go in several directions-modifying the basic contractive condition or changing the ambient space. Concerning the first direction we mention Hardy-Rogers and Ćirić quasi-contraction type conditions (see [18]), so called weakly contractive conditions of Alber and Guerre-Delabrieer

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[1] and Rhoades [19] and altering distance functions used by Khan et al. [12] and Boyd and Wong [5].

In 2003, Kirk et al. [13] introduced cyclic representation and cyclic contraction in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings and has been further used by several authors to obtain various fixed point results for not necessary continuous mappings in different spaces (see, e.g., [3, 7–11, 14–17] and others).

On the other hand, Sedghi et al. [20] in 2012 introduced the notion of S-metric spaces which generalized G-metric spaces and D^*-metric spaces. In [20] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

Recently, Gupta [8] introduced the concept of cyclic contraction in S-metric spaces and proved some fixed theorems in the said spaces which are proper generalizations of the results of Sedghi et al. [20].

The concept of C-class functions was introduced by Ansari [2] which actually covers a large class of contractive conditions.

In this article, we generalize the results of Gupta [8] (IJAA, 3 (2) (2013), 119–130) by using the concept of C-class functions.

2. PRELIMINARIES

One of the amazing generalizations of the Banach's contraction principle was initiated by Kirk et al. [13] via cyclic contraction.

Definition 1 ([13]). Let X be a nonempty set, $m \in \mathbb{N}$ and let $f\colon X \to X$ be a self-mapping. Then $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f, if

a) A_i, $i = 1, 2, \dots, m$ are nonempty subsets of X;
b) $f(A_1) \subset A_2, f(A_2) \subset A_3, \dots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

They proved the following fixed point result.

Theorem 1 ([13]). *Let (X, d) be a complete metric space, $f\colon X \to X$ and let $X = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to f. Suppose that f satisfies the following condition:*

$$d(fx, fy) \leq \psi(d(x, y)), \tag{2}$$

for all $x \in A_i$, $y \in A_{i+1}$, $i \in \{1, 2, \dots, m\}$, where $A_{m+1} = A_1$ and $\psi\colon [0, \infty) \to [0, \infty)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then f has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Notice that although a contraction is continuous, cyclic contraction need not be. This is one of the important observation of this theorem.

In 2010, Păcurar and Rus [15] introduced the following notion of cyclic weaker φ-contraction.

Definition 2 ([15])**.** Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ be closed nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. An operator $f\colon X \to X$ is called a cyclic weaker φ-contraction if

(1′) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f;
(2′) there exists a continuous, nondecreasing function $\varphi\colon [0,1) \to [0,1)$ with $\varphi(t) > 0$, for $t \in (0,1)$ and $\varphi(0) = 0$ such that

$$d(fx, fy) \le d(x,y) - \varphi(d(x,y)), \tag{3}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$, where $A_{m+1} = A_1$.

They proved the following result.

Theorem 2 ([15])**.** *Suppose f is a cyclic weaker φ-contraction on a complete metric space (X,d). Then f has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.*

We need the following definitions and lemmas in the sequel.

Definition 3 ([20])**.** Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:

(S1) $S(x,y,z) = 0$ if and only if $x = y = z$;
(S2) $S(x,y,z) \le S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 1 ([20])**.** Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.

Example 2 ([20])**.** Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Example 3 ([21])**.** Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$, for all $x, y, z \in \mathbb{R}$, is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Lemma 1 ([20, Lemma 2.5])**.** *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$, for all $x, y \in X$.*

Lemma 2 ([20, Lemma 2.12])**.** *Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x,x,y)$ as $n \to \infty$.*

Definition 4 ([20])**.** Let (X,S) be an S-metric space.

(1″) A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2″) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

(3″) The S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.

Every S-metric on X defines a metric d_S on X by

$$d_S = S(x, x, y) + S(y, y, x) \ \forall\, x, y \in X. \tag{4}$$

Let τ be the set of all subsets A of X with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X. Also, a nonempty subset A in the S-metric space (X, S) is S-closed if $\bar{A} = A$.

Lemma 3 ([8, Lemma 8]). *Let (X, S) be an S-metric space and A is a nonempty subset of X. Then A is said to be S-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

Definition 5 ([20]). Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that

$$S(Tx, Ty, Tz) \le L\, S(x, y, z), \tag{5}$$

for all $x, y, z \in X$. If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point.

Definition 6 ([2]). A mapping $F\colon [0, \infty) \times [0, \infty) \to R$ is called a C-class function if it is continuous and satisfies following axioms:

(i) $F(s, t) \le s$,
(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

An extra condition on F is that $F(0, 0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 4 ([2]). Define a function $F\colon [0, \infty) \times [0, \infty) \to R$ by

(i) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$,
(ii) $F(s, t) = ms, 0 < m < 1$, $F(s, t) = s \Rightarrow s = 0$,
(iii) $F(s, t) = \frac{s}{(1+t)^r}$, $r \in (0, \infty)$, $F(s, t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$,
(iv) $F(s, t) = \frac{\log(t+a^s)}{1+t}$, $a > 1$, $F(s, t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$,
(v) $F(s, t) = \frac{\ln(1+a^s)}{2}$, $a > e$, $F(s, 1) = s \Rightarrow s = 0$,
(vi) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0, \infty)$, $F(s, t) = s \Rightarrow t = 0$,
(vii) $F(s, t) = s \log_{t+a} a$, $a > 1$, $F(s, t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$,
(viii) $F(s, t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right)$, $F(s, t) = s \Rightarrow t = 0$,
(ix) $F(s, t) = s\beta(s)$, where $\beta\colon [0, \infty) \to [0, 1)$ and is continuous, $F(s, t) = s \Rightarrow s = 0$,
(x) $F(s, t) = s - \left(\frac{t}{k+t}\right)$, $F(s, t) = s \Rightarrow t = 0$,
(xi) $F(s, t) = s - \varphi(s)$, $F(s, t) = s \Rightarrow s = 0$, here $\varphi\colon [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$,
(xii) $F(s, t) = sh(s, t)$, $F(s, t) = s \Rightarrow s = 0$, here $h\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(s, t) < 1$ for all $t, s > 0$,

(xiii) $F(s,t) = s - \left(\frac{2+t}{1+t}t\right)$, $F(s,t) = s \Rightarrow t = 0$,

(xiv) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$,

(xv) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\phi\colon [0,\infty) \to [0,\infty)$ is a upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$,

(xvi) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$,

(xvii) $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t} \,\mathrm{d}\,x$, where Γ is the Euler Gamma function.

Then F are elements of $\mathcal{C}$.

Definition 7 ([2])**.** A function $\psi\colon [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(ψ_1) ψ is non-decreasing and continuous function,

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1 ([2])**.** We denote Ψ the class of all altering distance functions.

Definition 8 ([2])**.** A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be an ultra altering distance function, if it is continuous, non-decreasing such that $\varphi(t) > 0$, for $t > 0$ and $\varphi(0) \geq 0$.

Remark 2 ([2])**.** We denote Φ_u the class of all ultra altering distance functions.

3. Main Results

In this section, we shall establish some fixed point theorems for cyclic contraction in the setting of complete S-metric spaces. First, we shall prove the following lemma.

Lemma 4. *Let (X,S) be a complete S-metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, x_{n+1}). \tag{6}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}_{n\in\mathbb{N}}$ with $n(k) > m(k) > k$ of positive integers such that the following four sequences tend to ε for $k \to \infty$:

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)}),\ S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}),\ S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}),$$
$$S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}).$$

Proof. If suppose that $\{x_n\}$ is a sequence in X satisfying condition (6) which is not Cauchy, then there exists $\varepsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{7}$$

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{8}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (7). Then

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{9}$$

Now, using (8), (S2) and Lemma 1, we have

$$\begin{aligned}
\varepsilon &\leq S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\
&= S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\
&= 2S(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\
&\leq \varepsilon + 2S(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}) \quad \text{(by (9))}.
\end{aligned} \tag{10}$$

Letting $k \to +\infty$ in equation (10) and using (6), we get

$$\lim_{k\to\infty} S(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{11}$$

Again, with the help of (S2) and Lemma 1, we have

$$\begin{aligned}
S(x_{m(k)}, x_{m(k)}, x_{n(k)}) &\leq 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) \\
&\quad +S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) \\
&\leq 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) \\
&\quad +2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) \\
&\quad +S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}).
\end{aligned} \tag{12}$$

Also, with the help of (S2) and Lemma 1, we have

$$\begin{aligned}
S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) &\leq 2S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) \\
&\quad +S(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) \\
&= 2S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) \\
&\quad +S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}).
\end{aligned} \tag{13}$$

Letting $k \to +\infty$ in equation (13) and using (6), (11) and (12), we get

$$\lim_{k\to\infty} S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{14}$$

Again, with the help of (8), (S2) and Lemma 1, we have

$$\begin{aligned}
\varepsilon &\leq S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\
&= S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) \\
&\quad +S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}).
\end{aligned} \tag{15}$$

Letting $k \to +\infty$ in equation (15) and using (6) and (9), we get

$$\lim_{k\to\infty} S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) = \varepsilon. \tag{16}$$

Also, note that with the help of (S2) and Lemma 1, we have

$$\begin{aligned}
S(x_{m(k)}, x_{m(k)}, x_{n(k)}) &\leq 2S(x_{m(k)}, x_{m(k)}, x_{m(k)+1}) \\
&\quad +S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}) \\
&= 2S(x_{m(k)+1}, x_{m(k)+1}, x_{m(k)}) \\
&\quad +S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}).
\end{aligned} \tag{17}$$

Again, note that with the help of (S2) and Lemma 1, we have

$$\begin{aligned}
S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) &\leq 2S(x_{m(k)+1}, x_{m(k)+1}, x_{m(k)}) \\
&\quad +S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&= 2S(x_{m(k)+1}, x_{m(k)+1}, x_{m(k)}) \\
&\quad +S(x_{m(k)}, x_{m(k)}, x_{n(k)}).
\end{aligned} \tag{18}$$

Letting $k \to +\infty$ in equation (18) and using (6), (11) and (17), we get

$$\lim_{k\to\infty} S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) = \varepsilon. \tag{19}$$

This completes the proof. □

Theorem 3. *Let (X, S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^m A_i$ and $f\colon Y \to Y$. Suppose that:*

- a1) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;*
- a2) *there exists a continuous, nondecreasing function $\varphi\colon [0, +\infty) \to [0, +\infty)$ with $\varphi(t) > 0$, for $t > 0$ and such that for any $(x, y, z) \in A_i \times A_i \times A_{i+1}$, $i = 1, 2, \dots, m$, with $A_{m+1} = A_1$,*

$$S(fx, fy, fz) \leq F\Big(S(x, y, z), \varphi(S(x, y, z))\Big), \tag{20}$$

where $F \in \mathcal{C}$.

Then f has a unique fixed point $u \in \bigcap_{i=1}^m A_i$.

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$, $n = 0, 1, 2, \dots$. We shall prove that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_{n+2}) = 0. \tag{21}$$

If for some k, we have $\lim_{k\to\infty} S(x_{k+1}, x_{k+1}, x_{k+2}) = 0$, then equation (21) follows immediately. So, we can assume that $S(x_{n+1}, x_{n+1}, x_{n+2}) > 0$, for all n. From the condition a1), we observe that for all n, there exists $i = i_n \in \{1, 2, \dots, m\}$ such that $(x_{n+1}, x_{n+1}, x_{n+2}) \in A_i \times A_i \times A_{i+1}$. Then applying condition (20) for $x = y = x_n$ and $z = x_{n+1}$ to obtain

$$\begin{aligned}
S(x_{n+1}, x_{n+1}, x_{n+2}) &= S(fx_n, fx_n, fx_{n+1}) \\
&\leq F\Big(S(x_n, x_n, x_{n+1}), \varphi\big(S(x_n, x_n, x_{n+1})\big)\Big) \\
&\leq S(x_n, x_n, x_{n+1}),
\end{aligned} \tag{22}$$

for $n \in \mathbb{N}$.

Thus the sequence $\{S(x_{n+1}, x_{n+1}, x_{n+2})\}$ is decreasing and bounded from the below, thus there exists a real number $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_{n+2}) = r. \tag{23}$$

Then from (21), taking the limit as $n \to +\infty$, we get

$$r \leq F\big(r, \varphi(r)\big), \tag{24}$$

so $r = 0$ or $\varphi(r) = 0$, therefore $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_{n+2}) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the space (X, S). Suppose that this is not the case. Then, using Lemma 4, we get that there exists $\varepsilon > 0$ and two sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of positive integers such that $n(k) > m(k) > k$ and sequences in Lemma 4 tend to ε when $k \to +\infty$.

Elements $\{x_{m(k)}\}$ and $\{x_{n(k)-1}\}$ might not lie in adjacently labelled sets A_i and A_{i+1}. However, for all k, there exists $j(k) \in \{1, 2, \ldots, p\}$ such that $n(k) - 1 - m(k) + j(k) \equiv l(p)$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)-1}$ lie in adjacently labelled sets A_i and A_{i+1} for certain $i \in \{1, 2, \ldots, p\}$. To simplify the method, we will suppose that already $(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \in A_i \times A_i \times A_{i+1}$. Applying condition (20) for $x = y = x_{m(k)}$ and $z = x_{n(k)-1}$, we obtain

$$\begin{aligned} S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) &= S(fx_{m(k)}, fx_{m(k)}, fx_{n(k)-1}) \\ &\leq F\Big(S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}), \\ &\qquad \varphi\big(S(x_{m(k)}, x_{m(k)}, x_{n(k)-1})\big)\Big). \end{aligned} \tag{25}$$

On taking the limit as $k \to +\infty$ in (25), we get

$$\varepsilon \leq F\big(\varepsilon, \varphi(\varepsilon)\big), \tag{26}$$

so $\varepsilon = 0$ or $\varphi(\varepsilon) = 0$, that is, $\varepsilon = 0$ which is a contraction.

Thus $\{x_n\}$ is a Cauchy sequence. Since (X, S) is complete and Y is closed, it follows that the sequence $\{x_n\}$ converges to some $u \in Y$ and since $Y = \bigcup_{i=1}^{m} A_i$, so $u \in \bigcup_{i=1}^{m} A_i$. We will prove that u is a fixed point of f.

Using inequality (20) for $x = y = u$ and $z = x_{n+1}$ (which is possible since u belongs to each A_i), we obtain that

$$\begin{aligned} S(u, u, fu) &\leq 2S(u, u, x_{n+2}) + S(fu, fu, x_{n+2}) \\ &= 2S(u, u, x_{n+2}) + S(fu, fu, fx_{n+1}) \\ &\leq 2S(u, u, x_{n+2}) + F\Big(S(u, u, x_{n+1}), \varphi\big(S(u, u, x_{n+1})\big)\Big). \end{aligned}$$

On letting $n \to +\infty$ in (27) and using property of F and φ, we get

$$S(u, u, fu) \leq 0, \quad \text{that is,} \quad S(u, u, fu) = 0. \tag{27}$$

Thus, $u = fu$. Hence u is a fixed point of f. Now, we show that the fixed point of f is unique.

Suppose that there exists $u_1 \in Y$ such that $fu_1 = u_1$. Then from condition (20), we have

$$\begin{aligned} S(u,u,u_1) &= S(fu,fu,fu_1) \\ &\leq F\Big(S(u,u,u_1), \varphi\big(S(u,u,u_1)\big)\Big) \\ &\leq S(u,u,u_1), \end{aligned}$$

which implies that $S(u,u,u_1) = 0$. Hence $u = u_1$. Thus the fixed point of f is unique. This completes the proof. $\square$

If we take $F(s,t) = s - t$ in the Theorem 3, then we obtain the following result as corollary.

Corollary 1. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ be nonempty closed subsets of X, $Y = \cup_{i=1}^m A_i$ and $f\colon Y \to Y$. Suppose that:*

a1) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;*

a2) *there exists a continuous, nondecreasing function $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(t) > 0$, for $t > 0$ and such that for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1, 2, \dots, m$, with $A_{m+1} = A_1$,*

$$S(fx,fy,fz) \leq S(x,y,z) - \varphi(S(x,y,z)).$$

Then f has a unique fixed point $u \in \cap_{i=1}^m A_i$.

Remark 3. Corollary 1 extends the corresponding result of Păcurar and Rus [15] from complete metric space to the setting of complete S-metric space.

If we take $F(s,t) = k\,s$, where $0 < k < 1$ in the Theorem 3, then we obtain the following result as corollary.

Corollary 2. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^m A_i$, $f\colon Y \to Y$ an operator and $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Suppose that f satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1, 2, \dots, m$, with $A_{m+1} = A_1$,*

$$S(fx,fy,fz) \leq k\,S(x,y,z),$$

where $0 < k < 1$ is a constant. Then f has a unique fixed point $u \in \bigcap_{i=1}^m A_i$.

Remark 4. Corollary 2 extends the corresponding result of Sedghi et al. [20] for cyclic contraction.

If we take $F(s,t) = L\,s$, where $0 < L < 1$ and $A_1 = A_2 = \dots = A_m = X$ in the Theorem 3, then we obtain the following result as corollary.

Corollary 3 ([20])**.** *Let (X,S) be a complete S-metric space and $f\colon X \to X$ be a mapping such that for any $x, y, z \in X$,*

$$S(fx,fy,fz) \leq L\,S(x,y,z),$$

where $0 < L < 1$ is a constant. Then f has a unique fixed point in X.

Remark 5. Corollary 3 also extends the well-known Banach fixed point theorem [4] form complete metric space to the setting of complete S-metric space.

If we take $F(s,t) = \beta(s)\, s$, where $\beta\colon [0,+\infty) \to [0,1)$ is a continuous function, in the Theorem 3, then we obtain the following result as corollary.

Corollary 4. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^m A_i$, $f\colon Y \to Y$ an operator and $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Suppose that f satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1,2,\ldots,m$, with $A_{m+1} = A_1$,*

$$S(fx,fy,fz) \leq \beta\big(S(x,y,z)\big)\, S(x,y,z).$$

Then f has a unique fixed point $u \in \bigcap_{i=1}^m A_i$.

Theorem 4. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^m A_i$, $f\colon Y \to Y$ an operator and $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Suppose that f satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1,2,\ldots,m$, with $A_{m+1} = A_1$,*

$$\psi\Big(S(fx,fy,fz)\Big) \leq F\Big(\psi\big(M_f^1(x,y,z)\big), \varphi\big((M_f^1(x,y,z)\big)\Big), \tag{28}$$

where

$$M_f^1 = \max\Big\{S(x,y,z), S(fx,fx,x), S(fy,fy,y)\Big\},$$

$F \in \mathcal{C}$, $\psi \in \Psi$ and $\varphi \in \Phi_u$. Then f has a unique fixed point $v \in \bigcap_{i=1}^m A_i$.

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$, $n = 0,1,2,\ldots$. We shall prove that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_{n+2}) = 0. \tag{29}$$

If for some k, we have $\lim_{k\to\infty} S(x_{k+1}, x_{k+1}, x_{k+2}) = 0$, then equation (29) follows immediately. So, we can assume that $S(x_{n+1}, x_{n+1}, x_{n+2}) > 0$ for all n. From the condition $a1$), we observe that for all n, there exists $i = i_n \in \{1,2,\ldots,m\}$ such that $(x_{n+1}, x_{n+1}, x_{n+2}) \in A_i \times A_i \times A_{i+1}$. Then applying condition (28) for $x = y = x_n$ and $z = x_{n+1}$ and using Lemma 1 to obtain

$$\begin{aligned} \psi\Big(S(x_{n+1}, x_{n+1}, x_{n+2})\Big) &= S(fx_n, fx_n, fx_{n+1}) \\ &\leq F\Big(\psi\big(M_f^1(x_n, x_n, x_{n+1})\big), \varphi\big(M_f^1(x_n, x_n, x_{n+1})\big)\Big), \end{aligned} \tag{30}$$

where

$$\begin{aligned} & M_f^1(x_n, x_n, x_{n+1}) \\ = \; & \max\Big\{S(x_n, x_n, x_{n+1}), S(fx_n, fx_n, x_n), S(fx_n, fx_n, x_n)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{S(x_n,x_n,x_{n+1}),S(x_{n+1},x_{n+1},x_n),S(x_{n+1},x_{n+1},x_n)\Big\} \\
&= \max\Big\{S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_{n+1})\Big\} \\
&= S(x_n,x_n,x_{n+1}). \qquad (31)
\end{aligned}
$$

From equation (30) and (31), we obtain

$$
\begin{aligned}
\psi\Big(S(x_{n+1},x_{n+1},x_{n+2})\Big) &\le F\Big(\psi\big(S(x_n,x_n,x_{n+1})\big),\varphi\big(S(x_n,x_n,x_{n+1})\big)\Big) \\
&\le \psi\Big(S(x_n,x_n,x_{n+1})\Big). \qquad (32)
\end{aligned}
$$

Hence, we have

$$S(x_{n+1},x_{n+1},x_{n+2}) \le S(x_n,x_n,x_{n+1}).$$

Thus the sequence $\{S(x_{n+1},x_{n+1},x_{n+2})\}$ is decreasing and bounded from the below, thus there exists a real number $r \ge 0$ such that

$$\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+2}) = r. \qquad (33)$$

Then from (32), taking the limit as $n \to +\infty$, we get

$$\psi(r) \le F\big(\psi(r),\varphi(r)\big), \qquad (34)$$

so either $\psi(r) = 0$ or $\varphi(r) = 0$, by the property of ψ, we have $r = 0$. Therefore $\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+2}) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the space (X,S). From Lemma 4 and Theorem 3, we can easily show that $\{x_n\}$ is a Cauchy sequence. Since (X,S) is complete and Y is closed, it follows that the sequence $\{x_n\}$ converges to some $v \in Y$ and since $Y = \cup_{i=1}^m A_i$, so $v \in \cup_{i=1}^m A_i$. We will prove that v is a fixed point of f.

Using inequality (28) for $x = y = v$ and $z = x_{n+1}$ (which is possible since v belongs to each A_i) and using Lemma 1, we obtain that

$$
\begin{aligned}
&\psi\Big(S(v,v,fv)\Big) \\
\le\ & 2S(v,v,x_{n+2}) + S(fv,fv,x_{n+2}) \\
=\ & 2S(v,v,x_{n+2}) + S(fv,fv,fx_{n+1}) \\
\le\ & 2S(v,v,x_{n+2}) + F\Big(\psi\big(M_f^1(v,v,x_{n+1})\big),\varphi\big(M_f^1(v,v,x_{n+1})\big)\Big), \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
M_f^1(v,v,x_{n+1}) &= \max\Big\{S(v,v,x_{n+1}),S(fv,fv,v),S(fv,fv,v)\Big\} \\
&= \max\Big\{S(v,v,x_{n+1}),S(v,v,fv),S(v,v,fv)\Big\}. \qquad (36)
\end{aligned}
$$

On letting $n \to +\infty$ in equation (36), we get

$$M_f^1(v,v,x_{n+1}) \to S(v,v,fv). \qquad (37)$$

On letting $n \to +\infty$ in equation (35) and using continuity of F and (37), we get

$$\begin{aligned} \psi\Big(S(v,v,fv)\Big) &\leq F\Big(\psi\big(S(v,v,fv)\big),\big(S(v,v,fv)\big)\Big) \\ &\leq \psi\Big(S(v,v,fv)\Big), \end{aligned} \tag{38}$$

which implies that $S(v,v,fv)=0$. Thus, $v=fv$. Hence v is a fixed point of f. The uniqueness of the fixed point of f can be proved in the same way as in Theorem 3. This completes the proof. □

If we take $F(s,t)=s-t$ in the Theorem 4, then we obtain the following result as corollary.

Corollary 5. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y=\bigcup_{i=1}^m A_i$, $f\colon Y \to Y$ an operator and $Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Suppose that f satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i=1,2,\ldots,m$, with $A_{m+1}=A_1$,*

$$\psi\Big(S(fx,fy,fz)\Big) \leq \psi\Big(M_f^1(x,y,z)\Big)-\varphi\Big((M_f^1(x,y,z)\Big),$$

where M_f^1 is as in Theorem 4, $\psi \in \Psi$ and $\varphi \in \Phi_u$. Then f has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Remark 6. If we take $F(s,t)=s-t$, $\max\{S(x,y,z),S(fx,fx,x),S(fy,fy,y)\} = S(x,y,z)$ and $\psi(t)=t$, for all $t \geq 0$ in the Theorem 4, then we obtain the generalization of corresponding result due to Păcurar and Rus [15].

Theorem 5. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y=\cup_{i=1}^m A_i$, $f\colon Y \to Y$ an operator and $Y=\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Suppose that f satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i=1,2,\ldots,m$, with $A_{m+1}=A_1$,*

$$\psi\Big(S(fx,fy,fz)\Big) \leq F\Big(\psi\big(M_f^j(x,y,z)\big),\varphi\big((M_f^j(x,y,z)\big)\Big),\ j=2,3, \tag{39}$$

where

$$M_f^2=\max\Big\{S(x,y,z),S(fx,fx,x),S(fy,fy,y),S(fz,fz,z),S(fy,fy,z)\Big\}$$

and

$$M_f^3=\max\Big\{S(x,y,z),S(fx,fx,x),S(fy,fy,z),\tfrac{1}{2}[S(fx,fx,z)+S(fy,fy,x)]\Big\},$$

$F \in \mathcal{C}$, $\psi \in \Psi$ and $\varphi \in \Phi_u$. Then f has a unique fixed point $u \in \bigcap_{i=1}^m A_i$.

Proof. The proof follows from Theorem 4. □

Now, as common applications of fixed point theorems we provide some corollaries for integral type contraction (taking $A_1 = A_2 = \cdots = A_m = X$).

Denote Φ the set of functions $\phi\colon [0, +\infty) \to [0, +\infty)$ satisfying the following hypothesis:

($\mathcal{H}1$) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;
($\mathcal{H}2$) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \phi(s)\, ds > 0$.

Corollary 6. *Let (X, S) be a complete S-metric space. Let $f\colon X \to X$ be a mapping satisfying the following inequality:*

$$\int_0^{S(fx,fy,fz)} \psi(s)\,\mathrm{d}\,s \leq \beta(S(x,y,z)) \int_0^{S(x,y,z)} \psi(s)\,\mathrm{d}\,s,$$

for all $x, y, z \in X$, where β is as in Example 4 and $\psi \in \Phi$. Then f has a unique fixed point in X.

Proof. Follows from Corollary 4 by taking

$$t = \int_0^t \psi(s)\,\mathrm{d}\,s.$$

$\square$

Remark 7. If we take $\beta(S(x, y, z)) = k$, where $0 < k < 1$ is a constant, in the Corollary 6, then it extends Theorem 2.1 of Branciari [6] from complete metric space to the setting of complete S-metric space.

Now, we give some examples in support of our results.

Example 5. Let $X = [0, 1]$ and $f\colon X \to X$ be given by $f(x) = \frac{x}{8}$. Let $A_1 = A_2 = \cdots = A_m = [0, 1]$. Define the function $S\colon X^3 \to [0, \infty)$ by $S(x, y, z) = \max\{x, y, z\}$ for all for all $x, y, z \in X$, then S is an S-metric on X. Now, define the function $F(s, t) = \frac{s}{1+t}\colon [0, \infty) \to [0, 1)$ and $\varphi(t) = 1$. Consider all the cases and the general case if $x \geq y \geq z$ for all $x, y, z \in X$. It is clear that $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f.

(1) Now, we have

$$\begin{aligned} S(fx, fy, fz) &= S\Big(\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big) \\ &= \max\Big\{\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big\} \\ &= \frac{x}{8} \leq \frac{x}{1+1} = \frac{\max\{x, y, z\}}{1 + \varphi\big(\max\{x, y, z\}\big)} \\ &= \frac{S(x, y, z)}{1 + \varphi\big(S(x, y, z)\big)}. \end{aligned}$$

Clearly, all the conditions of Theorem 3 are satisfied and $x = 0$ is the unique fixed point of f.

(2) Again, consider the inequality of Corollary 2, we have

$$\begin{aligned} S(fx, fy, fz) &= S\Big(\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big) \\ &= \max\Big\{\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big\} = \frac{x}{8}, \end{aligned}$$

or

$$\frac{x}{8} \leq k\, S(x, y, z) = k \max\{x, y, z\} = k\, x,$$

or

$$k \geq \frac{1}{8}.$$

If we take $0 < k < 1$, then all the conditions of Corollary 2 are satisfied and $x = 0 \in \bigcup_{i=1}^{m} A_i$ is a unique fixed point of f.

Example 6. Let $X = [0, 1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Suppose $A_1 = [0, 1]$, $A_2 = [0, \frac{1}{2}]$, $A_3 = [0, \frac{1}{4}]$, $A_4 = [0, \frac{1}{8}]$ and $Y = \bigcup_{i=1}^{4} A_i$. Define $f\colon Y \to Y$ such that $f(x) = \frac{x}{3}$, for all $x \in Y$. Define the function $F(s, t) = s - t\colon [0, \infty) \to [0, 1)$ and $\varphi(t) = \frac{t}{2}$, for all $t > 0$. Without loss of generality, we assume that $x \geq y \geq z$, for all $x, y, z \in Y$. Then

$$S(fx, fy, fz) = \max\{fx, fy, fz\} = \max\Big\{\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\Big\} = \frac{x}{3},$$

$$S(x, y, z) = \max\{x, y, z\} = x,$$

and

$$\varphi(S(x, y, z)) = \frac{x}{2}.$$

Now, consider the inequality of Corollary 1, we have

$$S(fx, fy, fz) = \frac{x}{3} \leq x - \frac{x}{2} = \frac{x}{2}$$

or

$$\frac{1}{3} \leq \frac{1}{2},$$

which is true. Thus, all the conditions of Corollary 1 are satisfied and $u = 0 \in \bigcup_{i=1}^{4} A_i$ is a unique fixed point of f.

Example 7. Let $X = [0, 1]$ and $S\colon X^3 \to \mathbb{R}_+$ be given by

$$S(x, y, z) = \begin{cases} |x - z| + |y - z|, & \text{if } x, y, z \in [0, 1), \\ \\ 1, & \text{if } x = 1 \text{ or } y = 1 \text{ or } z = 1, \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space.

If a mapping $f\colon X \to X$ is given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x, y, z \in [0,1), \\ \\ \frac{1}{6}, & \text{if } x = y = z = 1, \end{cases}$$

and $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, 1]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to f. Now, define the function $F(s,t) = s - t\colon [0,\infty) \to [0,1)$ and $\varphi(t) = \frac{t}{5}$, for all $t > 0$. Without loss of generality, we assume that $x \geq y \geq z$, for all $x, y, z \in X$. Indeed, consider the following cases.

Case I: If $x, y \in [0, \frac{1}{2}]$, $z \in [\frac{1}{2}, 1)$ or $z \in [0, \frac{1}{2}]$, $x, y \in [\frac{1}{2}, 1)$. Then

$$\begin{aligned} S(fx, fy, fz) &= S\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big) = 0 \\ &\leq S(x,y,z) - \varphi\left(S(x,y,z)\right). \end{aligned}$$

Thus, the inequality of Corollary 1 is trivially satisfied.

Case II: If $x, y \in [0, \frac{1}{2}]$ and $z = 1$. Then

$$S(fx, fy, fz) = S\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\Big) = \frac{2}{3},$$
$$S(x,y,z) = 1$$

and

$$\varphi\left(S(x,y,z)\right) = \frac{1}{5}.$$

Consequently,

$$\begin{aligned} S(fx, fy, fz) &= \frac{2}{3} \leq S(x,y,z) - \varphi\left(S(x,y,z)\right) \\ &= 1 - \frac{1}{5} = \frac{4}{5}, \end{aligned}$$

which is true. Thus, all the conditions of Corollary 1 are satisfied.

Case III: If $x, z \in [0, \frac{1}{2}]$ and $y = 1$. Then

$$S(fx, fy, fz) = S\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right) = \frac{1}{3},$$
$$S(x,y,z) = 1$$

and

$$\varphi\left(S(x,y,z)\right) = \frac{1}{5}.$$

Consequently,

$$\begin{aligned} S(fx, fy, fz) &= \frac{1}{3} \leq S(x,y,z) - \varphi\left(S(x,y,z)\right) \\ &= 1 - \frac{1}{5} = \frac{4}{5}, \end{aligned}$$

which is true. Thus, all the conditions of Corollary 1 are satisfied.

Case IV: If $y, z \in [0, \frac{1}{2}]$ and $x = 1$. Then

$$S(fx, fy, fz) = S\Big(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\Big) = \frac{1}{3},$$
$$S(x, y, z) = 1$$

and

$$\varphi\left(S(x, y, z)\right) = \frac{1}{5}.$$

Consequently,

$$\begin{aligned} S(fx, fy, fz) &= \frac{1}{3} \leq S(x, y, z) - \varphi\left(S(x, y, z)\right) \\ &= 1 - \frac{1}{5} = \frac{4}{5}, \end{aligned}$$

which is true. Thus, all the conditions of Corollary 1 are satisfied.

Considering all the above cases, we conclude that the inequality used in Corollary 1 remains valid for φ and f constructed in the above example and consequently by applying Corollary 1, f has a unique fixed point (which is $v = \frac{1}{2} \in A_1 \cap A_2$).

Example 8. Let $X = [0, 1]$. We define $S \colon X^3 \to \mathbb{R}_+$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$. Suppose $A_1 = [0, 1]$, $A_2 = [0, \frac{1}{4}]$, $A_3 = [0, \frac{1}{28}]$, $A_4 = [0, \frac{1}{868}]$ and $Y = \bigcup_{i=1}^{4} A_i$. Consider the mapping $f \colon Y \to Y$ such that $f(x) = \frac{x^2}{1+3x}$ for all $x \in Y$. It is clear that $Y = \cup_{i=1}^{4} A_i$ is a cyclic representation of Y with respect to f. Further, consider the functions $\psi, \varphi \colon [0, +\infty) \to [0, +\infty)$ given by $\psi(t) = t$ and $\varphi(t) = \frac{t}{1+2t}$ for all $t \geq 0$ and define $F(s, t) = s - t \colon [0, \infty) \to [0, 1)$. Without loss of generality, we assume that $x \geq y \geq z$ for all $x, y, z \in Y$. Then, we have

$$S(fx, fy, fz) = \max\left\{\frac{x^2}{1+3x}, \frac{y^2}{1+3y}, \frac{z^2}{1+3z}\right\} = \frac{x^2}{1+3x},$$
$$S(x, y, z) = \max\{x, y, z\} = x,$$
$$S(fx, fx, x) = \max\left\{\frac{x^2}{1+3x}, \frac{x^2}{1+3x}, x\right\} = x,$$
$$S(fy, fy, y) = \max\left\{\frac{y^2}{1+3y}, \frac{y^2}{1+3y}, y\right\} = y,$$
$$S(fz, fz, z) = \max\left\{\frac{z^2}{1+3z}, \frac{z^2}{1+3z}, z\right\} = z,$$

$$S(fy, fy, z) = \max\left\{\frac{y^2}{1+3y}, \frac{y^2}{1+3y}, z\right\} = z,$$
$$S(fx, fx, z) = \max\left\{\frac{x^2}{1+3x}, \frac{x^2}{1+3x}, z\right\} = z,$$
$$[S(fy, fy, x) = \max\left\{\frac{y^2}{1+3y}, \frac{y^2}{1+3y}, x\right\} = x.$$

On the other hand,

$$\begin{aligned} M_f^1(x,y,z) &= \max\Big\{S(x,y,z), S(fx,fx,x), S(fy,fy,y)\Big\} \\ &= \max\{x, x, y\} = x, \end{aligned}$$

$$\begin{aligned} M_f^2(x,y,z) &= \max\Big\{S(x,y,z), S(fx,fx,x), S(fy,fy,y), \\ &\quad S(fz,fz,z), S(fy,fy,z)\Big\} \\ &= \max\{x, x, y, z, z\} = x \end{aligned}$$

and

$$\begin{aligned} M_f^3(x,y,z) &= \max\Big\{S(x,y,z), S(fx,fx,x), S(fy,fy,z), \\ &\quad \frac{1}{2}[S(fx,fx,z) + S(fy,fy,x)]\Big\} \\ &= \max\left\{x, x, z, \frac{1}{2}(z+x)\right\} = x. \end{aligned}$$

Result Analysis:

(1) Consider the inequality (28), we have

$$\begin{aligned} \psi\Big(S(fx,fy,fz)\Big) &= S(fx,fy,fz) = \frac{x^2}{1+3x} \\ &\le \psi\Big(M_f^1(x,y,z)\Big) - \varphi\Big(M_f^1(x,y,z)\Big) \\ &= x - \frac{x}{1+2x} = \frac{2x^2}{1+2x}, \end{aligned}$$

that is,

$$\frac{x^2}{1+3x} \le \frac{2x^2}{1+2x},$$

which is true. Hence

$$\psi\Big(S(fx,fy,fz)\Big) \le \psi\Big(M_f^1(x,y,z)\Big) - \varphi\Big(M_f^1(x,y,z)\Big),$$

holds true. Thus, the inequality (28) is satisfied, as well as the other assumptions of Theorem 4. We deduce that f has a unique fixed point $u \in A_1 \cap A_2 \cap A_3 \cap A_4 = \{0\}$.

(2) Consider the inequality (38) for $j = 2$, we have

$$\begin{aligned}\psi\Big(M_f^2(x,y,z)\Big) - \varphi\Big(M_f^2(x,y,z)\Big) &= \psi(x) - \varphi(x)\\ &= x - \frac{x}{1+2x} = \frac{2x^2}{1+2x} \geq \frac{x^2}{1+3x}\\ &= \psi\Big(S(fx,fy,fz)\Big).\end{aligned}$$

Hence

$$\psi\Big(S(fx,fy,fz)\Big) \leq \psi\Big(M_f^2(x,y,z)\Big) - \varphi\Big(M_f^2(x,y,z)\Big),$$

holds true. Thus, the inequality (38) for $j = 2$ is satisfied, as well as the other assumptions of Theorem 5. We deduce that f has a unique fixed point $u \in A_1 \cap A_2 \cap A_3 \cap A_4 = \{0\}$.

(3) Consider the inequality (38) for $j = 3$, we have

$$\begin{aligned}\psi\Big(M_f^3(x,y,z)\Big) - \varphi\Big(M_f^3(x,y,z)\Big) &= \psi(x) - \varphi(x)\\ &= x - \frac{x}{1+2x} = \frac{2x^2}{1+2x} \geq \frac{x^2}{1+3x}\\ &= \psi\Big(S(fx,fy,fz)\Big).\end{aligned}$$

Hence

$$\psi\Big(S(fx,fy,fz)\Big) \leq \psi\Big(M_f^3(x,y,z)\Big) - \varphi\Big(M_f^3(x,y,z)\Big),$$

holds true. Thus, the inequality (38) for $j = 3$ is satisfied, as well as the other assumptions of Theorem 5. We deduce that f has a unique fixed point $u \in A_1 \cap A_2 \cap A_3 \cap A_4 = \{0\}$.

4. Conclusion

In this paper, we study cyclic contraction in the setting of S-metric space using C-class function and establish some unique fixed point theorems for various cyclic contraction in the framework of complete S-metric spaces. Also we give some examples in support of our results. Our results extend, generalize and modify several results from the existing literature (see, e.g., [3,9,14,15,20] and many others) to the setting of complete S-metric spaces. The results also generalize the corresponding results of Gupta [8].

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