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THE PRE-LIMIT OF A REAL-VALUED FUNCTION

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Abstract. 1. In [1] S. Banach shown the existence of very known Banach linear shift-invariant functionals defined on the real vector space of all bounded real-valued functions on the semi-axis $t \geq 0$ and especially on the space of all real bounded sequences. In [2] G. G. Lorentz defined, by Banach shift-invariant functionals, the class of almost convergent sequences. In [3] almost convergence was extended to real-valued functions on the semi-axis $t \geq 0$. In [4] almost convergence was extended to bounded sequences in a real normed space.

2. This paper is devoted to a class of functions defined on the semi-axis $t \geq 0$ which are near to the functions f having $\lim_{t\to\infty} f(t)$. The paper is organized as follows. First, for a sufficiently large a (written $a > a_0$ for some a_0) by Ω we denote the real vector space of all functions defined on $[0, +\infty\rangle$ and bounded on $[a, +\infty\rangle$. Next, we will show the existence of a family of functionals defined on the space Ω. By these functionals we define the notion of pre-limit of a function $f \in \Omega$ and investigate the family of all these functions. Further, we will show a theorem characterizing a function having the pre-limit. Also we show another theorem which is very applicable, though it contains a new restrictive condition. Finally, to make the idea of pre-limit a little clearer, we give several examples functions having pre-limit.

1. A new family of functionals

Let us choose a double sequence $x = (\xi_k^n)$, $\xi_k^n \geq 0$ $(k = 1, 2, \ldots, n;$ $n = 1, 2 \ldots)$ and fix it. Then the functional $p_x \equiv p$ defined on the space Ω by

$$(1) \qquad p(f) = \overline{\lim_{t\to\infty}} \left\{ \overline{\lim_{n\to\infty}} \frac{1}{n} \left| \sum_{k=0}^{n-1} f(t + \xi_k^n) \right| \right\}, \quad f \in \Omega$$

corresponds to x.

The functional p is seen to by real-valued and it satisfies the conditions

$$p(f) \geq 0,\ p(af) = |a| p(f),\ p(f+g) \leq p(f) + p(g)\ (a \in R;\ f, g \in \Omega);$$

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that is, p is a symmetric convex functional on the space Ω. According to a corollary of Hahn-Banach theorem (see also [5], Exercise 2, p. 187) there exists a nontrifial linear functional L on the space Ω such that

$$(2) \qquad |L(f) \leq p(f), \quad f \in \Omega.$$

Next let Ω_0 be the space of all functions $f \in \Omega$ having $\lim_{t\to\infty} f(t) = 0$. Also, for some $s \in R$ ($s \neq 0$) let us define the function g by $g(t) = s$, $t \geq 0$. Then $g \in \Omega \backslash \Omega_0$ and $p(g) = |s| > 0$. Notice also that clearly we have

$$(3) \qquad p(f) = L(f) = 0, \quad f \in \Omega_0$$

Now, to extend the functiolnal L to the space spanned by Ω_0 and $\{g\}$ (that is, the space $\Omega_0 \cup \{g\}$), the value $L(g)$ we can choose arbitrarily in the segment $[-p(g), p(g)]$; that is, we can extend the functional L in a such way that it has distinct values at $g \in \Omega$. In other words, the functional L satisfying the above conditions is not unique.

Indeed, we can take the value $L(g)$ arbitrarily in the segment $[k, K]$, where

$$k = \sup_{f\in\Omega_0}\{-p(f+g)\}, \quad K = \inf_{f\in\Omega_0}\{p(f+g)\}$$

since $L(f) = 0$, $\forall f \in \Omega_0$ (see, for example, [6], p. 222). Further, by (1), we have $p(f+g) = p(g)$ since

$$\lim_{t\to\infty}[f(t) + g(t)] = \lim_{t\to\infty} g(t) = s.$$

So, we can take the value $L(g)$ arbitrarily in the segment $[-p(g), p(g)]$.

We shall now show the following lemma.

Lemma 1.17. *Let X be a real linear space and $p : X \to R$ a functional satisfying the conditions*

$$p(x) \geq 0,\ p(ax) = |a|p(x),\ p(x+y) \leq p(x) + p(y)\ (a \in R;\ x, y \in X).$$

Then for any $x_0 \in X$ there exists a linear functional L on X such that

$$(\forall x \in X)|L(x)| \leq p(x), \quad L(x_0) = p(x_0).$$

Proof. Cleraly, the set $X_0 = \{\alpha x_0,\ \alpha \in R\}$ is a subspace of the space X and L_0, defined by

$$L_0(\alpha x_0) = \alpha p(x_0) \quad (\alpha \in R)$$

is a linear functional on X_0 satisfying the condition

$$|L_0(\alpha x_0)| = |\alpha p(x_0)| = |\alpha| p(x_0) = p(\alpha x_0) \quad (\alpha \in R)$$

By a version of Hahn-Banach theorem (see [5], theorem 11.2, p. 181) there exists a linear functional L on X extending L_0 and satisfying the condition

$$(\forall x \in X) \ \ |L(x)| \leq p(x).$$

Also we have

$$L(x_0) = L_0(x_0) = 1 \cdot p(x_0) = p(x_0),$$

which completes the proof.

Denoting now by $\Pi^{(x)}$ the family of functionals satisfying the above conditions, for each $s \in R$, we obtain

$$(4) \qquad (\forall L \in \Pi^{(x)})\ L(x - s) = 0 \text{ iff } p_x(f - s) = 0, \quad f \in \Omega.$$

Indeed, $p_x(f - s) = 0$ clearly implies $L(f - s) = 0$, $\forall L \in \Pi^{(x)}$. Also, the implication

$$(\forall L \in \Pi^{(x)})\ \ L(f - s) = 0 \Rightarrow p_x(f - s) = 0$$

is equivalent to the implication

$$p_x(f - s) > 0 \Rightarrow (\exists L \in \Pi^{(x)})\ \ L(f - s) \neq 0$$

which, by the lemma proved before, is valid. So, (4) must be true.

Because the sequence $x = (\xi_k^n)$, $\xi_k^n \geq 0$ contained in (1) is arbitrary, we have shown the following theorem.

Theorem 1.1. *For any sequence* $x = (\xi_k^n)$, $\xi_k^n \geq 0$ $(k = 1, 2, \dots, n;$ $n = 1, 2, \dots)$ *there exists a family* $\Pi^{(x)}$ *of nontrivial functionals* L *defined on the space* Ω *such that for all* $a, b \in R$, *all* $s \in R$ *and all* $f, g \in \Omega$ *the following assertions are valid*

1° $L(af + bg) = aL(f) + bL(g)$,

2° $|L(f)| \leq p_x(f)$,

3° $(\forall L \in \Pi^{(x)})\ L(f - s) = 0$ *iff* $p_x(f - s) = 0$.

2. The pre-limit of a real-valued function

Having the results obtained before we can proceed to investigation the family of all functions $f \in \Omega$ to which all functionals from the theorem 1 assign same value.

Definition 2.2. *Let* $f \in \Omega$. *Then* $f(t)$ *has pre-limit* s *as* $t \to +\infty$ *if for at least one family* $\Pi^{(x)}$ *and for at least one number* $s^{(x)} \equiv s$ *the following assertion*

$$(5) \qquad (\forall L \in \Pi^{(x)}\ L(f - s) = 0$$

is valid.

Notice that, by the definiton 1, in general, it is possible that a function $f \in \Omega$ has distinct pre-limits which are determined by distinct sequences $x = (\xi_k^n)$. Also, it is clear that the pre-limit of $f(t)$ is a generalization of the usual limit of $f(t)$ as $t \to +\infty$.

Further, we can show that by a sequence $x = (\xi_k^n)$ is uniquely determined the pre-limit of a function $f \in \Omega$. Indeed, suppose s' and s'' are any two pre-limits of a function $f \in \Omega$ which are determined by same sequence $x = (\xi_k^n)$ and let us define the functions g and h by

$$g(t) = s' \quad \text{and} \quad h(t) = s'' \ \ (t \geq 0).$$

Then, by (5), we have

$$(\forall L \in \Pi^{(x)})\ L(h-g) = L(f-g) - L(f-h) = L(f-s') - L(f-s'') = 0$$

which, by (4) and (1), implies

$$p(h-g) = |s'' - s'| = 0 \quad \text{and} \quad s' = s''$$

Theorem 2.1. *Let* $f \in \Omega_0$.

1° *If for at least one double sequence* $x = (\xi_k^n)$, $\xi_k^n \geq 0$ $(k = 1, 2, \ldots, n-1;$ $n = 1, 2, \ldots)$ *and some* $s \in R$

$$\lim_{t\to+\infty}\left\{\lim_{n\to\infty}\frac{1}{n}\left|\sum_{k=0}^{n-1}[f(t+\xi_k^n) - s]\right|\right\} = 0 \tag{6}$$

holds, then pre-$\lim\limits_{t\to+\infty} f(t) = s$.

2° *If pre-*$\lim\limits_{t\to+\infty} f(t) = s$, *then for at least one double sequence* $x = (\xi_k^n)$

$$\lim_{t\to+\infty}\left\{\overline{\lim}_{n\to\infty}\frac{1}{n}\left|\sum_{k=0}^{n-1}[f(t+\xi_k^n) - s]\right|\right\} \tag{7}$$

holds.

Proof. Let the condition (6) is true. Then, by (1), (4) and (5), we have pre-$\lim\limits_{t\to+\infty} f(t) = s$; so, the condition (6) is sufficient.

Conversely, let pre-$\lim\limits_{t\to+\infty} f(t) = s$. Then for some $x = (\xi_k^n)$ $(\xi_k^n \geq 0)$, by (5), (4) and (1), we have

$$\overline{\lim_{t\to+\infty}}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\left|\sum_{k=0}^{n-1}[f(t+\xi_k^n) - s]\right|\right\},$$

which means that

$$\lim_{t\to+\infty}\left\{\overline{\lim_{n\to\infty}}\frac{1}{n}\left|\sum_{k=0}^{n-1}[f(t+\xi_k^n) - s]\right|\right\},$$

so, the condition (7) is necessary which completes the proof.

Applying now the theorem 2.1 we will show the following applicable theorem though it contains a new restrictive condition.

Theorem 2.2. *Let* $f \in \Omega$ *be a Riemann integrable function on each segment* $[a, a+T]$ *for* $T > 0$ *and* $a > a_0$. *If for at least one* $T(> 0)$

$$\frac{1}{T}\int_a^{a+T} f(t)dt \to s \quad as \quad a \to +\infty, \tag{8}$$

then pre-$\lim\limits_{t\to+\infty} f(t) = s$.

Proof. Suppose that the condition (8) is valid for some $T(>0)$. Let us choose the following fundamental sequence of partitions (P_n), where

$$P_n = \left\{ \frac{i}{n}T : \ i = 0, 1, 2, \dots, n \right\}, \quad n = 1, 2, \dots,$$

which subdivides the interval $[0, T]$ into n subintervals $[t_{k-1}^n, t_k^n]$ $(k = 1, 2, \dots, n;$ $n = 1, 2, \dots,)$ and let us choose arbitrarily the points $\xi_k^n \in [t_{k-1}^n, t_k^n]$ $(k = 1, 2, \dots, n;\ n = 1, 2, \dots)$. Because the function f is integrable on $[a, a+T]$ we have

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(a+\xi_k^n)\frac{T}{n} = \int_a^{a+T} f(t)dt$$

or

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f(a+\xi_k^n) = \frac{1}{T}\int_a^{a+T} f(t)dt.$$

Since (8) is true, we have

$$\lim_{a\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)dt = \lim_{a\to+\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(a+\xi_k^n) \right\} = s.$$

Hence we have

$$\lim_{a\to+\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} [f(a+\xi_k^n) - s] \right\} = 0.$$

which implies

$$\lim_{a\to+\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\left|\sum_{k=1}^{n} [f(a+\xi_k^n) - s]\right| \right\} = 0.$$

Now, letting $i = k-1$, we have

$$\lim_{a\to+\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\left|\sum_{i=0}^{n-1} [f(a+\xi_i^n) - s]\right| \right\} = 0.$$

which, by (6), means that pre-$\lim_{t\to+\infty} f(t) = s$ which completes the proof.

To make the idea of the pre-limit of a function a little clearer we give a number of examples.

Example 1. The function sine has pre-$\lim\limits_{t\to+\infty} \sin t = 0$. Indeed, for $T = 2\pi$, by (8), we have

$$\text{pre-}\lim_{t\to+\infty} \sin t = \lim_{a\to+\infty} \frac{1}{2\pi}\int_a^{a+2\pi} \sin t dt = \lim_{a\to+\infty} \frac{1}{2\pi}(-\cos a + \cos a) = 0.$$

Example 2. Let $f \in \Omega$ be defined by

$$f(t) = \operatorname{sgn}(\sin t), \quad t \geq 0.$$

Then clearly for all $a \geq 0$ we have

$$\frac{1}{2\pi}\int_a^{a+2\pi} f(t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = 0 \Rightarrow \text{pre-}\lim_{t\to+\infty} f(t) = 0.$$

Similarly, if $f(t) = \operatorname{sgn}(\cos t)$, then pre-$\lim_{t\to+\infty} f(t) = 0$.

Example 3. Let n be a positive integer and let $f \in \Omega$ be the periodic function with period n defined by

$$f(t) = [t], \quad 0 \leq t < n.$$

Then

$$\text{pre-}\lim_{t\to+\infty} f(t) = \lim_{a\to+\infty}\frac{1}{n}\int_a^{a+n} f(t)dt =$$

$$= \frac{1}{n}\int_0^n f(t)dt = \frac{0+1+2+\cdots+(n-1)}{n} = \frac{n(n-1)}{2n} = \frac{n-1}{2}.$$

Example 4. Let $f \in \Omega$ be defined by

$$f(t) = \begin{cases} 1, & t = n \\ 0, & t \neq n \end{cases} \quad (n = 0, 1, 2, \dots).$$

Then for all a and T ($a \geq 0$, $T > 0$ we have $\int_a^{a+T} f(t)dt = 0$ which implies

$$\text{pre-}\lim_{t\to+\infty} f(t) = \lim_{a\to+\infty}\frac{1}{T}\int_a^{a+T} f(t)dt = 0.$$

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