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Certain integral representations involving hypergeometric functions in two variables

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Abstract. Various integral representations of hypergeometric functions have been introduced and investigated due to their important applications in divers fields. In this article, we define some new Euler-type integral representations for the Horn's functions of two variables G_1, G_2, G_3 and H_1.

1. Introduction

Integral representations of hypergeometric functions have found applications in divers fields such as mathematics, physics, statistics, and engineering. Praveen et al. [21-29] studied several properties, various relations, different integral representations and many extensions of the hypergeometric functions. Hasanov et al. [12] studied some of the properties of the Horn type second-order double hypergeometric function H_2^* involving integral representations, differential equations, and generating functions. Choi et al. [9] introduced certain integral representations for Srivastava's triple hypergeometric functions H_A, H_B and H_C. Younis and Bin-Saad [18, 19] establish several integral representations and operational relations involving quadruple hypergeometric functions $X_i^{(4)} (i = 38, 40, 45, 48, 50)$. Younis and Nisar [20] introduce new integral representations of Euler-type for Exton's hypergeometric functions of four variables D_1, D_2, D_3, D_4 and D_5. Recently, several interesting and useful integral representations of many special functions have been investigated (see, e.g., [2-8, 11, 14]).

For our purpose, we begin by recalling Horn's hypergeometric functions of two variables $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 defined by (see

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[10, 13]):

$$G_1(a,b,c;x,y)=\sum_{m,n=0}^{\infty}(a)_{m+n}(b)_{n-m}(c)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},$$

(1) $$G_2(a,b,c,d;x,y)=\sum_{m,n=0}^{\infty}(a)_{m}(b)_{n}(c)_{n-m}(d)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$G_3(a,b;x,y)=\sum_{m,n=0}^{\infty}(a)_{2n-m}(b)_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_1(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}(c)_{n}}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_2(a,b,c,d;e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m}(c)_{n}(d)_n}{(e)_m}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_3(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n}}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_4(a,b;c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n}}{(c)_{m}(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_5(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_{n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_6(a,b,c;x,y)=\sum_{m,n=0}^{\infty}(a)_{2m-n}(b)_{n-m}(c)_{n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_7(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}(b)_{n}(c)_n}{(d)_{m}}\frac{x^m}{m!}\frac{y^n}{n!},$$

where $(a)_m$ is the well known Pochhammer symbol given by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}1, & (m=0),\\ a(a+1)\cdots(a+m-1), & (m\in\mathbb{N}:=\{1,2,\ldots\}).\end{cases}$$

Appell [1, 17] defined four hypergeometric functions of two variables denoted these by F_1, F_2, F_3 and F_4. We present here the definition of one of these functions

$$F_3(a,b,c,d;e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m}(b)_{n}(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

The aim of this paper is to present several new integral representations of Euler-type involving Horn's functions G_1, G_2, G_3 and H_1 in terms of hypergeometric functions of two variables.

2. Main results

Theorem 2.1. *Each of the following integral representations holds true:*

$$G_1(a,b,c;x,y) = \frac{\Gamma(a+a')(S-T)^a(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}} \int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{a'-1}}{(\alpha-T)^{a+a'}} \times G_2\left(a+a', 1-a', b, c; \frac{(S-T)(\alpha-R)x}{(S-R)(\alpha-T)}, \frac{-y}{(R-T)(S-\alpha)}\right) \mathrm{d}\,\alpha, \quad \left(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ T<R<S\right), \tag{2}$$

$$G_1(a,b,c;x,y) = \frac{4M_1^{a'}M_2^{b}\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')} \cdot \int_0^\infty\int_0^\infty \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a'-\frac{1}{2}}}{\left(1+M_1\sinh^2\alpha\right)^{a+a'}} \times \frac{\cosh\beta\left(\sinh^2\beta\right)^{b-\frac{1}{2}}}{\left(1+M_2\sinh^2\beta\right)^{b+b'}} \cdot H_2\Big(c, 1-a', a+a', b+b'; b'; -\frac{x\,\mathrm{csch}^2\alpha\,\mathrm{csch}^2\beta}{M_1M_2}, \frac{M_2 y}{\left(1+M_1\sinh^2\alpha\right)\left(1+M_2\sinh^2\beta\right)}\Big) \mathrm{d}\,\alpha\,\mathrm{d}\,\beta, \quad \Big(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(b')>0,\ M_1>0,\ M_2>0\Big), \tag{3}$$

$$G_1(a,b,c;x,y) = \frac{\Gamma(a+b)\Gamma(c+c')}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')} \cdot \int_0^1\int_0^1 \alpha^{c-1}(1-\alpha)^{c'-1}\beta^{a-1}(1-\beta)^{b-1} \times H_7\Big(1-c', \frac{a+b}{2}, \frac{a+b+1}{2}; 1-c-c'; \frac{\alpha\beta x}{(1-\alpha)^2(\beta-1)}, \frac{4(\alpha-1)\beta(1-\beta)y}{\alpha}\Big) \mathrm{d}\,\alpha\,\mathrm{d}\,\beta, \quad \left(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c)>0,\ \mathrm{Re}(c')>0\right), \tag{4}$$

$$G_1(a,b,c;x,y) = \frac{4\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (\sin^2\alpha)^{a+b-\frac{1}{2}}(\cos^2\alpha)^{c-\frac{1}{2}}(\sin^2\beta)^{b-\frac{1}{2}}(\cos^2\beta)^{a-\frac{1}{2}} \times \left[1 - x\cos^2\alpha\cot^2\beta - \frac{1}{4}y\sin^2\alpha\tan^2\alpha\sin^2 2\beta\right]^{-(a+b+c)} \mathrm{d}\,\alpha, \quad \left(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c)>0\right). \tag{5}$$

Proof. To prove the result in equality (2) asserted in Theorem 2.1, let $\mho$ denote the right-hand side of the equality (2). Then from the definition of Horn's function G_2 in (1), we get

$$\begin{aligned}\mho = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} &\frac{(a+a')_m (b)_{n-m} (c)_{m-n}}{(a)_{-n}} \times \\ &\times \frac{(S-T)^{a+m}(R-T)^{a'-n}}{(S-R)^{a+a'+m-1}} \cdot \\ &\cdot \int_R^S \frac{(\alpha-R)^{a+m-1}(S-\alpha)^{a'-n-1}}{(\alpha-T)^{a+a'+m}} \times \frac{x^m}{m!}\frac{y^n}{n!}\,\mathrm{d}\,\alpha.\end{aligned} \tag{6}$$

Applying the integral representation of the Beta function (see, e.g., [10, p. 10, (14)])

$$B(a,b) = \frac{(S-T)^a (R-T)^b}{(S-R)^{a+b-1}} \int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,\mathrm{d}\,\alpha,$$
$$(T < R < S,\ \mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0)\,.$$

in (6), we have

$$\mho = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} \frac{(a+a')_m (b)_{n-m} (c)_{m-n} B(a+m+n, a'-n)}{(a)_{-n}}. \tag{7}$$

Now, applying well known beta function (see, e.g., [15, 16])

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (7), we get the required result. On the same way, we find the results (3)–(5). □

A similar argument as in the proof of the Theorem 2.1. we can give the following theorems without proof.

Theorem 2.2. *The following integral representations holds:*

$$\begin{aligned}&G_2\,(a,b,c,d;x,y) = \\ &= \frac{\Gamma(b+d)\Gamma(c+c')}{\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)} \int_0^\infty \int_0^\infty \frac{\alpha^{c-1}}{(1+\alpha)^{c+c'}} \frac{\beta^{d-1}}{(1+\beta)^{b+d}} \times \\ &\quad \times F_3\left(a, \frac{c+c'}{2}, b+d, \frac{c+c'+1}{2}; c'; \frac{\beta x}{\alpha(1+\beta)}, \frac{4\alpha y}{(1+\alpha)^2\beta}\right) \mathrm{d}\,\alpha\,\mathrm{d}\,\beta,\end{aligned} \tag{8}$$
$$\left(\mathrm{Re}(b) > 0,\ \mathrm{Re}(c) > 0,\ \mathrm{Re}(c') > 0,\ \mathrm{Re}(d) > 0\right),$$

$$
\begin{aligned}
G_2\left(a,b,c,d;x,y\right) = {} & \frac{(1+M_1)^a(1+M_2)^b\Gamma(a+d)\Gamma(b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\cdot \\
& \cdot \int_0^1\int_0^1 \frac{\alpha^{a-1}\left(1-\alpha\right)^{d-1}}{\left(1+M_1\alpha\right)^{a+d}}\frac{\beta^{b-1}\left(1-\beta\right)^{c-1}}{\left(1+M_2\beta\right)^{b+c}}\cdot \\
& \cdot G_3\left(b+c,a+d;\frac{(1+M_1)\alpha\left(1-\alpha\right)\left(1+M_2\beta\right)x}{\left(1+M_1\alpha\right)^2\left(1-\alpha\right)},\right. \\
& \left.\frac{(1+M_2)\beta\left(1-\beta\right)\left(1+M_1\alpha\right)y}{\left(1+M_2\beta\right)^2\left(1-\alpha\right)}\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,
\end{aligned} \tag{9}
$$

$$
\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c)>0,\ \mathrm{Re}(d)>0,\ M_1>-1,\ M_2>-1\right),
$$

(10)

$$
\begin{aligned}
G_2\left(a,b,c,d;x,y\right) = {} & \frac{\Gamma(a+a')\Gamma(c+d)}{\Gamma(a)\Gamma(a')\Gamma(c)\Gamma(d)}\cdot \\
& \cdot \int_0^\infty\int_0^\infty \left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{a'-1}\left(e^{-\beta}\right)^c\left(1-e^{-\beta}\right)^{d-1}\times \\
& \times H_6\left(1-a',a+a',b;\frac{e^{-(\alpha-\beta)}\left(1-e^{-\beta}\right)x}{\left(1-e^{-\alpha}\right)^2},\frac{\left(e^{-\alpha}-1\right)e^{-\beta}y}{\left(1-e^{-\beta}\right)}\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,
\end{aligned}
$$

$$
\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ \mathrm{Re}(c)>0,\ \mathrm{Re}(d)>0\right),
$$

Theorem 2.3. *The following integral representations hold:*

$$
\begin{aligned}
G_3\left(a,b;x,y\right) = {} & \frac{\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')(S_1-R_1)^{a+a'-1}(S_2-R_2)^{b+b'-1}}\cdot \\
& \cdot \int_{R_1}^{S_1}\int_{R_2}^{S_2}\left(\alpha-R_1\right)^{a-1}\left(S_1-\alpha\right)^{a'-1}\left(\beta-R_2\right)^{b-1}\left(S_2-\beta\right)^{b'-1} \\
& \times G_2\left(b+b',a+a',1-a',1-b';\frac{\left(S_1-\alpha\right)\left(\beta-R_2\right)^2x}{\left(S_2-R_2\right)\left(\alpha-R_1\right)\left(S_2-\beta\right)},\right)\cdot \\
& \cdot\left(\frac{\left(\alpha-R_1\right)^2\left(S_2-\beta\right)y}{\left(S_1-R_1\right)\left(S_1-\alpha\right)\left(\beta-R_2\right)}\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,
\end{aligned} \tag{11}
$$

$$
\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(b')>0,\ R_1<S_1,\ R_2<S_2\right),
$$

(12)

$$
\begin{aligned}
G_3\left(a,b;x,y\right) = {} & \frac{\Gamma(a+a')}{2^{a+a'-2}\Gamma(a)\Gamma(a')}\int_{-1}^{1}\frac{\left[(1+\alpha)^2\right]^{a'-\frac12}\left[(1-\alpha)^2\right]^{a-\frac12}}{\left(1+\alpha^2\right)^{a+a'}}\times \\
& \times G_3\left(1-a',b;-\left(\frac{1+\alpha}{1-\alpha}\right)^2x,\left(\frac{1-\alpha}{1+\alpha}\right)^4y\right)\mathrm{d}\,\alpha,
\end{aligned}
$$

$$
\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0\right).
$$

$$G_3\,(a,b;x,y)=\frac{4(1+M_1)^a(1+M_2)^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}\cdot$$

$$\cdot\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\alpha\right)^{a-\frac{1}{2}}\left(cos^2\alpha\right)^{a'-\frac{1}{2}}}{\left(1+M_1\sin^2\alpha\right)^{a+a'}}\frac{\left(\sin^2\beta\right)^{b-\frac{1}{2}}\left(\cos^2\beta\right)^{b'-\frac{1}{2}}}{\left(1+M_2\sin^2\beta\right)^{b+b'}}\cdot$$

$$\cdot\,H_2\left(a+a',1-a',\frac{b+b'}{2},\frac{b+b'+1}{2};b;\right)\cdot$$

(13)
$$\cdot\left(-\frac{(1+M_1)^2y\sin^2\alpha\tan^2\alpha\cot^2\beta}{(1+M_2)\left(1+M_1\sin^2\alpha\right)},\right)\cdot$$

$$\cdot\left(\frac{(1+M_2)^2x\csc^2\alpha\sin^4\beta\left(1+M_1\sin^2\alpha\right)}{(1+M_1)\left(1+M_2\sin^2\beta\right)^2}\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,$$

$$\Big(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ \mathrm{Re}(b)>0,$$

$$\mathrm{Re}(b')>0,\ M_1>-1,\ M_2>-1\Big).$$

$$G_3\,(a,b;x,y)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a'-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

(14)
$$\times\,H_6\left(b,1-a',a+a';-\left(\frac{1+2\alpha}{1-2\alpha}\right)x,-\frac{(1-2\alpha)^2\,y}{2\,(1+2\alpha)}\right)\mathrm{d}\,\alpha,$$

$$\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0\right),$$

Theorem 2.4. *The following integral representations hold:*

$$H_1\,(a,b,c;d;x,y)=\frac{4M_1^aM_2^b\Gamma(2d)}{\Gamma(a)\Gamma(2d-a)}\cdot$$

$$\cdot\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\alpha\right)^{a-\frac{1}{2}}\left(\cos^2\alpha\right)^{2d-a-\frac{1}{2}}}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)^{2d}}\frac{\left(\sin^2\beta\right)^{b-\frac{1}{2}}\left(\cos^2\beta\right)^{c-\frac{1}{2}}}{\left(\cos^2\beta+M_2\sin^2\beta\right)^{b+c}}\cdot$$

(15)
$$\cdot\,H_3\left(b+c,d+\frac{1}{2};2d-a;\frac{M_2ycot^2\alpha\sin^2 2\beta}{4M_1\left(\cos^2\beta+M_2\sin^2\beta\right)^2},\right.$$

$$\left.\left(\frac{M_1M_2x\sin^2 2\alpha\sin^2\beta}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)^2\cos^2\beta+M_2\sin^2\beta)}\right)\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,$$

$$\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(2d-a)>0,\ M_1>0,\ M_2>0\right).$$

$$H_1\left(a,b,c;d;x,y\right)=\frac{\Gamma(a+a')\Gamma(b+c)}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(c)}\cdot$$

$$\cdot\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{a-1}}{(1+\alpha)^{a+a'}}\frac{\beta^{b-1}}{(1+\beta)^{b+c}}\times \tag{16}$$

$$\times H_4\left(b+c,a+a';a',d;\frac{\beta y}{\alpha\left(1+\beta\right)^2},\frac{\alpha\beta x}{\left(1+\alpha\right)\left(1+\beta\right)}\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,$$

$$\big(\mathrm{Re}(a)>0,\ \mathrm{Re}(a')>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c)>0\big),$$

$$H_1\left(a,b,c;d;x,y\right)=\frac{\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(c)\Gamma(b)\Gamma(b')}\cdot$$

$$\cdot\int_0^{\infty}\int_0^{\infty}\left(e^{-\alpha}\right)^{b'}\left(1-e^{-\alpha}\right)^{b-1}\left(e^{-\beta}\right)^{a}\left(1-e^{-\beta}\right)^{c-1}\times$$

$$\times H_2\left(1-b',a+c,\frac{b+b'}{2},\frac{b+b'+1}{2};d;xe^{(\alpha-\beta)}\left(e^{-\alpha}-1\right),\right)\cdot \tag{17}$$

$$\cdot\left(4ye^{-(\alpha-\beta)}\left(e^{-\alpha}-1\right)\left(1-e^{-\beta}\right)\right)\mathrm{d}\,\alpha\,\mathrm{d}\,\beta,$$

$$\big(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(b')>0,\ \mathrm{Re}(c)>0\big),$$

$$H_1\left(a,b,c;d;x,y\right)=\frac{\Gamma(b+c)}{2^{b+c-1}\Gamma(b)\Gamma(c)}\int_{-1}^{1}\left(1+\alpha\right)^{b-1}\left(1-\alpha\right)^{c-1}\times$$

$$\times H_5\left(b+c,a;d;\frac{(1+\alpha)(1-\alpha)}{4}y,\frac{(1+\alpha)}{2}x\right)\mathrm{d}\,\alpha, \tag{18}$$

$$\left(\mathrm{Re}(b)>0,\ \mathrm{Re}(c)>0\right).$$

Conclusion

We concluding our investigation by remarking that in our present investigation, we have presented some new Euler-type integral representations for the Horn's functions of two variables G_1, G_2, G_3 and H_1. As above-noted, special functions and their integral representations have appeared in and been applied to a diversity of research subjects such as statistics, theoretical physics, theory of group representations, and number theory. In this connection, the newly introduced Euler-type integral representations presented here and the other uninvestigated ones (if any) are hoped and believed to have potential applications to various research subjects including the abovementioned ones.

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