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On p-topological groups

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Abstract. In this paper, we introduce the notions of p-topological group and p-irresolute topological group which are generalizations of the notion topological group. We discuss the properties of p-topological groups with illustrative examples and we provide a connected p-topological group on any group G whose cardinality is not equal to 2. Also, we prove that translations and inversion in p-topological group are p-homeomorphism and demonstrate that every p-topological group is p-homogenous which leads to check whether a topology on a group satisfies the conditions of p-topological group or not.

1. introduction

Topological group is a mathematical structure on a set which is defined by underlying two distinguished structures on that set namely group and a topology. While merging two distinguished structures, the way of approach is keeping one structure as fundamental and the other one as deciding factor. We come across so many types of mathematical structures binded together in this way such as ring, field, vector space, algebra, normed linear spaces, etc. Sophus Lie built up the vast theory of those topological groups which he called continuous groups and are also known as Lie groups [12]. He studied the case, for a group, how it is possible to define a topology such that the binary operations, group multiplication and inversion are continuous. Then he define continuous group, a group having continuous binary operation which the basic idea of topological groups and Lie groups emerged. A topological group in modern notion is defined as, a group together with a topology such that the binary operations - multiplication and inversion are continuous. Based on this, some generalizations of topological groups such as paratopological groups, semitopological groups and quasitopological groups are defined. In a finite group, all the above mentioned generalizations coincide [1].

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The concepts of S-topological group and s-topological group were discussed in [10] and the theory of almost topological group was initiated in [11]. In this paper, we discuss some more generalizations which are defined based on pre-open sets and present a new generalization of topological group called p-topological group.

2. Preliminaries

Throughout this paper, the triple $(G, \cdot, \tau)$ denotes a group $(G, \cdot)$ together with a topology τ. For $x, y \in G$ we write xy instead of $x \cdot y$. For any $x \in G$, x^{-1} denotes the inverse of x in G. Let $A, B \subseteq G$, then $AB = \{ab : a \in A, b \in B\}$. The notion of pre-open sets of a topology was introduced by Mashhour et. al in [2] and the notion of semi-open sets was defined by N. Levine in [14]. The notion of regular open sets was introduced by K. Kuratowski [8]. Let X be a topological space. For a set A, interior of A is denoted by $int(A)$ and the closure of A is denoted by $cl(A)$. A subset A of X is said to be **pre-open** [2] (respectively, **semi-open** [14] and **regular open** [11]) if $A \subseteq int(cl(A))$ (respectively, $A \subseteq cl(int(A))$ and $A = int(cl(A))$). The largest pre-open set contained in A is termed as **pre-interior** of A [17] and the smallest pre-closed set containing A is called as **pre-closure** of A [17]. Pre-interior and pre-closure of A are denoted by $pint(A)$ and $pcl(A)$. For a set S, the power set of S is denoted by $\mathcal{P}(\mathcal{S})$ and for a topology τ, the collection of pre-open sets is denoted by τ_p. Union of any collection of pre-open sets is pre-open and intersection of two pre-open sets need not be pre-open but intersection of an open set with a pre-open set is pre-open. The complement of a pre-open set is called pre-closed [2].

Definition 1. Let X and Y be topological spaces. A mapping $f : X \mapsto Y$ is said to be **continuous** (respectively, semi-continuous [14], pre-continuous [2]) if for each open neighbourhood V of $f(x)$, there exists an open (respectively, semi-open, pre-open) neighbourhood U of $x \in X$ such that $f(U) \subseteq V$. A mapping f is said to be **almost continuous** [15] (respectively, **pre-irresolute** [6]) if for each regular open (respectively, pre-open) neighbourhood V of $f(x)$, there exists an open (respectively, pre-open) neighbourhood U of x such that $f(U) \subseteq V$.

Definition 2. Let G be a group and τ be a topology on G. Then the pair (G, τ) is said to be **topological group** [3] (respectively, s**-topological group** [10], **almost topological group** [11]) if multiplication and inversion are continuous (respectively, semi-continuous, almost continuous). The pair (G, τ) is said to be **paratopological group** [3] (respectively, **semitopological group**) if multiplication is continuous (respectively, semi-continuous). Let $x, y \in G$ then (G, τ) is called a s**-topological group** [10] if for each open neighbourhood U of xy^{-1} there exist semi-open neighbourhoods V of x and W of y such that $VW^{-1} \subseteq U$.

Lemma 1 ([7]). *Let $(X_i)_{i\in I}$ be a family of topological spaces and $\emptyset \neq A_i \subseteq X_i$ for each $i \in I$. Then, $\prod_{i\in I} A_i$ is pre-open in $\prod_{i\in I} X_i$ if and only if A_i is pre-open in X_i for each $i \in I$ and A_i is non-dense for only finitely many $i \in I$.*

Lemma 2 ([5]). *Let X be a topological space and $A \subseteq X$. Then:*

(i) $int(A) \subseteq pint(A)$;
(ii) $pcl(A) \subseteq cl(A)$.

Definition 3. A topological space X is said to be:

(i) **submaximal** [7] if every dense subset of X is open;
(ii) **p-regular** [13] if for each closed set F of X and each point $x \in X\setminus F$, there exist disjoint pre-open sets U and V such that $F \subseteq U$ and $x \in V$.

Lemma 3 ([7]). *For a topological space (X, τ) the following conditions are equivalent:*

(i) *X is submaximal.*
(ii) *Every pre-open set is open.*

Lemma 4 ([13]). *For a topological space X the following are equivalent:*

(i) *X is p-regular.*
(ii) *For each $x \in X$ and each open set U of X containing x, there exists a pre open set V of x such that $pcl(V) \subset U$.*
(iii) *For each closed set F of X, $\cap\{pcl(V) : F \subset V, V$ is pre-open in $X\} = F$.*
(iv) *For each subset A of X and each open set U of X such that $A\cap U \neq \emptyset$, there exists a pre-open set V of X such that $A \cap V \neq \emptyset$ and $pcl(V) \subseteq U$.*
(v) *For each nonempty set A of X and each closed set F of X such that $A\cap F = \emptyset$, there exist pre-open sets V, W of X such that $A\cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.*

3. p-topological groups and their basic properties

In this section, we introduce the concept of p-topological group and investigate its basic properties with illustrative examples.

Definition 4. A pair (G, τ) is said to be p**-topological group**, for each $x, y \in G$:

- for each open neighbourhood U of xy, there exist pre-open neighbourhoods V of x and W of y such that $VW \subseteq U$,
- for each open neighbourhood I of x^{-1} there exists pre-open neighbourhood S of x such that $S^{-1} \subseteq I$.

In other words, multiplication and inversion mappings are pre-continuous.

We observe that

- Any group with partition topology is a trivial example of $p2$-topological group.
- Every finite left (right) topological group is p-topological group. Since the basis of topology on finite left (right) topological group is cosets of a subgroup (partition topology) [1] and for the partition topology τ on any set X, τ_p is $\mathcal{P}(\mathcal{X})$ and so every subset of X is pre-open.
- Every topological group is p-topological group, but converse need not be true. The following Examples 3.2, 3.4, 3.5 are all a p-topological group but not a topological group.

One may ask the question that, if there is any topology τ for a group G whose $\tau_p \neq \mathcal{P}(\mathcal{X})$ such that satisfies the conditions of p-topological group. An example is given below which answers this.

Example 1. Consider the group $G = (\mathbb{Z}_3, \oplus)$ with the topology $\tau = \{\emptyset, \{1,2\}, G\}$. For the topology τ, $\tau_p = \{\emptyset, \{1\}, \{2\}, \{0,1\}, \{0,2\} \{1,2\}, G\}$.

Multiplication:

$x \oplus y$	open neighbourhoods of $(x \oplus y)$	x	y	pre-open neighbourhood of x	pre-open neighbourhood of y	pre-open neighbourhood of x $\oplus$ pre-open neighbourhood of y
0	G	0	0	G	G	G
		1	2			
		2	1			
1	G, $\{1,2\}$	0	1	$\{0,1\}$	$\{1\}$	$\{1,2\}$
		1	0	$\{1\}$	$\{0,1\}$	
		2	2	$\{2\}$	$\{0,2\}$	
2	G, $\{1,2\}$	0	2	$\{0,2\}$	$\{2\}$	$\{1,2\}$
		1	1	$\{1\}$	$\{0,1\}$	
		2	0	$\{2\}$	$\{0,2\}$	

Inverse:

x^{-1}	open neighbourhoods of x^{-1}	x	pre-open neighbourhood of x	Inverse of pre-open neighbourhood of x
0	G	0	G	G
1	G, $\{1,2\}$	2	$\{1,2\}$	$\{1,2\}$
2	G, $\{1,2\}$	1	$\{1,2\}$	$\{1,2\}$

Thus $(G, \oplus, \tau)$ is a p-topological group. Let us change the topology on G and check whether p-topological group or not.

Example 2. Consider the group $G = (\mathbb{Z}_3, \oplus)$ with a topology $\tau = \{\emptyset, \{0,1\}, G\}$. For given τ, $\tau_p = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{1,2\}, G\}$. Now, consider $1 \in G$ which can be written as $1 = 2 \oplus 2$. Let $\{0,1\}$ be an open neighbourhood of 1. Then there does not exist pre-neighbourhoods U, V of 2 such that $UV \subseteq \{0,1\}$. Thus (G, τ) is not a p-topological group. We see some specialization of the p-topological group mentioned in Example 3.2 as follows.

A finite group with indiscrete topology is the only connected finite topological group. But in the case of p-topological group we can provide some more connected topologies. For example, the above mentioned p-topological group is connected. We can define a topology τ on G, for any group G such that (G, τ) is connected p-topological group as follows.

Example 3. Let G be any group having cardinality greater than 2 with the topology $\tau = \{\emptyset, G \setminus \{e\}, G\}$. Now, τ_p is $\mathcal{P(G)} \setminus \{e\}$. Let $x \in G$ and U be an open neighbourhood of x. Then $x = ab$ for some $a, b \in G$ and $x = y^{-1}$ for some $y \in G$.
Case 1: Suppose $U = G$, then G itself is the pre-open neighbourhood of a, b and y respectively.
Case 2: Suppose $U = G \setminus \{e\}$, then $x \neq e$ and $\{y\}$ is a pre-open neighbourhood of x^{-1} such that $\{y\}^{-1} = \{x\} \subset U$ and so inversion is pre-continuous. We discuss the pre-continuity of multiplication as follows:

If $a, b \neq e$, then $\{a\}$, $\{b\}$ are pre-open neighbourhoods of a and b such that $\{a\} . \{b\} \subset U$.

If either a or b is identity, then the other element should be x. $\{e, z\}$ where $z \neq y$ and $\{x\}$ are pre-open neighbourhoods of e and x such that $\{e, z\} . \{x\} \subset U$.

Thus, (G, τ) is a connected p-topological group.

As in the above example, we can provide more connected p-topological groups for an infinite group as follows :

Example 4. Let G be an infinite group and $a \in G$ be an arbitrary non-identity element with the topology $\tau = \{\emptyset, G \setminus \{e, a\}, G\}$. Then $\tau_p = \mathcal{P(G)} \setminus \{\{e\}, \{a\}, \{e, a\}\}$. Let $x = bc$ for some $b, c \in G$ and U be an open neighbourhood of x. If $U = G$ then (G, τ) is a p-topological group. Let us assume that $U = G \setminus \{e, a\}$. Then $x \notin \{e, a\}$ and the cases follows:
Case 1: If $b, c \notin \{e, a\}$, then $\{b\}, \{c\}$ are pre-open neighbourhoods of b and c such that $\{b\} . \{c\} = \{x\} \subset U$.
Case 2: If $b = c = a$, then $\{a, y_1\}$, where $y_1 \notin \left\{e, a^{-1}, y_1^{-1}, ay_1^{-1}, y_1^{-1}a\right\}$ is a pre-open neighbourhood of b, c such that $\{a, y_1\} . \{a, y_1\} \subset U$.
Case 3: Suppose either b or $c \in \{e, a\}$.

If either b or c is e, then the other element should be x. Without loss of generality, let us fix $b = e$. Then $\{e, y_2\}$, where $y_2 \notin \left\{x^{-1}, ax^{-1}, x^{-1}a\right\}$ and $\{x\}$ are pre-open neighbourhoods of b and c such that $\{e, y_2\} . \{x\} \subset U$.

If $b = a$, then $c = a^{-1}x$. Now, $\{a, y_3\}$ and $\left\{a^{-1}x, y_4\right\}$, where $y_3 \notin \left\{x^{-1}a, ax^{-1}a, y_4^{-1}, ay_4^{-1}\right\}$ and $y_4 \notin \left\{a^{-1}, e\right\}$ are pre-open neighbourhoods of b and c respectively such that $\{a, y_3\} . \left\{a^{-1}x, y_4\right\} \subset U$.

If $c = a$, then $b = xa^{-1}$. Now, $\left\{xa^{-1}, y_5\right\}$ and $\{a, y_6\}$, where $y_5 \notin \left\{a^{-1}, e\right\}$ and $y_6 \notin \{ax^{-1}, ax^{-1}a, y_5^{-1}, y_5^{-1}a\}$ are pre-open neighbourhoods of b and c respectively such that $\{a, y_3\} . \left\{a^{-1}x, y_4\right\} \subset U$.

Thus, multiplication is pre-continuous on G. Now, the pre-continuity of inversion follows:

Case 1: If $x^{-1} \neq a$, then $\{x^{-1}\}$ is a pre-open neighbourhood of x^{-1} such that $\{x^{-1}\}^{-1} = \{x\} \subset U$.
Case 2: If $x^{-1} = a$, then $\{x^{-1}, y\}$ where $y \neq x$ is a pre-open neighbourhood of x^{-1} such that $\{x^{-1}, y\}^{-1} = \{x, y^{-1}\} \subset U$.

Hence (G, τ) is a connected p-topological group. By the above example, We can see that, in an infinite group G, for any $n \in \mathbb{N}$ we can provide n number of connected topologies $\tau_i, i = 1, 2, 3, \ldots, n$ such that (G, τ_i) is p-topological group.

Example 5. Consider the group of real numbers under usual addition $(\mathbb{R}, +)$ with the lower limit topology. Since the multiplication is continuous on $\mathbb{R}$, it is pre-continuous. But the inversion is not pre-continuous. Indeed, let $[0, a) : a \in \mathbb{R}^+$ be an open neighbourhood of 0. The inverse of 0 is itself. There does not exists any pre-open neighbourhood of 0 whose inverse is contained in $[0, a)$.

Proposition 1. *Let A be an open set in p-topological group. Then for each $x \in G$,*

(i) $xcl(A) \subseteq cl(xA)$.
(ii) $cl(A)x \subseteq cl(Ax)$.
(iii) $[cl(A)]^{-1} \subseteq cl(A^{-1})$.
(iv) $int(xA) \subseteq xint(A)$.
(v) $int(Ax) \subseteq int(A)x$.
(vi) $int(A^{-1}) \subseteq [int(A)]^{-1}$.

The proof is trivial by the facts that, A is open $\Leftrightarrow$ $intA = A$ and $pcl(A^{-1}) \subseteq cl(A^{-1})$ (Lemma 2.4 (ii)). Though the result is trivial an interesting fact is as follows.

In the above proposition, the set A cannot be assumed to be pre-open and the reverse containment need not be hold. Let $G = (\mathbb{Z}_4, \oplus)$ with $\tau = \{\emptyset, \{0,1\}, \{2\}, \{0,1,2\}, \{3\}, \{0,1,3\}, \{2,3\}, G\}$. Then $\tau_p = \mathcal{P}(\mathbb{Z}_4)$ and so (G, τ) is a p-topological group. Let $x = 1 \in G$ and $\{3\} \in \tau$ then $xA = Ax = \{0\}$ and $A^{-1} = \{1\}$.

(i) $cl(xA) = \{0,1\}$, $xcl(A) = \{0\} \Rightarrow cl(xA) \nsubseteq xcl(A)$.
(ii) $cl(Ax) = \{0,1\}$, $cl(A)x = \{0\} \Rightarrow cl(Ax) \nsubseteq cl(A)x$.
(iii) $cl(A^{-1}) = \{0,1\}$, $[cl(A)]^{-1} = \{1\} \Rightarrow cl(A^{-1}) \nsubseteq [cl(A)]^{-1}$.
(iv) $int(xA) = \emptyset$, $xint(A) = \{0\} \Rightarrow xint(A) \nsubseteq int(xA)$.
(v) $int(Ax) = \emptyset$, $int(A)x = \{0\} \Rightarrow int(A)x \nsubseteq int(Ax)$.
(vi) $int(A^{-1}) = \emptyset$, $[int(A)]^{-1} = \{1\} \Rightarrow [int(A)]^{-1} \nsubseteq int(A^{-1})$.

Proposition 2. *Let G be a p-topological group and A be an open set in G. Then for any $x \in G$, xA and Ax are pre-open.*

Proof. Let $b \in xA$, then $b = xa$ for some $a \in A$. Now, $a = x^{-1}b$ and by the pre-continuity of multiplication there exist pre-open sets U and V of x^{-1}

and b such that $UV \subseteq A$ which implies $b \in V \subseteq xA$. Hence xA is pre-open. Similarly we can prove that Ax is pre-open. $\square$

In the above result, the openness of A cannot be extended to pre-openess. Indeed, consider (G, τ) in Example 3.2 which is a p-topological group. Let $\{1\} \in \tau_p$ and $2 \in G$. Then $2 \oplus \{1\} = \{1\} \oplus 2 = \{0\} \notin \tau_p$.

Proposition 3. *Let G be a p-topological group. Then A is pre-open if and only if A^{-1} is pre-open.*

Proof. Let $A \in \tau_p$, then there exists an open set O in G such that $A \subseteq O \subseteq cl(A)$. Now, $A^{-1} \subseteq O^{-1} \subseteq (cl(A))^{-1}$. Since inversion is pre-continuous, we have O^{-1} is pre-open and $(cl(A))^{-1}$ is pre-closure of A^{-1}. By using Lemma 2.4 (ii), $A^{-1} \subseteq O^{-1} \subseteq int(cl(O^{-1})) \subseteq (cl(A))^{-1} \subseteq cl(A^{-1})$. Hence there exists an open set $int(cl(O^{-1}))$ such that $A^{-1} \subseteq int(cl(O^{-1})) \subseteq cl(A^{-1})$ and so A^{-1} is pre-open. Proof of the converse is similar. $\square$

By using Lemma 2.3 and Proposition 3.9, Definition 3.1 can be rewrite in short as: A pair (G, τ) is said to be **p-topological group** if for each $x, y \in G$ and for each open neighbourhood U of xy^{-1} there exist pre-open sets V of x and W of y such that $VW^{-1} \subseteq U$.

Proposition 4. *Let A be any closed subset of a p-topological group. Then for any $x \in G$, xA and Ax are pre-closed.*

Proof. Let $b \in pcl(xA)$. Let $c = x^{-1}b$ and W be an open neighbourhood of c. Then by the pre-continuity of the multiplication, there exist pre-open sets U and V of x^{-1} and b in G, respectively such that $UV \subset W$. Since $b \in pcl(xA)$ we have $V \cap xA \neq \emptyset$. Let $d \in V \cap xA$, then $x^{-1}d \in A \cap UV \subseteq A \cap W$ which implies the nonempty set $A \cap W$. Thus c is a limit point of A. Since A is closed we have $c \in A$. Now $b = xc$ and so $b \in xA$. By the above argument, $pcl(xA) \subseteq xA$ and since $xA \subseteq pcl(xA)$ is trivial we have $xA = pcl(xA)$. Hence xA is pre-closed. Proof of Ax is similar. $\square$

Theorem 1. *Let A be any subset of a p-topological group G. Then:*

(i) $pcl(xA) \subseteq xcl(A)$.
(ii) $xpcl(A) \subseteq cl(xA)$.
(iii) $xint(A) \subseteq pint(xA)$.
(iv) $int(xA) \subseteq xpint(A)$.
(v) $pcl(xcl(A)) \subseteq xcl(A)$.
(vi) $xint(A) \subseteq pint(xint(A))$.
(vii) $(pcl(A))^{-1} \subset cl(A^{-1})$.

Proof. (i) Let $b \in pcl(xA)$ and $c = x^{-1}b$ in G. Let W be an open neighbourhood of c. By the pre2-continuity of the multiplication, there exist pre-open sets U and V of x^{-1} and b, respectively such that $UV \subseteq W$. Since $b \in pcl(xA)$, there exists some d in G such that $d \in (xA) \cap V$ which implies that $d \in A \cap UV \subseteq A \cap W$. We have the nonempty set

$A \cap W$ which implies arbitrary open neighbourhood of c intersects A. Thus, $c \in cl(A) \Rightarrow xc = b \in xcl(A)$.

(ii) Consider $b \in xpcl(A)$, then $b = xa$ for some $a \in pcl(A)$. Let W be an open neighboruhood of b. Since multiplication is pre-continuous, there exist pre-open sets U and V of x and a such that $UV \subseteq W$. Since $a \in pcl(A)$, we have the nonempty set $A \cap V$. This means that there exists an element $c \in A \cap V$ which implies $xc \in xA \cap UV \subseteq (xA) \cap W$. We have, $(xA) \cap W$ which is nonempty and so every open neighbourhood of b intersects xA. Thus, $b \in cl(xA)$.

(iii) Let $b \in xint(A)$. Then $b = xa$ for some $a \in int(A)$. Now, by the pre-continuity of multiplication, there exist pre-open sets U and V of x^{-1} and b such that $UV \subseteq int(A)$. Then, $x^{-1}V \subseteq UV \subseteq int(A) \subseteq A$ which implies that $V \subseteq xA$. Since V is a pre-open neighbourhood of b we have $b \in pint(xA)$. Thus, $xint(A) \subseteq pint(xA)$.

(iv) Let $b \in int(xA)$. Then $b = xa$ for some $a \in A$. We know that multiplication is pre-continuous. Then there exist pre-open neighbourhoods U and V of x and a such that $UV \subseteq int(A)$. Now, $xV \subseteq UV \subseteq int(xA) \subseteq xA$ implies that $xV \subseteq xpint(A)$. Since $b = xa \in xV$, we have $b \in xpint(A)$. Thus, $int(xA) \subseteq xpint(A)$.

(v) Let $b \in pcl(xcl(A))$ and W be an open neighbourhood of c where $c = x^{-1}b$. Now, by the pre-continuity of multiplication there exist pre-open sets U and V of x^{-1} and b such that $UV \subseteq W$ and so $V \subset xW$. Since $b \in pcl(xcl(A))$, we have $V \cap (xcl(A)) \neq \emptyset$ which implies $xW \cap x(cl(A)) \neq \emptyset$. Thus, $W \cap cl(A) \neq \emptyset$ and so some points of W are limit points of A. Since W is an open neighbourhood of those points, we have $W \cap A \neq \emptyset$. Hence $c \in cl(A)$ and so $xc = b \in xcl(A)$.

(vi) Suppose $b \in xint(A)$. Then $b = xa$ for some $a \in int(A)$. Since multiplication is pre-continuous, there exist pre-open sets U and V of x^{-1} and b such that $UV \subseteq int(A)$. Now, $x^{-1}V \subseteq UV \subseteq int(A) \subseteq pint(A)$ implies that $V \subseteq xint(A)$. Since V is a pre-open neighbourhood of b in G such that $b \in V$ and $b \in pint(xint(A))$. Thus, $xint(A) \subseteq pint(xint(A))$.

(vii) Let $y \in [pcl(A)]^{-1}$ then $y = z^{-1}$ for some $z \in pcl(A)$. Let S be any open neighbourhood of y in G then by pre-continuity of the inversion mapping, there exists a pre-open set I of z in G such that $I^{-1} \subseteq S$. Since $z \in pcl(A)$ there exist some $x \in G$ such that $x \in A \cap I$ which implies that $x^{-1} \in A^{-1} \cap I^{-1} \subseteq A^{-1} \cap S$ and so $A^{-1} \cap S \neq \emptyset$. Hence, $y \in cl(A^{-1})$ and so $(pcl(A))^{-1} \subset cl(A^{-1})$. □

Theorem 2. *Let A and B be any subsets of a p-topological group G. Then $pcl(A)pcl(B) \subseteq cl(AB)$.*

Proof. Let $x \in pcl(A)pcl(B)$ and W be any open neighbourhood of x in G where $x = ab$ for some $a \in pcl(A)$ and $b \in pcl(B)$. By the pre-continuity of the multiplication, there exist pre-open sets U and V in G containing a and b, respectively such that $UV \subseteq W$. Since $a \in pcl(A)$ and $b \in pcl(B)$

there exist $c \in A \cap U$ and $d \in B \cap V$. Now $cd \in (AB) \cap (UV) \subseteq AB \cap W$ which implies that $AB \cap W \neq \emptyset$. Hence x is a limit point of AB and so $x \in cl(AB)$. $\square$

Theorem 3. *Let G and H be p-topological groups with H is submaximal and f be a pre-irresolute homomorphism at identity e_G. Then f is pre-irresolute.*

Proof. Let $x \in G$ and V be a pre-open set in H containing $f(x) = y$. Since H is submaximal, by Lemma 2.6, V is open. By Proposition 3.8, left translation of an open set is pre-open, we have $y^{-1}V$ is pre-open in H containing e_H. Since f is pre-irresolute at identity e_G, there exists pre-open set U in G containing e_G such that $f(U) \subset y^{-1}V$. Given that f is homomorphism and so $f(xU) = f(x)f(U) \subseteq V$. Hence f is pre-irresolute. $\square$

One may remind that, A bijective mapping $f : X \mapsto Y$ is **p1-homeomorphism** [4] if f is pre-continuous and $f(A)$ is pre-open for every open set A of X.

Theorem 4. *Let G be a p-topological group and $a \in G$. Then for all $x \in G$*

(i) *The mappings $\lambda_a(x) = ax$ and $\rho_a(x) = xa$ are p-homeomorphism.*
(ii) *Inversion mapping is p-homeomorphism.*

Proof. (i) Let $x \in G$ and U_1 be an open set containing ax. Since multiplication is pre-continuous, for each open neighbourhood U_1 of ax there exist pre-open neighbourhoods V_1 and W_1 of a and x such that $V_1W_1 \subseteq U_1$ which implies $aW_1 \subseteq U_1$ and so $\lambda_a(W_1) \subseteq U_1$. Thus, λ_a is pre-continuous. Let $x \in G$ and U_2 be an open neighbourhood of x. The element x can be written as $x = a^{-1}ax$. Since each left translation is pre-continuous, there exist pre-open sets V_2 and W_2 of a^{-1} and ax such that $V_2W_2 \subseteq U_2$. Hence each left translation is p-homeomorphism. The proof of $\rho_a(x)$ is similar.

(ii) Let I_1 be an open neighbourhood of x^{-1}. Since G is p-topological group, for each open neighbourhood I_1 of x^{-1} there exists pre-open neighbourhood S_1 of x such that $S_1^{-1} \subseteq I_1$. Thus, inversion mapping is pre-continuous. Let I_2 be an open neighbourhood of x. Since inversion is pre-continuous there exists pre-open neighbourhood S_2 of x^{-1} such that $S_2^{-1} \subseteq I_2$. Hence inversion is p-homeomorphism. $\square$

Definition 5. A topological space X is said to be a **p-homogeneous space** if for any $x, y \in X$, there exists a p-homeomorphism f such that $f(x) = y$.

Theorem 5. *E0very p-topological group G is p-homogeneous.*

Proof. Let $a, b \in G$ and $c = ba^{-1}$. By Theorem 3.14, each translation in p-topological group is p-homeomorphism. Thus we have $\lambda_c(a) = ca = ba^{-1}a = b$. Hence, G is p-homogeneous. $\square$

The reason behind Theorem 3.16 is that, it is harder to decide, whether a topology on a group G satisfies the required conditions of p-topological group or not by checking pre-continuity on each elements. In a homogeneous space, all points behave in the same way. This observation suggests that, at first we have to define a basis at the identity element e. Then move the basis by means of translations to obtain a pre-open base at each element of the group G.

Theorem 6. *Let G be a p-topological group and H be a subgroup of G.*

(i) *If H contains a nonempty open set, then H is pre-open in G.*
(ii) *If H is open, then it is pre-closed.*
(iii) *If H is open, then it is a p-topological group.*

Proof. (i) Suppose the subgroup H contains a nonempty open set U. By Proposition 3.14, each translation is p-homeomorphism, so we have Ua is pre-open in G for each $a \in H$. Therefore $H = \cup_{a \in H} Ua$ is pre-open in G.

(ii) Let H be an open subgroup of G. Then $\gamma = \{Ha_i : a_i \in G\}$ is the family of all right cosets of H which is disjoint pre-open covering of G. Thus, $G = \cup_{a_i \in G} Ha_i$ and so $Ha_i = \left(\cup_{a_j \neq a_i \in G} Ha_j\right)^c$. Therefore every element of γ is both pre-open and pre-closed. In particular, $H = He$ is pre-closed in G.

(iii) We have to show that for each $x, y \in H$ and for each open neighbourhood W of xy^{-1} in H, there exist pre-open sets $U \subseteq H$ of x and $V \subseteq H$ of y such that $UV^{-1} \subseteq W$. Since G is p-topological group, there exist pre-open sets A of x and B of y such that $AB^{-1} \subseteq W$. Since H is open, the sets $U = H \cap A$ and $V = H \cap B$ are pre-open. Thus, $UV^{-1} \subseteq AB^{-1} \subseteq W$. $\square$

One may note that, by Lemma 2.3, For a family of topological spaces $\{X_i\}_{i=1}^n$, a set $A \subseteq \prod_{i=1}^n X_i$ is pre-open $\Leftrightarrow A = \prod_i^n A_i$ where A_i is pre-open in each component X_i.

Theorem 7. Let G be a p-topological group and $\mathfrak{U}$ an open base at the identity e of G. Let $\mathfrak{U}_p$ be the pre-open base at e corresponding to $\mathfrak{U}$. Then:

(i) for every $U \in \mathfrak{U}$, there is an element $V \in \mathfrak{U}_p$ such that $V^2 \subset U$.
(ii) for every $U \in \mathfrak{U}$, there is an element $V \in \mathfrak{U}_p$ such that $V^{-1} \subset U$.
(iii) for every $U \in \mathfrak{U}$, and every $x \in U$ there is $V \in \mathfrak{U}_p$ such that $xV \subset U$ ($Vx \subset U$).
(iv) for every $U \in \mathfrak{U}$ and $x \in G$, there is $V \in \mathfrak{U}_p$ such that $xVx^{-1} \subset U$.
(v) for $U, V \in \mathfrak{U}$, there is $W \in \mathfrak{U}_p$ such that $W \subset U \cap V$.

Proof. (i) Let $U \in \mathfrak{U}$. Then U is an open neighbourhood of e. We know that $e = e.e$. Since G is a p-topological group, the mapping $(x, y) \mapsto xy$ is pre-continuous and so there exists pre-neighbourhoods P and Q of e such that PQ is contained in U. Let V be the smallest pre-neighbourhood among P and Q and so there exists $V \in \mathfrak{U}_p$ such that $V^2 \subset U$.

(ii) Let $U \in \mathfrak{U}$. Then U is an open neighbourhood of e. We know that the inverse of e is itself. Since G is a p-topological group, the mapping $x \mapsto x^{-1}$

is pre-continuous and so there exists a pre-neighbourhood V of e such that $V^{-1} \subset U$.

(iii) Let $U \in \mathfrak{U}$ and $x \in U$. We know that $x = x.e$ $(x = ex)$. Since G is a p-topological group, the mapping $(x, y) \mapsto xy$ is pre-continuous and so there exist pre-neighbourhoods P of x and Q of e such that $PQ(QP)$ is contained in U. So for all $x \in U$, there is a $V \in \mathfrak{U}_{\mathfrak{p}}$ such that $xV \subset U$ $(Vx \subset U)$.

(iv) Let $U \in \mathfrak{U}$ and $x \in G$. We know that $xex^{-1} = e$. Since G is a p-topological group, each translation is a p-homeomorphism of G and so the map $e \mapsto xex^{-1}$ is a p-homomorphism of G. Hence for every $U \in \mathfrak{U}$ and $x \in G$, there is a $V \in \mathfrak{U}_{\mathfrak{p}}$ such that xVx^{-1} is contained in U.

(v) Since $\mathfrak{U}$ is an open base at e, for each $U, V \in \mathfrak{U}$, there is $W \in \mathfrak{U}$ such that $W \subset U \cap V$. Since $W \in \mathfrak{U}$ we have $W \in \mathfrak{U}_{\mathfrak{p}}$. □

Theorem 8. *Let G be a p-topological group with base $\mathfrak{B}_e$ at the identity element e such that for each $U \in \mathfrak{B}_e$ there is a symmetric open neighbourhood V of e such that $V^2 \subset U$. Then G satisfies the axiom of p-regularity at e.*

Proof. Let U be an open set containing the identity e. By assumption, there is a symmetric open neighbourhood V of e such that $V^2 \subset U$. We have to prove that $pcl(V) \subset U$. Let $x \in pcl(V)$. The set xV is a pre-open neighbourhood of x, which implies $xV \cap V \neq \emptyset$. Therefore, there exist points $a, b \in V$ such that $b = xa$ and so $x = ba^{-1} \in VV^{-1} = VV \subset U$. Thus $pcl(V) \subseteq U$. By Lemma 2.6 (ii), G satisfies p-regularity at e. □

4. p-IRRESOLUTE TOPOLOGICAL GROUPS AND PRE-CONNECTEDNESS

In this section, we discuss the independency of p-topological group from other generalization concepts of topological group. We also explore pre-connectedness properties of p-irresolute topological group.

Example 6. Consider the group $G = (\mathbb{Z}_n, \oplus)$ with the topology $\tau = \{\emptyset, \{0\}, \mathbb{Z}_n\}$.

- Suppose if we consider $\mathbb{Z}_n$ as an open set, then $Z_n \times Z_n$ is a semi-open set in $\mathbb{Z}_n \times \mathbb{Z}_n$ and its image under multiplication is $\mathbb{Z}_n$. Also, $\mathbb{Z}_n$ is a semi-open set in $\mathbb{Z}_n$ and its image under inversion is $\mathbb{Z}_n$. Suppose if we consider $\{0\}$ as an open set, then $\{(0, 0), (1, n-1), (2, n-2), \ldots, (n-1, 1)\}$ is a semi-open set in $\mathbb{Z}_n \times \mathbb{Z}_n$ and its image under multiplication is $\{0\}$. Also, $\{0\}$ is a semi - open set in $\mathbb{Z}_n$ and its image under inversion is $\{0\}$. Thus, (G, τ) is a S-topological group. But (G, τ) is not a s-topological group, since the preimage of an open set $\{0\}$ under multiplication $\{(0, 0), (1, n-1), (2, n-2), \ldots, (n-1, 1)\}$ is not a product of two semi - open sets in G.
- Since $\tau_p = \tau$, to be a p-topological group multiplication need to be continuous. But multiplication is not continuous in (G, τ). Thus, (G, τ) is not a p-topological group.

- (G,τ) is also an almost topological group since the only regular open set in G is itself.

Theorem 9. *Let G be a group and τ be a topology on G such that at least one singleton is open in G then (G,τ) is an s-topological group if and only if τ is discrete.*

Proof. Let τ be discrete then each singleton is open. Since an open set is semi-open, then each subset is semi-open and thus (G,τ) is s-topological group. Conversely, suppose (G,τ) is s-topological group having one singleton as open. Since each open set is semi-open and (By Lemma 3.1, [10]) translation of a semi-open set is semi-open. Thus, each singleton is semi-open and so each singleton has a nonempty interior and so each singleton is open which implies that τ is discrete. □

By considering (G,τ) in Example 4.1, the above result need not be true in S-topological and almost topological groups.

Example 7. Consider the group $G = (\mathbb{Z}_n, \oplus)$ with the topology $\tau = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \ldots, \mathbb{Z}_n\}$.

- (G,τ) is an almost topological group, since the only regular open set in τ is Z_n we can take Z_n itself as the open set required to obtain almost continuity for both multiplication and inverse.
- (G,τ) is not a p-topological group, since $\{0\}$ is open but its 0 translation $1 \oplus \{0\} = \{1\}$ is not pre-open in G.
- The inverse image of an open set $\{0\}$ is $\{(0,0),(1,n-1),(2,n-2),\ldots,(n-1,1)\}$ which is semi-open and inverse of $\{0\}$ is $\{0\}$. Hence (G,τ) is a S-topological group.
- Consider the open set $\{0\}$ which is singleton, by Theorem 9, since τ is not discrete, (G,τ) is not an s-topological group.

Example 8. Consider (G,τ) in Example 3.2, which is a p-topological group.

- Since the only regular open set in G is itself, we have (G,τ) is an almost topological group.
- Since every s-topological group is S-topological we have (G,τ) is not a s-topological group.

Definition 6. The pair (G,τ) is said to be **p-irresolute topological group** if multiplication and inversion mappings are *pre*-irresolute.

A topological space X is said to be **pre-connected** [16] if X cannot be written as the union of two disjoint nonempty pre-open sets.

Example 9. Consider $X = \{1,2\}$ with $\tau = \{\emptyset, \{1\}, X\}$ which is connected. Now, $\tau = \tau_p$ and so connected implies pre-connected. Thus, (X,τ) is pre-connected.

Example 10. Consider $\mathbb{R}$ with usual topology which is connected. Here $\mathbb{Q}$ and $\mathbb{Q}^c$ are disjoint pre-open sets $[\mathbb{Q} \subset int(cl(\mathbb{Q})), \mathbb{Q}^c \subset int(cl(\mathbb{Q}^c))]$ whose union is $\mathbb{R}$. Hence $\mathbb{R}$ with usual topology is not pre-connected.

Theorem 10. *Let G be a p-irresolute topological group and H be a subgroup of G. If H, G/H are pre-connected, then G is pre-connected.*

Proof. Let us assume that $G = U \cup V$ where U and V are disjoint nonempty pre-open sets. Since H is pre-connected, each coset of H is either a subset of U or a subset of V. Thus, the relation

$$\begin{aligned} G/H &= \{xH : xH \subset U\} \cup \{xH : xH \subset V\} \\ &= \{xH : x \in U\} \cup \{xH : x \in V\}. \end{aligned}$$

It expresses G/H as the union of disjoint nonempty pre-open sets which is a contradiction to pre-connectedness of G/H. Thus, G is pre-connected. $\square$

Theorem 11. *Let G be a pre-connected p-irresolute topological group and e be its identity element. If V is any pre-open neighbourhood of e, then G is generated by V.*

Proof. Let V be a pre-open neighbourhood of e. For each $n \in \mathbb{N}$, we denote V^n by the set of elements of the form $v_1 \cdot v_2 \cdots v_n$ where each $v_i \in V$. Let $U = \cup_{n=1}^{\infty} V^n$. Since G is pre-connected, suppose if we prove U is pre-open and pre-closed, we have $G = U$ and so G is generated by V. Since each V^n is pre-open and arbitrary union of pre-open sets is pre-open, U is pre-open. Let us prove that U is pre-closed. Let $x \in pcl(U)$. Since xV^{-1} is a pre-open neighbourhood of x, it must intersect U. Thus, let $y \in U \cap xV^{-1}$. Since $y \in xV^{-1}$ then $y = xv^{-1}$ for some $v \in V$. Since $y \in U$ then $y \in V^n$ for some $n \in \mathbb{N}$ which implies $y = v_1 v_2 \cdots v_n$ with each $v_i \in V$. Now, we have $x = v_1 v_2 \cdots v_n v$ and so $x \in V^{n+1} \subseteq U$. Hence U is pre-closed. Since G is pre-connected and U is pre-open and pre-closed we have $U = G$. Thus, G is generated by V. $\square$

Theorem 12. *If G is a pre-connected, p-irresolute topological group and N, a discrete invariant subgroup of G, then $N \subseteq Z(G)$, where $Z(G)$ denotes the center of the group G.*

Proof. Suppose the invariant subgroup $N = \{e\}$, then the result is trivial. Suppose that the subgroup N is non-trivial. Let $x \in N$ be an arbitrary element of G distinct from the identity element e. Since the group N is discrete, we can find an open neighbourhood U of x in G such that $U \cap N = \{x\}$. Since every open set is pre-open and by definition of p-irresolute topological group, there exist a pre-open neighbourhood V of e and a pre-open neighbourhood Vx of x in G such that $(Vx)V^{-1} \subset U$. Let $a \in V$ be arbitrary. Since N is an invariant subgroup of G, we have $aN = Na$ which implies that $ax \in Na$ and so $axa^{-1} \in N$. It is also clear that $axa^{-1} \in VxV^{-1} \subset U$. Therefore, $axa^{-1} \in U \cap N = \{x\}$ which implies $axa^{-1} = x$.

Thus, $ax = xa$ for each $a \in V$. Since the group G is pre-connected, V^n with $n \in \mathbb{N}$ covers the group G. Thus, every element $b \in G$ can be written in the form $b = a_1 a_2 \cdots a_n$ where $a_1, a_2, \ldots, a_n \in V$ and $n \in \mathbb{N}$. Since x commutes with every element of V, we have

$$\begin{aligned} bx &= a_1 a_2 \cdots a_n x \\ &= a_1 a_2 \cdots x a_n \\ &\vdots \\ &= a_1 x a_2 \cdots a_n \\ &= x a_1 a_2 \cdots a_n \\ &= xb. \end{aligned}$$

Hence the element $x \in N$ is in the center of group G. Since x is an arbitrary element of N, we proved that the center of G contains N. □

5. Conclusion

Topological groups mostly deals with an infinite set alone by an assumption in separation. To overcome this, some generalizations of topological groups are defined but they did not attain similar properties to topological group. By defining p-topological group, we reach a space which has properties close relevant to topological group.

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