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INITIAL SEGMENTS IN BCC-ALGEBRAS

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Abstract. The role of initial segments in BCC-algebras is described.

1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [8]) defined a class of algebras of type $(2,0)$ called *BCK-algebras* which generalize the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (cf. [9]). K. Iséki posed an interesting problem whether the class of BCK-algebras is a variety. That problem was solved by A. Wroński [11] who proved that BCK-algebras do not form a variety. In connection with this problem, Y. Komori [10] introduced the notion of BCC-algebras, and W. A. Dudek (cf. [2], [3]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [7], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun (cf. [4]) considered the fuzzification of ideals in BCC-algebras. They proved that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing a counterexample.

Any BCC-algebra (similarly as a BCK-algebra) may be treatment as a partially ordered groupoid with a some smallest element. All BCK-ideals and all BCC-ideals are ideals in the sense of ordered sets, but not conversely.

In this paper we describe the role of initial segments in BCC-algebras and find the criterion under which the initial segment is a BCC-ideal.

2. Preliminaries

By an algebra $\mathbf{G}=(G,\cdot,0)$ we mean a non-empty set G together with a binary multiplication denoted by juxtaposition and a distinguished element 0. Dots we use only to avoid repetitions of brackets. For example, the formula $((x\cdot y)\cdot(z\cdot y))\cdot(x\cdot z)=0$ will be written as $(xy\cdot zy)\cdot xz=0$.

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An algebra $(G,\cdot,0)$ is called *a BCC-algebra* if it satisfies the following conditions:

$$(1) \qquad (xy \cdot zy) \cdot xz = 0,$$
$$(2) \qquad xx = 0,$$
$$(3) \qquad 0x = 0,$$
$$(4) \qquad x0 = x,$$
$$(5) \qquad xy = yx = 0 \quad \textit{implies } x = y.$$

The above definition of a BCC-algebra is a dual form of the ordinary definition (cf. [1], [10]). In our convention *any BCK-algebra is a BCC-algebra*, but not conversely. A BCC-algebra which is not a BCK-algebra is called *proper*. Note that (cf. [2]) *a BCC-algebra is a BCK-algebra iff it satisfies*

$$(6) \qquad xy \cdot z = xz \cdot y$$

or

$$(7) \qquad (x \cdot xy)y = 0.$$

Similarly as in the case of BCK-algebras, any BCC-algebra may be viewed as a partially ordered set with the order $\leq$ defined by:

$$(8) \qquad x \leq y \textit{ iff } xy = 0.$$

This natural BCC-order has the following properties:

$$(9) \qquad xy \leq x,$$
$$(10) \qquad xy \cdot zy \leq xz,$$
$$(11) \qquad x \leq y \textit{ implies } xz \leq yz \textit{ and } zy \leq zx$$

(cf. Proposition 2 in [2]). Moreover, one can prove (cf. [5]) that every non-empty set G partially ordered by the relation ρ may be treatment as a BCK-algebra $(G,\cdot,0)$, where 0 is the smallest element of G and $xy = 0$ for $x\rho y$, and $xy = x$ otherwise. We say that a BCK-algebra with such defined a multiplication has *the trivial structure*.

A BCC-algebra lineary ordered by the relation (8) is called *a BCC-chain* or *a BCK-chain* if it is a BCK-algebra.

A non-empty subset A of a BCC-algebra G is called *a BCK-ideal of* G iff (i) $0 \in A$ and (ii) $y, xy \in A$ *imply* $x \in A$. Obviously, if A is a BCK-ideal of G and $y \in A$ then $x \in A$ for every $x \leq y$. A subset B of G is called *a BCC-ideal* (cf. [6], [7]) iff (i) $0 \in B$ and (ii) $y, xy \cdot z \in B$ *imply* $xz \in B$. Any BCC-ideal is clearly a BCK-ideal, but not conversely. The converse holds in BCK-algebras. Moreover, any BCC-ideal induces a some congruence, but there are congruences which are not induced by such ideals (cf. [7]).

3. Initial segments

For any fixed elements $a \leq b$ of a BCC-algebra G the set

$$[a,b] = \{x \in G : a \leq x \leq b\} = \{x \in G : ax = xb = 0\}$$

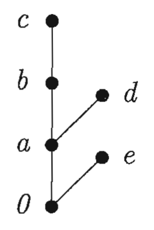
is called *the segment* of G. Note that the segment

$$[0,b] = \{x \in G : x \leq b\} = \{x \in G : xb = 0\},$$

called *initial*, is de facto the left annihilator of b. Since $[0,b]$ has two elements only in the case when $b \in G$ is an atom of G, then from result obtained in [6] follows that a BCC-algebra in which all initial segments have at most two elements has the trivial structure.

Example 3.1. An algebra $G = \{0,a,b,c,d,e\}$ defined by the table

·	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0



is a proper BCC-algebra (cf. [6]). Its initial segments have the form $[0,a] = \{0,a\}$, $[0,b] = \{0,a,b\}$, $[0,c] = \{0,a,b,c\}$, $[0,d] = \{0,a,d\}$, $[0,e] = \{0,e\}$. All these segments are BCK-chains.

On the other hand, BCC-algebras defined in [2] by Tables 8, 9 and 10 are BCC-chains with 3 as the greatest element. Since these BCC-algebras are proper, then its are not BCK-chains. A BCC-algebra defined in [2] by Table 14 is an example of a minimal proper BCC-algebra which coincides with some its initial segment. It is not a BCC-chain because elements 1 and 2 are incomparable. Algebras defined by Tables 11, 12, 13 and 15 are minimal proper BCC-algebras which are a set-theoretic union of two different BCK-chains. (By the way note that Table 15 is printed with the misprint. Namely $12 = 1$ must be replaced by $12 = 0$.)

Proposition 3.2. *Every initial segment of a BCC-algebra is a BCC-subalgebra.*

Proof. Obviously $0 \in [0,c]$. If $x,y \in [0,c]$, then $x \leq c$ and $y \leq c$, which by (11) and (9) implies $xy \leq cy \leq c$. Thus $xy \in [0,c]$, which proves that $[0,c]$ is a BCC-subalgebra. □

Proposition 3.3. *The set-theoretic union of any two initial segments of a given BCC-algebra is a BCC-subalgebra.*

Proof follows directly from (9). □

Proposition 3.4. *A BCC-algebra containing at least two initial segments $[0,x]$ and $[0,y]$ such that $[0,x] \cap [0,y] = \{0\}$ and $xy \neq x$ is proper.*

Proof. Assume a contrary that G is a BCK-algebra in which $[0,x]\cap[0,y]=\{0\}$ for some $x\neq y$. Then $x\cdot xy\leq y$ (by (7)) and $x\cdot xy\leq x$ (by (6) or (9)). Thus $x\cdot xy\in[0,x]\cap[0,y]=\{0\}$. Hence $x\leq xy$. This with (9) gives $xy=x$, which is a contradiction. Thus G cannot be a BCK-algebra. □

As a consequence of Proposition 1 from [2] we obtain

Corollary 3.5. *Every BCC-chain containing at most three elements is a BCK-chain.* □

In general, initial segments are not BCC-ideals.

Example 3.6. It is easily to verify that an algebra defined by the following table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	b	0

is a BCC-algebra isomorphic to a proper lineary ordered BCC-algebra given by Table 9 in [2]. Since $ba\in[0,a]$ and $a<b$, then $b\notin[0,a]$. Thus $[0,a]$ is not a BCK-ideal. Of course, it is not also a BCC-ideal.

Proposition 3.7. *An initial segment $[0,c]$ of a BCC-algebra G is a BCC-ideal if and only if for all $x,z\in G$*

$$xc\cdot z\leq c \ \textit{ implies } xz\leq c. \tag{12}$$

Proof. Assume that the above implication holds. If $xy\cdot z$ and y are in $[0,c]$, then $xy\cdot z\leq c$ and $y\leq c$. But $0\in[0,c]$ and $y\leq c$ imply (by (11)) $xc\cdot z\leq xy\cdot z$. Thus $xc\cdot z\leq c$, which by the assumption gives $xz\leq c$. Hence $xz\in[0,c]$, i.e. $[0,c]$ is a BCC-ideal.

The converse is obvious. □

Corollary 3.8. *If $[0,c]$is a BCC-ideal of G, then for every $x\in G$*

$$xc\leq c \ \textit{ implies } x\leq c. \tag{13}$$

Corollary 3.9. *If a non-trivial segment $[0,c]$ is a BCK-ideal or a BCC-ideal of G, then $xc\neq c$ for all non-zero $x\in G$.*

Proof. Let $[0,c]$, where $c\neq 0$, be a BCK-ideal. If $xc=c$ for some $x\in G$, then $xc\in[0,c]$ and, in the consequence, $x\leq c$, which is a contradiction since in this case we obtain $0=xc=c$. □

Corollary 3.10. *If $(xc\cdot z)c=xz\cdot c$ holds for all $x,z\in G$, then $[0,c]$ is a BCC-ideal.* □

If a BCC-algebra G satisfies the identity

$$(xy\cdot z)y=xz\cdot y, \tag{14}$$

then, of course, all initial segments are BCC-ideals. Since, for $z=0$ this identity has the form

$$xy\cdot y=xy, \tag{15}$$

and for $z = xy$ it implies $(x \cdot xy)y = 0$, then, by (7), a BCC-algebra satisfying (14) is *a positive implicative* BCK-algebra. Obviously in any positive implicative BCK-algebra (i.e. in a BCK-algebra satisfying (15)) the condition (14) holds. Thus for BCK-algebras conditions (14) and (15) are equivalent. For BCC-algebras this statement is not true. There are proper BCC-algebras in which holds only (14) (cf. [2]). BCC-algebras satisfying (15) are called *positive implicative.*

From the above remarks follows

Proposition 3.11. *A BCC-algebra satisfying* (14)*is a positive implicative BCK-algebra in which all initial segments are BCK-ideals.* □

Proposition 3.12. *A BCC-algebra in which all initial segments have at most two elements is a positive implicative BCK-algebra. Initial segments of such BCK-algebra are BCK-ideals.*

Proof follows from Lemma 1, Theorem 3 and Corollary 8 in [6]. □

Proposition 3.13. *If* $[0, c]$ *is a two-elements BCC-ideal, then* $xc \cdot z = c$ *implies* $xz = c$.

Proof. Indeed, by Proposition 3.7, from $xc \cdot z = c$ follows $xz \leq c$. But $xz = 0$, by (11) and (9), gives $xc \cdot z = 0$, which is a contradiction. Therefore must be $xz = c$. □

Proposition 3.14. $[0, c]$ *is a BCC-ideal if and only if the relation* $\sim$ *defined by*

$$x \sim y \quad \textit{iff} \quad xy \leq c \quad \textit{and} \quad yx \leq c$$

is a congruence.

Proof. It is clear that the above relation is reflexive and symmetric. If $[0, c]$ is a BCC-ideal, $x \sim y$ and $y \sim z$, then $xy, yx, yz, zy \in [0, c]$. This, by (11) and (10), gives $xz \cdot c \leq xz \cdot yz \leq xy \leq c$. Thus $xz \cdot c \leq c$, which, by Corollary 3.8, implies $xz \leq c$. Similarly $zx \cdot c \leq zx \cdot yx \leq zy \leq c$ implies $zx \leq c$. Hence $\sim$ is also transitive.

Let now $x \sim y$ and $u \sim v$. Since $xu \cdot yu \leq xy \leq c$ and $yu \cdot xu \leq yx \leq c$, then $xu \sim yu$. On the other hand, from $(yu \cdot c) \cdot yv \leq (yu \cdot vu) \cdot yv = 0 \leq c$ and Proposition 3.7 follows $yu \cdot yv \leq c$. In the similar way from $(yv \cdot c) \cdot yu \leq (yv \cdot uv) \cdot yu = 0$ follows $yv \cdot yu \leq c$. Thus $yu \sim yv$, which by transitivity of $\sim$ gives $xu \sim yv$. Hence $\sim$ is a congruence.

Conversely, let $\sim$ be a congruence determined by the segment $[0, c]$. Since $w \sim 0$ iff $w \in [0, c]$, then $[0, c] = \{w \in G : w \sim 0\}$. Thus $xy \cdot z, y \in [0, c]$ imply $0 \sim xy \cdot z \sim x0 \cdot z \sim xz$, which proves that $[0, c]$is a BCC-ideal. □

Let $C_x = \{y \in G : y\rho x\}$, where ρ is an arbitrary congruence on a BCC-algebra G. The family $\{C_x : x \in G\}$ gives a partition of G which is denoted by G/ρ. For $x, y \in G$, we define $C_x * C_y = C_{xy}$. Since ρ has the substitution property, the operation $*$ is well-defined. But in general, $(G/\rho, *, C_0)$ is not a BCC-algebra (cf. [10]). It is a BCC-algebra only in the case when a congruence ρ is determined by a BCC-ideal (Theorem 3.5 in [7]). Thus the following statement is true.

Proposition 3.15. *If $\sim$ is a congruence defined in Proposition 3.14, the $G/\sim$ is a BCC-algebra.* □

In the same way as Proposition 3.7 we can prove

Proposition 3.16. *An initial segment $[0, c]$ of a BCC-algebra G is a BCK-ideal if and only if (13) holds for every $x \in G$.* □

Corollary 3.17. *A two-element segment $[0, c]$ is a BCK-ideal if and only if $xc \neq c$ for every $x \in G$.*

Proof. If $xc \neq c$ and $xc \leq c$, then $xc = 0$. Thus $[0, c]$ is a BCK-ideal. The converse statement follows from Corollary 3.9. □

Corollary 3.18. *If $xc \cdot c = xc$ for every $x \in G$, then $[0, c]$ is a BCK-ideal.* □

Corollary 3.19. *Initial segments of a positive implicative BCC-algebra are BCK-ideals.* □

In general, initial segments of a positive implicative BCC-algebra are not BCC-ideals. As an example we may consider a subalgebra $S = \{0, a, b, e\}$ from Example 3.1. This subalgebra is positive implicative (it is isomorphic to a BCC-algebra defined by Table 12 in [2]), but $[0, e]$ is not a BCC-ideal since $be \cdot a \in [0, e]$ and $ba \notin [0, e]$.

Proposition 3.20. *A finite BCC-chain of a positive implicative BCC-algebra is a BCK-chain with the trivial structure.*

Proof. (by induction) For BCC-chains containing at least two elements our statement is obviously true. If $[0, c]$ has $n + 1 \geq 3$ elements, then there exists $y \in [0, c]$ such that $[y, c]$ has only two elements. Thus $[0, y]$ has n elements and, by the assumption, has the trivial structure. Since by Corollary 3.19 it is also a BCK-ideal, then for every $x \in [0, y]$ from $cx \in [0, y]$ follows $c \in [0, y]$, which is impossible because $y < c$. Thus $cx \notin [0, y]$, i.e. $y < cx \leq c$. Hence $cx = c$ for every $x < c$. This completes the proof. □

Corollary 3.21. *A finite lineary ordered BCC-algebra is positive implicative if and only if it has the trivial structure.* □

4. Constructions

In this section we give several methods of construction of BCC-algebras with given BCC-chains. Some general methods of constructions of proper BCC-algebras one can find in [3]. First we observe that

Proposition 4.1. *Any finite BCK-chain may be extended to a proper BCC-chain.*

The proof is based on the observation that any two-elements BCK-chain may be extended to three-elements BCK-chain with the trivial structure. Any three-elements BCK-chain may be extended to a proper BCC-chain by the following construction, which is a special case of the construction used in Proposition 3 from [2].

Construction A. *Let $(G,\cdot,0)$ be a finite BCK-chain containing at least three elements and let c be its maximal element. Then $G\cup\{d\}$, where $d\notin G$, with the operation*

$$xy=\begin{cases} xy & x,y\in G\\ 0 & x\in G\cup\{d\},\ y=d\\ d & x=d,\ y=0\\ c & x=d,\ y\in G\end{cases}$$

is a proper BCC-chain.

Obtained BCC-chain is proper since $(d\cdot dy)y\neq 0$for any $0<y<c$.

As a simple consequence of Corollary 3 from [2] we obtain

Construction B. *Let $(G,\cdot,0)$ be a finite proper BCC-chain. Then $G\cup\{d\}$, where $d\notin G$, with the operation*

$$xy=\begin{cases} xy & x,y\in G\\ 0 & x\in G\cup\{d\},\ y=d\\ d & x=d,\ y\in G\end{cases}$$

is a proper BCC-chain.

From these two constructions follows

Corollary 4.2. *Any finite proper BCC-chain may be extended to at least two non-isomorphic proper BCC-chains of the same order.* □

Basing on the Construction B one can prove

Corollary 4.3. *Any initial segment $[0,c]$ is isomorphic to a maximal ideal of some BCC-algebra.*

Let $\{G_i\}_{i\in I}$ be a non-empty family of BCC-chains (or BCC-algebras) such that $G_i\cap G_j=\{0\}$ for any distinct $i,j\in I$. In $\{G_i\}_{i\in I}$ we define a new multiplication identifying it with a multiplication in any G_i, and putting $xy=x$ if belongs to distinct G_i. Direct computations shows that the union $\bigcup_{i\in I}G_i$ is a BCC-algebra. It is called *the disjoint union of* $\{G_i\}$ and is denoted by $\sum_{i\in I}G_i$. The BCC-algebra G_i is called *a component of* $\sum_{i\in I}G_i$. It is easily to shown that *any component G_i is a BCC-ideal of $\sum_{i\in I}G_i$.*

In general case where $\{G_i\}_{i\in I}$ is an arbitrary non-empty family of BCC-chains (BCC-algebras), we consider $\{G_i\times\{i\}\}_{i\in I}$ and identify all $(0_i\,,i)$, where 0_i is a constant of G_i. By identifying each $x_i\in G_i$ with $(x_i\,,i)$, the assumption of the definition mentioned above is satisfied. Consequently, we can define the disjoint union of an arbitrary family of BCC-chains.

Let $\prod G_i$ be the direct product of a non-empty family of BCC-algebras G_i. For any fixed $i\in I$, let x_i be an element of $\prod G_i$ such that $x_i(j)=0$ for any $i\neq j$ and $x_i(i)=x\in G_i$. Then $G_i^*=\{x_i : x\in G_i\}$ is a subalgebra of $\prod G_i$, which is naturally isomorphic to G_i. If $i\neq j$, then $x_ix_j=x_i$ and $G_i^*\cap G_j^*=\{0\}$. Hence $\bigcup_{i\in I}G_i^*=\sum_{i\in I}G_i^*$, and in the consequence, $\bigcup_{i\in I}G_i$ is a subalgebra of $\prod G_i$. Since $\bigcup G_i^*$ is isomorphic to $\sum G_i$, we obtain

Proposition 4.4. *$\sum G_i$ is a subdirect product of G_i.* □

By the identification G_i with G_i^* we get

Corollary 4.5. *$\sum G_i$ is the minimal subalgebra of $\prod G_i$ containing all G_i.* □

It is clear that if in the above construction al G_i are BCK-algebras then $\sum G_i$ and $\prod G_i$ are BCK-algebras. If at least one BCC-algebra G_i is proper then $\sum G_i$ and $\prod G_i$ are also proper BCC-algebras.

5. References

[1] W.A. Dudek: *On BCC-algebras*, Logique et Analyse **129-130** (1990), 103 – 111.

[2] W.A. Dudek: *On proper BCC-algebras*, Bull. Inst. Math. Academia Sinica **20** (1992), 137 – 150.

[3] W.A. Dudek: *On constructions of BCC-algebras*, Selected Papers on BCK- and BCI-algebras 1 (1992), 93 – 96.

[4] W.A. Dudek and Y. B. Jun: *Fuzzy BCC-ideals in BCC-algebras*, Math. Montisnigri **10** (1999), in print.

[5] W.A. Dudek and R. Rousseau: *Set-theoretic relations and BCH-algebras with trivial structure*, Zbornik Rad. Prirod.-Mat. Fak. Univ. Novi Sad, ser. Mat. **25.1** (1995), 75 – 82.

[6] W.A. Dudek and X.H. Zhang: *On atoms in BCC-algebras*, Disscusiones Math., Algebra and Stoch. Methods **15** (1995), 81 – 85.

[7] W.A. Dudek and X.H. Zhang: *On ideals and congruences in BCC-algebras*, Czech. Math. J. **48** (123) (1998), 21 – 29.

[8] Y. Imai and K. Iséki: *On axiom system of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19 – 22.

[9] K. Iséki and S. Tanaka: *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978), 1 – 26.

[10] Y. Komori: *The class of BCC-algebras is not a variety*, Math. Japon. **29** (1984), 391 – 394.

[11] A. Wroński: *BCK-algebras do not form a variety*, Math. Japon. **28** (1983), 211 – 213.

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