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ON p-SEMIGROUPS

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Abstract. Generalizing the notion of an anti-inverse semigroup, we introduce the notion of a p-semigroup, for arbitrary $p \in N$. We prove that every p-semigroup is covered by groups, classes of which are completely described.

1. Introduction

In papers [1] and [4], anti-inverse semigroups are introduced and investigated. Recall that a semigroup $(S,+)$ is called anti-inverse if for every $a \in S$, there exists $b \in S$, such that $a = b + a + b$ and $b = a + b + a$.

In the present paper we generalize the foregoing notion, defining a p-semigroup $p \in N$. For $p = 1$ we obtain anti-inverse semigroups. We investigate general properties of the new class. These are described by a number of identities, depending on p. We also prove that each element and particular classes of elements of a p-semigroup have their own unity.

We prove that every p-semigroup is covered by groups. For the converse, we present complete description of classes of groups, union of which gives a p-semigroup. It turns out that the mentioned characterization depends on some divisibility properties of the integer p. Main results about anti-inverse semigroups are obtained from the corresponding theorems about p-semigroups, for $p = 1$.

As proved in [1], each anti-inverse semigroup is covered by groups, which are cyclic of order 1, cyclic of order 2, Klein or quaternion groups. Groups covering p-semigroups belong to a wider class. In addition to the mentioned groups it contains cyclic groups of order n ($n \in N$) and direct product of cyclic groups and generalized quaternion groups.

2. Results

Let $(S,+)$ be a semigroup and $p \in N$. The relation τ_p on $(S,+)$ is introduced by:

$$x\tau_p y \Longleftrightarrow x + py + x = y \wedge py + x + py = x.$$

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If $x\tau_p y$ for $x, y \in S$, then py is called the p-**element** of element x.

Definition 1. *A semigroup $(S, +)$ will be called a* p-**semigroup** *if each element has its p-element.*

Let Π_p denote the class of all p-semigroups, i.e.,

$$S \in \Pi_p \Longleftrightarrow (\forall x \in S)(\exists y \in S)(x\tau_p y).$$

Lemma 1. *Let $x\tau_p y$ in semigroup S. Then:*
1^o $2x = (p+1)y$, 2^o $py + x = (2p+1)x + p^2 y$, 3^o $(4p+1)x = x$.

Proof. 1^o $2x = py + x + py + x = py + y = (p+1)y$.

$$\begin{aligned} 2^o\ py + x &= p(x + py + x) + x \\ &= x + py + x + x + py + x + \cdots + x + py + x + x \\ &= x + p(py + 2x) = x + p(py + (p+1)y) = x + p(2p+1)y \\ &= x + p(p+1)y + p^2 y = x + p(2x) + p^2 y = (2p+1)x + p^2 y. \\ 3^o\ x &= py + x + py = (2p+1)x + p^2 y + py = (2p+1)x + p(p+1)y \\ &= (2p+1)x + p(2x) = (4p+1)x. \ \square \end{aligned}$$

Lemma 2. *Let $x, y \in S$, $p \in N$ and $(4p+1)x = x$ and $2x = (p+1)y$. Then $(p+1)y = (p+1)(2p+1)y$.*

Proof. Straightforward. $\square$

Lemma 3. *Let $x, y \in S$, $p \in N$ and $(\forall a \in S)$ $(4p+1)a = a$. Then*

$$2x = (p+1)y \Longrightarrow y = (2p^2 + 2p + 1)y.$$

Proof. Since $(2p^2+2p+1)y = (p+1)y + (2p^2+p)y$, from the conditions and Lemma 2. we have

$$\begin{aligned} (2p^2 + 2p + 1)y &= (p+1)(2p+1)y + (2p^2 + p)y \\ &= p(4p+1)y + (3p+1)y = py + (3p+1)y \\ &= (4p+1)y = y. \ \square \end{aligned}$$

Theorem 1. *Let S be a semigroup. Then*

$$\begin{aligned} S \in \Pi_p \quad \Longleftrightarrow \quad & (\forall x \in S)(\exists y \in S)(2x = (p+1)y,\ py + x = (2p+1)x + p^2 y, \\ & (4p+1)x = x). \end{aligned}$$

Proof. Let $S \in \Pi_p$. Then by Lemma 1., the right side of the equivalence follows immediately. Conversely, let for arbitrary $x \in S$ and his existing $y \in S$ be

$$2x = (p+1)y,\ py + x = (2p+1)x + p^2 y \text{ i } (4p+1)x = x.$$

Then, by Lemma 3., we have

$$\begin{aligned} x + py + x &= x + (2p+1)x + p^2 y = (p+1)(2x) + p^2 y \\ &= (p+1)(p+1)y + p^2 y = (2p^2 + 2p + 1)y = y. \end{aligned}$$

Furthermore

$$\begin{aligned} py + x + py &= (2p+1)x + p^2y + py = (2p+1)x + p(p+1)y \\ &= (2p+1)x + p(2x) = (4p+1)x = x. \end{aligned}$$

So, $S \in \Pi_p$. □

Recall that a semigroup S is **regular** if for every $a \in S$ there exists $x \in S$ such that $a = axa$.

Corollary 1. *(i) Each p-semigroup is a regular semigroup.*

(ii) Each element x of a p-semigroup has its own unity e_x, where $e_x = 4px$.

(iii) In a p-semigroup S all elements y which are in relation τ_p to x have the same unity.

(iv) If in a p-semigroup $2px = e_x$ and $x\tau_p y$, then $py + x = x + p^2y$.

Proof. *(i)* From $(4p+1)x = x$ follows $x = x + (4p-1)x + x$ for every x.

(ii) From $(4p+1)x = x$ we have that $x + 4px = 4px + x = x$, so $e_x = 4px$.

(iii) By Lemma 3..

(iv) By Lemma 1.. □

Proposition 1. *Let $x\tau_p y$ in a semigroup S. Then:*

(i) $y + x = 3x + py$, *(ii)* $y + x + y = 5x$, *(iii)* $x + y + x = (3p+2)y$.

Proof. Straightforward. □

Theorem 2. *Let $x\tau_p y$ in a p-semigroup S. For arbitrary positive integer k we have:*

$$(i) \quad 2kx + y = y + 2kx, \qquad (ii) \quad x + 2ky = 2ky + x.$$

Proof. *(i)* $2kx + y = k(p+1)y + y$ (Lemma 1.) $= y + 2kx$.

(ii) Using the Proposition 1. *(i)* $2k$ times and Lemma 1., too, we have

$$\begin{aligned} 2ky + x &= (2k-1)y + y + x = (2k-1)y + 3x + py \\ &= (2k-1)y + x + 2x + py = (2k-1)y + x + (p+1)y + py \\ &= (2k-1)y + x + (2p+1)y = \cdots = x + 2k(2p+1)y \\ &= x + k(4p+1)y + ky = x + 2ky. \ \square \end{aligned}$$

Lemma 4. *Let $S \in \Pi_p$ and $x \in S$. Then*

$$x\tau_p x \Longrightarrow px = x \wedge 2x = e_x.$$

Proof. Straightforward. □

Lemma 5. *Let $x\tau_p y$ in a p-semigroup S and $k \in N$. Then $4ky = 8kx$.*

Proof. Straightforward. □

Lemma 6. *Let $x\tau_p y$ in semigroup S. Let k be the smallest positive integer for which $kx = e_x$. Then*

$$qx = e_x \Longrightarrow k \mid q.$$

Proof. Straightforward. □

Theorem 3. *Let S be a semigroup. Then*

$$S \in \Pi_p \Longleftrightarrow (\forall x \in S)(\exists y \in S)(2x = (p+1)y,\ 2x = 2(x+py),\ (4p+1)x = x).$$

Proof. Let $S \in \Pi_p$. Then for any $x \in S$, there exists $y \in S$ such that $x\tau_p y$. For such x and y, by Theorem 1., we have that $(2x = (p+1)y$ and $(4p+1)x = x$. Furthermore, $2(x+py) = x+py+x+py = y+py = 2x$.

Conversely, by supposition $2x = (p+1)y$ and $(4p+1)x = x$ and Lemma 3. we have $y = (2p^2+2p+1)y$. Let us prove that $e_x = e_y$. We have

$$\begin{aligned} e_x &= 4px = 2p(2x) = 2p(p+1)y = (2p^2+2p-1)y + y \\ &= (2p^2+2p-1)y + (4p+1)y = (2p^2+2p+1)y + (4p-1)y \\ &= y + (4p-1)y = 4py = e_y. \end{aligned}$$

By using supposition $2x = 2(x+py)$, we have

$$\begin{aligned} x+py+x &= x+py+x+e_x = x+py+x+4py \\ &= x+py+x+py+3py = 2(x+py)+3py = 2x+3py \\ &= (p+1)y+3py = y, \\ py+x+py &= e_y+py+x+py = 4px+py+x+py \\ &= (4p-1)x+x+py+x+py = (4p-1)x+2(x+py) \\ &= (4p-1)x+2x = x. \ \square \end{aligned}$$

Let T be a nonempty subset of the semigroup S. Let $[T]$ denote the subsemigroup of semigroup S generated by the set T. Let us denote by A_a the set of all p-elements of element a in p-semigroup S, i.e., $A_a = \{pb \mid a\tau_p b\}$.

Theorem 4. *Let $S \in \Pi_p$ and $a \in S$. Then for each subset $I_a \subset A_a$, $GI_a = [a \cup I_a]$ is a group.*

Proof. Let $x \in GI_a$. Then $x = x_1 + x_2 + \cdots + x_n$, where $x_i \in a \cup I_a (i = 1, 2, \cdots, n)$, and

$$x_i = \begin{cases} pa_i, & x_i \neq a,\ pa_i \in I_a \\ a, & x_i = a. \end{cases}$$

Consider x' which is of the following form $x' = x'_n + x'_{n-1} + \cdots + x'_1$, where

$$x'_i = \begin{cases} 3pa_i, & x_i \neq a, \quad pa_i \in I_a \\ (4p-1)a, & x_i = a,\ (i = 1, 2, \cdots, n). \end{cases}$$

It is clear that x' is from GI_a. Since for each $i = 1, 2, \cdots, n$ we have $x_i + x'_i = e_a$ and $x_i + e_a = x_i$, then $x + x' = e_a$. Similarly $x' + x = e_a$, so GI_a is a group. □

Corollary 2. *Each p-semigroup S has the following form $S = \bigcup_{a \in S} GI_a$.*

In other words, each p-semigroup is covered by groups.

Lemma 7. *Let $x\tau_p y$ in p-semigroup S. Then:*

(i) *If p is even number, then $p^2y = e_x$.*

(ii) *If p is odd number, then $p^2y = py$.*

Proof. (i) Since p is even number, we have $p^2y = \left(\frac{p}{2}\right)^2(4y) = \left(\frac{p}{2}\right)^2(8x)$ (Lemma 5.) $= \frac{p}{2}(4px) = e_x$.

(ii) We distinguish two cases: $p = 4p_1+1$ and $p = 4p_2+3$, where $p_1, p_2 \in N_0$. In the first case we have $p^2y = (4p_1+1)(py) = p_1(4py)+py = e_y+py = py$. In the second case we have $p^2y = (4p_2+3)py = p_2(4py)+3py = e_y+3py = 3py$. Since S is p-semigroup, then there is a $z \in S$ such that $y\tau_p z$. So we get

$$\begin{aligned} 3py &= p(2y) + py = p(p+1)z + py = p(4p_2+4)z + py \\ &= (p_2+1)(4pz) + py = e_z + py = py. \end{aligned}$$

Accordingly, $p^2y = py$. □

Lemma 8. *Let $pb \in A_a$, $I_a = \{pb\}$, where a is from the p-semigroup S. Then*

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\},$$

where k is the smallest positive integer such that $ka = e_a$.

Proof. Since $pb + a = (2p+1)a + p^2b$, by using Lemma 7. we have that $pb + a = (2p+1)a$ or $pb + a = (2p+1)a + pb$. If $pb + a = (2p+1)a$, then $2pb = 2p(a + pb + a) = 2p(a + (2p+1)a) = (p+1)(4pa) = e_a$. If $pb + a = (2p+1)a + pb$, then

$$\begin{aligned} 2pb &= p(b+b) = p(a+pb+a+b) = p(a+(2p+1)a+pb+b) \\ &= p(a+(2p+1)a+2a) = 2p^2a + 4pa = 2p^2a. \end{aligned}$$

From the previous relations we have that $3pb = pb$ or $3pb = 2pa^2 + pb$. Also $4pb = e_a$. Therefore each element from GI_a has one of the following forms: $e_a, ma, pb, na + pb$, where m and n are positive integers smaller than k. □

Theorem 5. *Let $pb, pc \in A_a$, $pb \neq pc$, $I_a = \{pb\}$, $I'_a = \{pc\}$, where a is from a p-semigroup S. Then:*

(i) *If $pc = ma + pb$ for some $m \in N$, then $GI_a = GI'_a$.*

(ii) *If $pc \neq ma+pb$ for every $m \in N$, then $GI_a \cap GI'_a = \{e_a, a, 2a, \cdots, (k-1)a\}$, where k is the smallest positive integer such that $ka = e_a$.*

Proof. (i) Let $pc = ma + pb$ for some $m \in N$. Then $pc \in GI_a$, so $GI'_a \subset GI_a$. Since for arbitrary $m \in N$ exists $l \in N$ such that $m < 4lp$, we have

$$pb = e_a + pb = (4lp-m)a + ma + pb = (4lp-m)a + pc,$$

so, $pb \in GI'_a$. Accordingly, $GI_a \subset GI'_a$.

(ii) Let $pc \neq ma+pb$ for every $m \in N$. Suppose that GI_a and GI'_a have, besides e_a and ta ($t \in N$ and $t \leq k-1$), some other common element, i.e. let

$m_1a+pb=m_2a+pc$. Then $(4p-m_2)a+m_1a+pb=(4p-m_2)a+m_2a+pc$, i.e. $(4p+m_1-m_2)a+pb=4pa+pc$. Consequently $pc=(4p+m_1-m_2)a+pb$, which is contradictory to the assumption that $pc\neq ma+pb$ for all $m\in N$. So, $GI_a\neq GI'_a$ and

$$GI_a\cap GI'_a=\{e_a,a,2a,\cdots,(k-1)a\}. \ \square$$

The following considerations are referred to p-semigroups, where p is an odd number.

Lemma 9. *Let $x\tau_p y$ in a p-semigroup and let p be an odd number. Then $2px=2py$.*

Proof. We distinguish two cases: $p=4p_1+1$ and $p=4p_2+3$ $(p_1,p_2\in N_0)$. If $p=4p_1+1$, then

$$2px=p(p+1)y=(4p_1+1+1)py=p_1(4py)+2py=e_x+2py=2py.$$

If $p=4p_2+3$, Then

$$2px=p(p+1)y=p(4p_2+3+1)y=(p_2+1)(4py)=e_x.$$

Since $y\tau_p z$ holds for some z from p-semigroup, then, similarly to the previous, we have $2py=e_y$. Since $e_x=e_y$ (Corollary 1. (iii)), we finally have $2px=2py$. $\square$

Proposition 2. *Let x be arbitrary element of a p-semigroup where p is an odd number. Then $p^2x=px$.*

Proof. If p is of the form $4p_1+1$ $(p_1\in N_0)$, then

$$p^2x=(4p_1+1)(px)=p_1(4px)+px=e_x+px=px.$$

Let p be of the form $4p_2+3$ $(p_2\in N_0)$. Since x is from p-semigroup, then there is a y such that $x\tau_p y$. Therefore $2px=p(p+1)y=p(4p_2+3+1)y=(p_2+1)(4py)=e_x$. Consequently

$$p^2x=(4p_2+3)(px)=p_2(4px)+2px+px=e_x+e_x+px=px. \ \square$$

Lemma 10. *Let $x\tau_p y$ in a p-semigroup where p is an odd number. Then $p(x+py)=px+py$.*

Proof. Straightforward. $\square$

From Lemma 1., Lemma 7. and Lemma 9. the next corollary follows immediately.

Corollary 3. *Let $x\tau_p y$ in a p-semigroup where p is an odd number. Then:*

$$(i)\quad py+x=(2p+1)x+py, \qquad (ii)\quad py+x=x+3py.$$

Corollary 4. *Let $x\tau_p y$ in a p-semigroup, p being an odd number and $m\in N$. Then*

$$py+(2m+1)x=(2p+2m+1)x+py.$$

Lemma 11. *Let x be an arbitrary element from a p-semigroup where p is an odd number. Then*

$$x\tau_p x \Longleftrightarrow 2x = e_x$$

Proof. Straightforward. □

Let a be an arbitrary element of a p-semigroup S, let p be an odd number and $a \notin A_a$. In further text we will study the group GI_a, where I_a has exactly one element pb, i.e., $I_a = \{pb\}$, respectively $GI_a = [\{a, pb\}]$. Further, we will denote by k the smallest positive integer such that $ka = e_a$.

Lemma 12. *Let a be an arbitrary element of p-semigroup, let p be an odd number and let k be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then $k \mid 4p$ and $k > 2$.*

Proof. Straightforward. □

Corollary 5. *Let a be an arbitrary element from a p-semigroup, let p be an odd number, let k be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then for some $k_1 \mid p$ we have $k = k_1$ or $k = 2k_1$ or $k = 4k_1$ and $k > 2$.*

Recall that a **generalized quaternion group** is $(Q_{8k,+})$, $k \in N$, where

$$Q_{8k} = \{e, a, 2a, \cdots, (4k-1)a, b, a+b, 2a+b, \cdots, (4k-1)a+b\}$$

and the following conditions are fulfilled: $4ka = e$, $2b = 2ka$, $3b = 2ka + b$, $4b = 4ka$, $b + 2ma = 2ma + b$ $(m \in N)$, $b + (2n+1)a = (2k+2n+1)a + b$ $(n \in N_0)$.

Theorem 6. *Let a be arbitrary element of a p-semigroup, let p be an odd number, let k be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then for $k = 4k_1$ $(k_1 \mid p)$ and $I_a = \{pb\}$ we have that*

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\}$$

i.e., it is the generalized quaternion group.

Proof. By corollary 3. (i) and Lemma 9. we have $pb+a = (2p+1)a+pb$, $2pb = 2pa$, $3pb = 2pa + pb$ i $4pb = e_a$. Therefore

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\}.$$

Let us prove that all elements from GI_a are different.

(1) Since k is the smallest positive integer such that $ka = e_a$, then $\{e_a, a, 2a, \cdots, (k-1)a\}$ is the cyclic subgroup of group GI_a, so $ma \neq na$ for different $m, n \leq k$.

(2) Let us prove that $pb \neq e_a$. Suppose that this is not true, i.e., $pb = e_a$. Then $b = a + pb + a = a + e_a + a = 2a$. Therefore $e_a = pb = p(2a) = 2pa$, so $k \mid 2p$, i.e., $4k_1 \mid 2p$, which is impossible because p is an odd number. Accordingly, $pb \neq e_a$.

(3) Let us prove that $pb \neq ma$, for arbitrary positive integer m smaller than k. Suppose that this is not true, i.e. that $pb = ma$ for some positive integer m. Then $a = pb + a + pb = ma + a + ma = (2m+1)a$. Furthermore $ka = (k-1)a + a = (k-1)a + (2m+1)a = ka + 2ma$. Since $ka = e_a$, then $2ma = e_a$. By assumption $a \notin A_a$, so $pb \neq a$. Therefore $2 < 2m < 2k$, thus $2m = k = 4k_1$. From the last relation we have $m = 2k_1$, so $pb = 2k_1a$. Furthermore $b = a + pb + a = a + 2k_1a + a = 2(k_1+1)a$, from which $pb = p(2(k_1+1)a) = 2(k_1+1)(pa)$. By using Corollary 4.1. we have $k_1 \mid p$, so k_1 is an odd number. Therefore $4 \mid 2(k_1+1)$, thus $pb = 2(k_1+1)(pa) = e_a$, what is impossible by (2). Thus, $pb \neq ma$ for any positive integer m smaller than k.

(4) Let us prove that $ma + pb \neq e_a$ for any positive integer m smaller than k. Suppose that this is not true i.e. let $ma+pb = e_a$ be for some positive integer m. Then

$$pb = e_a + pb = ma + pb + pb = ma + 2pb = ma + 2pa = (m+2p)a,$$

which is impossible by (3). Thus, $ma + pb \neq e_a$ for any positive integer m.

(5) Let us prove that $ma + pb \neq na$ if m and n are positive integers smaller than k. Suppose that $ma + pb = na$ for some positive integers m and n smaller than k. Then

$$pb = 4pa + pb = (4p-m)a + ma + pb = (4p-m)a + na = (4p+n-m)a,$$

which is impossible by (3). Thus, $ma + pb \neq na$ for any positive integers m and n smaller than k.

(6) Let us prove that $ma + pb \neq pb$ for any positive integer m smaller than k. Suppose that $ma + pb = pb$ be for some positive integer m smaller than k. Then

$$ma = ma + 4pb = ma + pb + 3pb = pb + 3pb = e_a,$$

which is in contradiction with the assumption that k is the smallest positive integer such that $ka = e_a$. Accordingly, $ma + pb \neq pb$ for any positive integer m smaller than k.

(7) It remains to prove that $ma + pb \neq na + pb$ if m and n are different positive integers smaller than k. Suppose the opposite, i.e., let $ma + pb = na + pb$ be for some positive integers m and n smaller than k. Then

$$na = na + 4pb = na + pb + 3pb = ma + pb + 3pb = ma,$$

which is impossible by (1). Thus, $ma + pb \neq na + pb$ if m and n are different positive integers smaller than k.

According to (1)-(7) all elements of set GI_a are mutually different. For $k = 4$, GI_a is the quaternion group, so, for $k > 4$, GI_a is the generalized quaternion group. □

Theorem 7. *Let a be an arbitrary element of a p-semigroup, let p be an odd number, let k be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then for $k = 2k_1$ $(k_1 \mid p)$ and $I_a = \{pb\}$ we have:*

(1) $GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\}$ *and* $GI_a \simeq C_{k_1} \times C_2 \times C_2'$, *where* C_{k_1} *is the cyclic group of order* k_1 *and* C_2 *and* C_2' *are the cyclic groups of order two, or*

(2) $GI_a = \{e_a, a, 2a, \cdots, (k-1)a\}$ *is the cyclic group of order* k.

Proof. By Corollary 5. we have that $k_1 \mid p$, so $k \mid 2p$. Consequently $2pa = e_a$. Since $2pa = 2pb$, then $2pb = e_a$, too. From equalities $2pa = e_a$ and $pb + a = (2p+1)a + pb$ we get $pb + a = a + pb$. By Lemma 8. we have

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\}.$$

First, let us prove that $pb = ma$ only if $m = k_1$. Suppose the opposite, i.e., let $pb = ma$ and $m \neq k_1$. First, let m be even number, i.e., $m = 2m_1$ ($m_1 \in N$). Then

$$\begin{aligned} b &= a + pb + a = a + ma + a = (m+2)a \quad \text{and} \\ pb &= p(m+2)a = p(2m_1+2)a = (m_1+1)(2pa) = (m_1+1)e_a = e_a. \end{aligned}$$

Since $pb = ma$, then $ma = e_a$. Since $m < k$, then $ma = e_a$ which is in contradiction with the assumption that k is the smallest positive integer such that $ka = e_a$. Accordingly, the even positive integer m for which $pb = ma$, does not exist. Let m be an odd number smaller than k and $m \neq k_1$, i.e., $m = 2m_2 + 1$ ($m_2 \in N$). Then $b = (m+2)a$ and $pb = p(m+2)a = p(2m_2 + 3)a = (m_2+1)(2pa) + pa = e_a + pa = pa$. Hence, $pb = pa = ma$. Since $k_1 \mid p$ and p is odd number, there exists $k_2 \in N_0$ so that $p = k_1(2k_2+1)$. Therefore $pa = k_1(2k_2+1)a = k_2(2k_1a) + k_1a = e_a + k_1a = k_1a$. From the last equality follows that $ma = k_1a$,whence $m = k_1$. So, if $pb = k_1a$, then $GI_a = \{e_a, a, 2a, \cdots, (k-1)a\}$ is the cyclic group of order k.

If $pb = e_a$, then $GI_a = \{e_a, a, 2a, \cdots, (k-1)a\}$ is the cyclic group of order k, too.

Let $pb \neq e_a$ and $pb \neq k_1a$. Let us prove that in this case $GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a+pb, 2a+pb, \cdots, (k-1)a+pb\}$ and $GI_a \simeq C_{k_1} \times C_2 \times C_2'$, where C_{k_1} is the cyclic group of order k_1 and C_2 and C_2' are the cyclic groups of order 2. By the assumption we have that $pb \neq e_a$ and $pb \neq k_1a$. We have proved that $pb \neq ma$ for $m \neq k_1$,too. We prove that other elements of GI_a are different in the same way as in Theorem 6..

Let us prove that $GI_a \simeq C_{k_1} \times C_2 \times C_2'$. Let

$$C_{k_1} = \{e_a, 2a, 4a, 6a, \cdots, 2(k_1-1)a\}, \quad C_2 = \{e_a, k_1a\} \ \text{and} \ C_2' = \{e_a, pb\}.$$

Immediately, it is clear that C_{k_1} is the cyclic group of order k_1, and C_2 and C_2' are the cyclic groups of order 2. Let us define function f of the set $C_{k_1} \times C_2 \times C_2'$ to set GI_a in the following way: $f(x,y,z) = x + y + z$. Since $a + pb = pb + a$, then GI_a is commutative group. Therefore

$$\begin{aligned} & f((x_1,y_1,z_1) + (x_2,y_2,z_2)) = f(x_1+x_2, y_1+y_2, z_1+z_2) \\ =\ & (x_1+x_2) + (y_1+y_2) + (z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) \\ =\ & f(x_1,y_1,z_1) + f(x_2,y_2,z_2), \end{aligned}$$

so, f is a homomorphism. Let us prove that each element of GI_a is an image of some element from the set $C_{k_1} \times C_2 \times C_2'$. Let x be from the set GI_a. If $x = e_a$, then $x = e_a + e_a + e_a = f(e_a, e_a, e_a)$. If $x = 2ma$ $(1 \leq m \leq k_1 - 1)$, then $x = 2ma + e_a + e_a = f(2ma, e_a, e_a)$. If $x = (2m-1)a$ $(1 \leq m \leq k_1)$, then

$$\begin{aligned} x &= (2m-1)a + e_a = (2m-1)a + 2k_1 a = (2m + k_1 - 1)a + k_1 a + e_a \\ &= f((2m + k_1 - 1)a, k_1 a, e_a). \end{aligned}$$

If $x = pb$, then $x = e_a + e_a + pb = f(e_a, e_a, pb)$. If $x = 2ma + pb$ $(1 \leq m \leq k_1 - 1)$, then $x = 2ma + e_a + pb = f(2ma, e_a, pb)$. If $x = (2m-1)a + pb$ $(1 \leq m \leq k_1)$, then

$$\begin{aligned} x &= (2m-1)a + e_a + pb = (2m-1)a + 2k_1 a + pb \\ &= (2m + k_1 - 1)a + k_1 a + pb = f((2m + k_1 - 1)a, k_1 a, pb). \end{aligned}$$

So, function f is "*on*".

Let us prove that the function f is "1 - 1". Let (x, y, z) be from the set $C_{k_1} \times C_2 \times C_2'$. Element (x, y, z) has one of the following forms: (e_a, e_a, e_a), (e_a, e_a, pb), $(e_a, k_1 a, e_a)$, $(e_a, k_1 a, pb)$, $(2ma, e_a, e_a)$, $(2ma, e_a, pb)$, $(2ma, k_1 a, e_a)$, $(2ma, k_1 a, pb)$, where $1 \leq m \leq k_1 - 1$. Furthermore:

$$\begin{aligned} & f(e_a, e_a, e_a) = e_a, \quad f(e_a, e_a, pb) = pb, \quad f(e_a, k_1 a, e_a) = k_1 a, \\ & f(e_a, k_1 a, pb) = k_1 a + pb, \quad f(2ma, e_a, e_a) = 2ma, \\ & f(2ma, e_a, pb) = 2ma + pb, \quad f(2ma, k_1 a, e_a) = (2m + k_1)a, \\ & f(2ma, k_1 a, pb) = (2m + k_1)a + pb. \end{aligned}$$

It is easy to prove that all elements on the right sides of equalities are different, so, function f is "1 - 1". Accordingly, f is an isomorphism. This proves the theorem. □

Theorem 8. *Let a be an arbitrary element of a p-semigroup, let p be an odd number, let $k \neq 1$ be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then for $k = k_1$ $(k_1 \mid p)$ and $I_a = \{pb\}$ we have that*

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a + pb, 2a + pb, \cdots, (k-1)a + pb\}$$

is the cyclic group generated by $a + pb$.

Proof. If $pb = e_a$, then $GI_a = \{e_a, a, 2a, \cdots, (k-1)a\}$ is the cyclic group generated by $a + pb$.

Let $pb \neq e_a$. Since $k \mid p$ and $ka = e_a$, then $pa = e_a$. From $2pa = 2pb$ and $pb + a = (2p+1)a + pb$, it follows that $2pa = 2pb = e_a$ and $pb + a = a + pb$. By Lemma 8.

$$GI_a = \{e_a, a, 2a, \cdots, (k-1)a, pb, a + pb, 2a + pb, \cdots, (k-1)a + pb\}.$$

Let us prove that all elements of GI_a are different.

First, let us prove that $pb \neq ma$ $(1 < m < p)$. Suppose that this is not true, i.e. let $pb = ma$. Then $b = a + pb + a = a + ma + a = (m+2)a$.

Further $pb = p(m+2)a = (m+2)e_a = e_a$, which is in contradiction with the assumption that $pb \neq e_a$. So, $pb \neq ma$. The proof that the other elements are different is the same as in Theorem 6.

Let us prove that GI_a is the cyclic group generated by $a+pb$. If m is an even number, then $mpb = e_a$, and if m is an odd number, then $mpb = pb$. Since $pb+a = a+pb$, then: $e_a = (2k)(a+pb)$, $pb = k(a+pb)$, $ma = m(a+pb)$ for even number m, $ma = (m+k)(a+pb)$ for odd number m, $ma+pb = (m+k)(a+pb)$ for even number m, $ma + pb = m(a+pb)$ for odd number m.

Therefore, GI_a is the cyclic group of order $2k$ generated by $a+pb$. □

Theorem 9. *Let a be an arbitrary element of a p-semigroup, let p be an odd number, $a \in A_a$ and $I_a = \{pb\}$. Then:*

(1) *If $a = pb = e_a$, then $GI_a = \{e_a\}$ is the cyclic group of order one;*

(2) *If $a = e_a$ and $pb \neq e_a$, then $GI_a = \{e_a, pb\}$ is the cyclic group of order two;*

(3) *If $a \neq e_a$ and $pb = e_a$, then $GI_a = \{e_a, a\}$ is the cyclic group of order two;*

(4) *If $a \neq e_a$ and $pb \neq e_a$, then $GI_a = \{e_a, a, pb, a+pb\}$ is the Klein group.*

Proof. (1) This proposition follows immediately.

(2) Let $a = e_a$ and $pb \neq e_a$. By Lemma 9. and Lemma 11. we have that $2pb = e_a$, so $GI_a = \{e_a, pb\}$.

(3) Let $a \neq e_a$ and $pb = e_a$. By Lemma 11. we have that $2a = e_a$, so $GI_a = \{e_a, a\}$.

(4) Let $a \neq e_a$ and $pb \neq e_a$. By Lemma 9. and Lemma 11. we have $2a = e_a$ and $2pb = e_a$. Using the Proposition 2. (i), we conclude that $pb+a = a+pb$. Thus, $GI_a = \{e_a, a, pb, a+pb\}$. □

Proposition 3. *Let x be an arbitrary element of a p-semigroup and let $p = 4p_1 + 3$ $(p_1 \in N_0)$. Then $2px = e_x$.*

Proof. Straightforward. □

Corollary 6. *Let $x\tau_p y$ in a p-semigroup and let $p = 4p_1 + 3$ $(p_1 \in N_0)$. Then $x+y = y+x$.*

Proof. Straightforward. □

Corollary 7. *Let a be an arbitrary element of p-semigroup, let $p = 4p_1 + 3$ $(p_1 \in N_0)$, let k be the smallest positive integer such that $ka = e_a$ and let $a \notin A_a$. Then, $k = k_1$ or $k = 2k_1$, for some $k_1 \mid p$ and $k > 2$.*

The following considerations are referred to p-semigroups in which p is an even number.

Lemma 13. *Let $x\tau_p y$ in a p-semigroup and let p be an even number. Then:*

(1) $2y = 4x$, (2) $2py = e_x$, (3) $py = 2px$, (4) $y = (2p+2)x$, (5) $x+y = y+x$, (6) $x+py = py+x$, (7) $p(x+py) = px$.

Proof. Let $x\tau_p y$.

(1) Since p is even number, by Theorem 2. we have $2y = y + x + py + x = y + py + 2x = 2x + 2x = 4x$.

(2) By (1) we have $2py = 4px = e_x$.

(3) Since p is even number, by (1) we have $py = \frac{p}{2}(2y) = \frac{p}{2}(4x) = 2px$.

(4) By (3) we get $y = x + py + x = x + 2px + x = (2p + 2)x$.

(5) By (4) we have $x + y = x + (2p + 2)x = (2p + 2)x + x = y + x$.

(6) Equality $x + py = py + x$ immediately follows from (5).

(7) Since p is an even number, by Theorem 2. we have $p(x+py) = px+p^2y$. By Lemma 7. (i) we have $p^2y = e_x$, so $p(x + py) = px$. □

Theorem 10. *Let S be a semigroup and let p be an even number. Then*

$$S \in \Pi_p \iff (\forall x \in S)((4p + 1)x = x).$$

Proof. Let $S \in \Pi_p$. By Theorem 1. we have $(\forall x \in S)((4p + 1)x = x)$.

Conversely, let $(\forall x \in S)((4p+1)x = x)$. Let us take that $y = (2p+2)x$ and let us prove that $x\tau_p y$. Let $p = 2p_1 (p_1 \in N)$. Then

$$\begin{aligned} x + py + x &= x + p(2p + 2)x + x = x + p(4p_1 + 2)x + x \\ &= p_1(4px) + (2p + 2)x = e_x + y = y. \end{aligned}$$

Furthermore

$$py + x + py = p(2p + 2)x + x + p(2p + 2)x = (p + 1)(4px) + x = e_x + x = x.$$

So, $x\tau_p y$. □

Theorem 11. *Let a be an arbitrary element of a p-semigroup, let p be an even number and let k be the smallest positive integer such that $ka = e_a$ and $I_a = \{pb\}$. Then $GI_a = \{e_a, a, 2a, \cdots, (k - 1)a\}$ is the cyclic group of order k.*

Proof. Straightforward. □

Let us denote by C_k the cyclic group of order k, by K_4 the Klein group, by $C_k \times C_2 \times C_2$ the direct product of cyclic groups and by Q_{8k} the generalized quaternion group of order $8k$, where $k \in N$. In particular for $k = 1$ we have that Q_8 is the quaternion group.

Let us define the classes of groups $\Gamma'_p, \Gamma''_p, \Gamma'''_p$ in the following way:

(i) For $p = 4p_1 + 1$ $(p_1 \in N_0)$ we define the class Γ'_p by

$$\begin{aligned} G \in \Gamma'_p \iff & (\exists k \in N)(k \mid p \wedge (G = C_k \vee G = C_{2k} \vee G = K_4 \\ \vee\ & G = C_k \times C_2 \times C_2 \vee G = Q_{8k})). \end{aligned}$$

(ii) For $p = 4p_2 + 3$ $(p_2 \in N_0)$ we define the class Γ''_p by

$$\begin{aligned} G \in \Gamma''_p \iff & (\exists k \in N)(k \mid p \wedge (G = C_k \vee G = C_{2k} \vee G = K_4 \\ \vee\ & G = C_k \times C_2 \times C_2)). \end{aligned}$$

(*iii*) For $p = 2p_3$ ($p_3 \in N$) we define the class Γ_p''' by

$$G \in \Gamma_p''' \iff (\exists k \in N)(k \mid p \ \wedge \ (G = C_k \ \vee \ G = C_{2k} \ \vee \ G = C_{4k})).$$

In each of the foregoing cases, if p is not of the given form, then the corresponding class is empty.

Lemma 14. *Let S be a semigroup which is a union of groups from the class Γ_p', i.e., $S = \bigcup\{G \mid G \in \Gamma_p'\}$. Then $S \in \Pi_p$.*

Proof. Let S be a semigroup, $p = 4p_1+1$ ($p_1 \in N_0$), $S = \bigcup\{G \mid G \in \Gamma_p'\}$ and $x \in S$. Depending on to which group the element x from the class Γ_p' belongs, we distinguish five cases.

1) Let $x \in C_k$ and $k \mid p$. Since

$$C_k = \{e, a, 2a, \cdots, (k-1)a\}$$

for some $a \in S$, we distinguish two cases.

a) If $x = e$, then for y we take that $y = e$.

b) Let $x = ma, 1 \leq m \leq k-1$. For y we take that $y = 2x = 2ma$. 2) Let $x \in C_{2k}$ and $k \mid p$. Since

$$C_{2k} = \{e, a, 2a, \cdots, (2k-1)a\}$$

for some $a \in S$, we distinguish two cases.

a) If $x = e$ for y we take that $y = e$.

b) Let $x = ma$, $1 \leq m \leq 2k-1$. For y we take that $y = 2x = 2ma$.

3) Let $x \in K_4$. Then for y we take that $y = x$.

4) Let $x \in C_k \times C_2 \times C_2'$ and $k \mid p$, where $C_k = \{e_1, a_1, 2a_1, \cdots, (k-1)a_1\}$, $C_2 = \{e_2, a_2\}$ and $C_2' = \{e_3, a_3\}$ for some $a_1, a_2, a_3 \in S$. We distinguish two cases:

a) Let x have one of the following forms: (e_1, e_2, e_3), (e_1, e_2, a_3), (e_1, a_2, e_3), (e_1, a_2, a_3). In all cases $2x = (e_1, e_2, e_3)$. For y we take that $y = 2x = (e_1, e_2, e_3)$.

b) Let x have one of the following forms: (ma_1, e_2, e_3), (ma_1, e_2, a_3), (ma_1, a_2, e_3), (ma_1, a_2, a_3), where $1 \leq m \leq k-1$. In all cases $2x = (2ma_1, e_2, e_3)$. For y we take that $y = 2x = (2ma_1, e_2, e_3)$.

5) Let $x \in Q_{8k}$, $k \mid p$, where

$$Q_{8k} = \{e, a, 2a, \cdots, (4k-1)a, b, a+b, 2a+b, \cdots, (4k-1)a+b\}$$

for some $a, b \in S$. It is known that for elements a and b from Q_{8k} the following equalities hold: $4ka = e$, $2b = 2ka$, $3b = 2ka + b$, $4b = e$, $b + 2ma = 2ma + b$ ($m \in N$), $b + (2n+1)a = (2k+2n+1)a + b$, ($n \in N_0$). Since $p = (2k_1+1)k$

and $p = 4p_1 + 1$ $(k_1, p_1 \in N_0)$, then:

$$2pa = 2(2k_1+1)ka = k_1(4ka) + 2ka = e + 2ka = 2ka = 2b,$$
$$2pa + b = 2b + b = 3b,$$
$$4pa = 2b + 2b = 4b = e,$$
$$b + (2n+1)a = (2k+2n+1)a + b = (2p+2n+1)a + b,$$
$$pb = (4p_1+1)b = p_1(4b) + b = e + b = b.$$

To prove that $x\tau_p y$ for some $y \in S$, we distinguish six cases.

a) If $x = e$ then for y we take that $y = e$.

b) If $x = 2ma$ $(m \in N)$ then for y we take that $y = 4ma + 2b$.

c) If $x = (2n+1)a$, $(n \in N_0)$ then for y we take that $y = (2p+4n+2)a+b$.

d) If $x = b$, then for y we take that $y = pa$.

e) If $x = 2ma + b$, $(m \in N)$ then for y we take that $y = (4m+p)a$.

f) If $x = (2n+1)a+b$, $(n \in N_0)$ then for y we take that $y = (p+4n+2)a$.

We have $x\tau_p y$ in all cases. For example, here is the proof for the case 5) c).

$$\begin{aligned}
x + py + x &= (2n+1)a + p((2p+4n+2)a+b) + (2n+1)a \\
&= (2n+1)a + p(2p+4n+2)a + pb + (2n+1)a \\
&= (2n+1)a + 2p^2a + n(4pa) + 2pa + b + (2n+1)a \\
&= (2n+1)a + (4p_1+1)(2pa) + e + 2pa \\
&\quad + b + (2n+1)a \\
&= (2n+1)a + 2p_1(4pa) + 2pa + 2pa + b + (2n+1)a \\
&= (2n+1)a + e + e + b + (2n+1)a = (2n+1)a \\
&\quad + (2p+2n+1)a + b = (2p+4n+2)a + b = y, \\
py + x + py &= p((2p+4n+2)a+b) + (2n+1)a \\
&\quad + p((2p+4n+2)a+b) \\
&= 2p^2a + n(4pa) + 2pa + pb + (2n+1)a + 2p^2a \\
&\quad + n(4pa) + 2pa + pb \\
&= (4p_1+1)(2pa) + e + 2pa + b + (2n+1)a \\
&\quad + (4p_1+1)(2pa) + e + 2pa + b \\
&= 2p_1(4pa) + 2pa + 2pa + b \\
&\quad + (2n+1)a + 2p_1(4pa) + 2pa + 2pa + b \\
&= e + e + b + (2n+1)a + e + e + b \\
&= b + (2n+1)a + b = (2p+2n+1)a + b + b \\
&= (2p+2n+1)a + 2pa = 4pa + (2n+1)a \\
&= e + (2n+1)a = x.
\end{aligned}$$

Hence, $x\tau_p y$. □

Lemma 15. *Let S be a semigroup which is a union of groups from the class Γ''_p, i.e., $S = \bigcup\{G \mid G \in \Gamma''_p\}$. Then $S \in \Pi_p$.*

Proof. Straightforward. □

Lemma 16. *Let S be a semigroup which is a union of groups from the class Γ'''_p, i.e., $S = \bigcup\{G \mid G \in \Gamma'''_p\}$. Then $S \in \Pi_p$.*

Proof. Let S be a semigroup, $p = 2p_3$, $(p_3 \in N)$, $S = \bigcup\{G \mid G \in \Gamma'''_p\}$ and $x \in S$. We distinguish two cases:

a) If $x = e$ then for y we take that $y = e$. Similarly to 1) a) of Lemma 14. we have that $x\tau_p y$.

b) If $x = ma$ $(1 \leq m \leq k-1) \vee 1 \leq m \leq 2k-1 \ \vee \ 1 \leq m \leq 4k-1)$ then for y we take that $y = 2mpa + 2ma$. Then we have

$$\begin{aligned} x + py + x &= ma + p(2mpa + 2ma) + ma = ma + 2mp^2a + 2mpa + ma \\ &= 2ma + mp_1(4pa) + 2mpa = 2mpa + 2ma = y, \\ py + x + py &= p(2mpa + 2ma) + ma + p(2mpa + 2ma) \\ &= 2mp^2a + 2mpa + ma + 2mp^2a + 2mpa \\ &= (mp + m)(4pa) + ma = e + ma = x. \end{aligned}$$

So, $x\tau_p y$. □

According to Theorem 6., Theorem 7., Theorem 8., Theorem 9., Theorem 11., Lemma 14., Lemma 15. and Lemma 16. we get the following theorem.

Theorem 12. *Let S be a semigroup. Then:*

1) For $p = 4p_1 + 1$ $(p_1 \in N_0)$

$$S \in \Pi_p \Longleftrightarrow S = \bigcup\{G \mid G \in \Gamma'_p\}.$$

2) For $p = 4p_2 + 3$ $(p_2 \in N_0)$

$$S \in \Pi_p \Longleftrightarrow S = \bigcup\{G \mid G \in \Gamma''_p\}.$$

3) For $p = 2p_3$ $(p_3 \in N)$

$$S \in \Pi_p \Longleftrightarrow S = \bigcup\{G \mid G \in \Gamma'''_p\}.$$

Theorem 12. can be presented in the following way.

Let S be a semigroup. Then

$$\begin{aligned} S \in \Pi_p \quad \Longleftrightarrow \quad & S = \bigcup\{G \mid G \in \Gamma'_p\} \ \vee \ S = \bigcup\{G \mid G \in \Gamma''_p\} \\ \vee \quad & S = \bigcup\{G \mid G \in \Gamma'''_p\}. \end{aligned}$$

As all p-semigroups from classes Γ'_p, Γ''_p and Γ'''_p are groups, finally, we give an example of a p-semigroup (for p odd) which is not a group.

+	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	b	d
b	b	c	e	a	d
c	c	b	a	e	d
d	d	d	d	d	d

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