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AN ESTIMATION OF APPROXIMATION FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we give an estimation of the method for finding approximative solution of ordinary differential equations of the first order and these equations systems by use of operators whitch satisfy condition

$$|\big(U(\psi_1)\big)(x) - \big(U(\psi_2)\big)(x)| \le A_0\|\psi_1 - \psi_2\|_C$$

Let $P_{h,a} = \{(x_1, x_2, \dots, x_k) : x_1 \in [x_0, x_0 + h], x_j \in [x_{j0} - a, x_{j0} + a], x_0, x_{j0} \in \mathbb{R}, j = 2, 3, \dots, k, h, a > 0\}$. Denote by AN^1 the class of functions $f(x_1, x_2, \dots, x_k)$ which continuous on $P_{h,a}$ and which satisfy Lipschitz's condition of the first order with constant A for variable $x_s, s \ge 2$:

$$\begin{aligned}|f(x_1, x_2, \dots, x_{s-1}, x_s'', x_{s+1}, \dots, x_k) - f(x_1, x_2, \dots, x_{s-1}, x_s', x_{s+1}, \dots, x_k)|\\ \le A|x_s'' - x_s'|.\end{aligned} \tag{9}$$

Let $\big(U_n\big)_{n\in\mathbb{N}}$ be the sequence of operators $U_n : C^{\infty}_{[x_0,x_0+h]} \to C_{[x_0,x_0+h]}$, $\forall n \in \mathbb{N}$ which satisfy condition:

$$\begin{aligned}&\big(\exists A_0 > 0\big)\big(\forall n \in \mathbb{N}\big)\big(\forall \psi_1, \psi_2 \in C^{\infty}_{[x_0,x_0+h]}\big)\big(\forall x \in [x_0, x_0 + h]\big)\\ &\big(U_n(\psi_1)\big)(x) - \big(U_n(\psi_2)\big)(x)| \le A_0\|\psi_1 - \psi_2\|_C.\end{aligned} \tag{10}$$

The base of the method of approximative solution of differential equations by use of linear operators was introduced by V.K.DZIADIK (see [1], [2]). In this paper the linear condition of operator U_n is changed by the condition (10). In this respect estimations of approximations for the solutions of differential equations of the first order and their sistems are obtained in a simplier form.

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Theorem 1. *Let the differential equation be of the form*

$$y' = f(x,y), \qquad y(x_0) = y_0, \tag{11}$$

where $f(x,y) \in AN^1$*, and operator* U_n*, which satisfies the condition (10), and let the operational equation*

$$\tilde{y}_n(x) = \big(U_n(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}_n(t)]dt)\big)(x) \tag{12}$$

has the solution on $[x_0, x_0+h]$. *Then the function* $\tilde{y}_n(x)$*, which is the solution of that equation, approximates on the segment* $[x_0, x_0+h]$ *the solution* $y(x)$ *of the equation (11) and the inequality*

$$\|y(x) - \tilde{y}_n(x)\|_C \le (1+\alpha)\,\|y(x) - \big(U_n(y)\big)(x)\|_C$$

is valid, where

$$\alpha = \begin{cases} \dfrac{\eta}{1-\eta} & ,\eta \overset{def}{=} A_0 A h < 1 \\ \infty & ,\eta \ge 1. \end{cases} \qquad \square$$

Proof :

$$|y(x) - \tilde{y}_n(x)| = \big|y(x) - \big(U_n(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}_n(t)]dt)\big)(x)\big|$$

$$= \Big|\big[y(x) - \big(U_n(y_0 + \int_{x_0}^{\xi} f[t, y(t)]dt)\big)(x)\big]$$

$$+ \big[\big(U_n(y_0 + \int_{x_0}^{\xi} f[t, y(t)]dt)\big)(x) - \big(U_n(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}_n(t)]dt)\big)(x)\big]\Big|$$

$$\le \big|y(x) - \big(U_n(y)\big)(x)\big| + \big|\big(U_n(y_0 + \int_{x_0}^{\xi} f[t, y(t)]dt)\big)(x)$$

$$- \big(U_n(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}_n(t)]dt)\big)(x)\big|$$

$$\le \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0\Big\|\int_{x_0}^{x} f[t, y(t)]dt - \int_{x_0}^{x} f[t, \tilde{y}_n(t)]dt\Big\|$$

$$= \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0\Big\|\int_{x_0}^{x} \big[f[t, y(t)] - f[t, \tilde{y}_n(t)]\big]dt\Big\|$$

$$\le \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0 \int_{x_0}^{x_0+h} \big\|f[t, y(t)] - f[t, \tilde{y}_n(t)]\big\|dt$$

$$\le \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0 A \int_{x_0}^{x_0+h} \big\|y(t) - \tilde{y}_n(t)\big\|dt$$

$$= \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0 A h\big\|y(x) - \tilde{y}_n(x)\big\|.$$

The previous performations are valid for $\forall x \in [x_0, x_0 + h]$ and therefore we also get that:

$$\big\|y(x) - \tilde{y}_n(x)\big\| \leq \big\|y(x) - \big(U_n(y)\big)(x)\big\| + A_0 Ah\big\|y(x) - \tilde{y}_n(x)\big\|,$$

that is

$$\big\|y(x) - \tilde{y}_n(x)\big\| \leq \frac{1}{1 - A_0 Ah}\big\|y(x) - \big(U_n(y)\big)(x)\big\|.$$

This proves the theorem. ■

In the previous theorem there is still the question of the existence of the solution of the operational equation (12). The answer to this question is given in the following theorem.

Theorem 2. (About the existence of the solution) *If in the conditions of the Theorem 1 the number h satisfies inequality*

$$h < \frac{1}{A_0 A}$$

than the operational equation (12) is solvable for every $x \in [x_0, x_0 + h]$, where the solution $\tilde{y}_n(x)$ is unique and we get it by using successive approximations. We also get that for $\forall j \in \mathbb{N}$

$$\big\|\tilde{y}_n^{(j)}(x) - \tilde{y}_n(x)\big\|_C \leq \frac{q^j}{1 - q}\big\|\tilde{y}_n^{(1)} - \tilde{y}_n^{(0)}\big\|_C,$$

where $\tilde{y}_n^{(0)}(x)$ – is an arbitrary beginning approximation of $\tilde{y}_n(x)$ (with value in $[y_0 - a, y_0 + a]$) and for $\tilde{y}_n^{(j)}(x)$ – j the approximation of $\tilde{y}_n(x)$ obtained from:

$$\tilde{y}_n^{(j)}(x) = \Big(U_n\big(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}_n^{(j-1)}(t)]dt\big)\Big)(x) \quad , \quad j = 1, 2, 3, \ldots$$

and

$$q = A_0 Ah. \quad \square$$

Proof: By the operational equation (12) we introduce operator

$$\big(\mathcal{A}(y)\big) = \Big(U_n\big(y_0 + \int_{x_0}^{\xi} f[t, y(t)]dt\big)\Big)(x)$$

and the class of continuous functions on the segment $[x_0, x_0 + h]$. Show that A is the operator of contraction.

Namely, for arbitrary continuous functions $y(x), \tilde{y}(x)$ on $[x_0, x_0 + h]$ (with values in $[y_0 - a, y_0 + a]$) we get

$$\begin{aligned}\|(\mathcal{A}(y))(x) - (\mathcal{A}(\tilde{y}))(x)\| &= \Big\|\Big(U_n\big(y_0 + \int_{x_0}^{\xi} f[t, y(t)]dt\big)\Big)(x) \\ &\quad - \Big(U_n\big(y_0 + \int_{x_0}^{\xi} f[t, \tilde{y}(t)]dt\big)\Big)(x)\Big\| \\ &\le A_0\Big\|\int_{x_0}^{x} f[t, y(t)]dt - \int_{x_0}^{x} f[t, \tilde{y}(t)]dt\Big\| \\ &\le A_0 \int_{x_0}^{x_0+h} \|f[t, y(t)] - f[t, \tilde{y}(t)]\|dt \\ &\le A_0 A h \|y(x) - \tilde{y}(x)\|.\end{aligned}$$

For $A_0 A h < 1$ operator $\mathcal{A}$ is the operator of contraction. From the theorem of the fixed point it follows that the operational equation (12) is solvable and that we get the approximate solution by use of succesive approximation method. According to the same theorem we easily get that

$$\|\tilde{y}_n^{(j)}(x) - \tilde{y}_n(x)\| \le \frac{q^j}{1-q}\|\tilde{y}_n^{(1)} - \tilde{y}_n^{(0)}\| \quad , \quad \forall j \in \mathbb{N}.$$

This proves the theorem. ■

We consider now the approximation of the solution of system of ordinary differential equations given in the form

$$(13) \qquad y_i' = f_i(x, y_1, y_2, \dots, y_k), \quad y_i(x_0) = y_i, \quad i = 1, 2, \dots, k$$

and let $f_i \in AN^1$ for $i = 1, 2, \dots, k$. We transform system (13) in the integral form

$$(14) \qquad y_i(x) = y_i + \int_{x_0}^{x} f_i[t, y_1(t), y_2(t), \dots, y_k(t)]dt.$$

Define functions $\tilde{y}_i(x)$ for $i = 1, 2, \dots, k$

$$(15) \qquad \tilde{y}_i(x) = \Big(U_n\big(y_i + \int_{x_0}^{\xi} f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \dots, \tilde{y}_k(t)]dt\big)\Big)(x)$$

Now, we can formulate the following theorem.

Theorem 3. *Let system of ordinary differential equations is in the form*

$$y_i' = f_i(x, y_1, y_2, \dots, y_k), \quad y_i(x_0) = y_i, \quad i = 1, 2, \dots, k\ ,$$

where $f_i \in AN^1, i = 1, 2, \dots, k$, and operator U_n, which satisfies the condition (10), and let the system of equations (7) has the solution on the segment $[x_0, x_0+h]$. Then the functions $\tilde{y}_i(x)$, $\quad i = 1, 2, \dots, k$, which are the solutions *of that systems, approximates on* $[x_0, x_0+h]$ *solutions* $y_i(x)$, $\quad i = 1, 2, \ldots, k$, *of the system of equations (14) and the inequality*

$$\sum_{i=1}^{k} \|y_i(x) - \tilde{y}_i(x)\| \le (1+\alpha) \sum_{i=1}^{k} \|y_i(x) - \big(U_n(y)\big)(x)\|$$

is valid, where

$$\alpha = \begin{cases} \dfrac{\eta}{1-\eta} & , \ \eta \overset{def}{=} A_0 A h < 1 \\ \infty & , \ \eta \ge 1. \end{cases} \qquad \square$$

Proof:

$$\begin{aligned}
&|y_i(x) - \tilde{y}_i(x)| = \big|y_i(x) - \big(U_n(y_i)\big)(x) + \big(U_n(y_i)\big)(x) - \tilde{y}_i(x)\big| \\
&\le \big|y_i(x) - \big(U_n(y_i)\big)(x)\big| + \big|\big(U_n(y_i)\big)(x) - \tilde{y}_i(x)\big| \\
&= \big|y_i(x) - \big(U_n(y_i)\big)(x)\big| + \Big|\Big(U_n\big(y_i + \int_{x_0}^{\xi} f_i[t, y_1(t), y_2(t), \ldots, y_k(t)]dt\big)\Big)(x) \\
&\quad - \Big(U_n\big(y_i + \int_{x_0}^{\xi} f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_k(t)]dt\big)\Big)(x)\Big| \\
&\le \big\|y_i(x) - \big(U_n(y_i)\big)(x)\big\| \\
&+ A_0 \Big\| \int_{x_0}^{x} \big(f_i[t, y_1(t), y_2(t), \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_k(t)]\big) dt \Big\| \\
&= \big\|y_i(x) - \big(U_n(y_i)\big)(x)\big\| \\
&+ A_0 \Big\| \int_{x_0}^{x} \big(f_i[t, y_1(t), y_2(t), \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), y_2(t), \ldots, y_k(t)] \\
&+ f_i[t, \tilde{y}_1(t), y_2(t), \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), y_3(t) \ldots, y_k(t)] \\
&+ f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), y_3(t) \ldots, y_k(t)] - \ldots - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_{k-1}(t), y_k(t)] \\
&+ f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_{k-1}(t), y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_k(t)]\big) dt \Big\| \\
&\le \big\|y_i(x) - \big(U_n(y_i)\big)(x)\big\| \\
&+ A_0 \Big\| \int_{x_0}^{x_0+h} \Big(\big|f_i[t, y_1(t), y_2(t), \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), y_2(t), \ldots, y_k(t)]\big| \\
&+ \big|f_i[t, \tilde{y}_1(t), y_2(t), \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), y_3(t) \ldots, y_k(t)]\big| \\
&+ \big|f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), y_3(t) \ldots, y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \tilde{y}_3(t) \ldots, y_k(t)]\big| + \ldots \\
&+ \big|f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_{k-1}(t), y_k(t)] - f_i[t, \tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_k(t)]\big|\Big) dt \Big\|
\end{aligned}$$

$$\leq \left\|y_i(x) - \left(U_n(y_i)\right)(x)\right\| + A_0 A \int_{x_0}^{x_0+h} \sum_{j=1}^{k} \|y_j(t) - \tilde{y}_j(t)\| dt$$

$$= \left\|y_i(x) - \left(U_n(y_i)\right)(x)\right\| + A_0 A h \sum_{j=1}^{k} \|y_j(x) - \tilde{y}_j(x)\|$$

Since the deducing is valid for any $x \in [x_0, x_0 + h]$ if follows that

$$\|y_i(x) - \tilde{y}_i(x)\| \leq \left\|y_i(x) - \left(U_n(y_i)\right)(x)\right\| + A_0 A h \sum_{j=1}^{k} \|y_j(x) - \tilde{y}_j(x)\| dt$$

Summing these inequalities for $i = 1, \ldots, k$ we obtain:

$$\sum_{i=1}^{k} \|y_i(x) - \tilde{y}_i(x)\| \leq \sum_{i=1}^{k} \left\|y_i(x) - \left(U_n(y_i)\right)(x)\right\| + k A_0 A h \sum_{j=1}^{k} \|y_j(x) - \tilde{y}_j(x)\| dt$$

hence

$$\sum_{i=1}^{k} \|y_i(x) - \tilde{y}_i(x)\| \leq \frac{1}{1 - k A_0 A h} \sum_{i=1}^{k} \left\|y_i(x) - \left(U_n(y_i)\right)(x)\right\|. \qquad \blacksquare$$

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