

Qualitative study of a third order rational system of difference equations

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ABSTRACT. This paper is concerned with the dynamics of positive solutions for a system of rational difference equations of the following form

$$u_{n+1} = \frac{\alpha u_{n-1}^2}{\beta + \gamma v_{n-2}}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}^2}{\beta_1 + \gamma_1 u_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial values $u_{-i}, v_{-i} \in (0, \infty)$, $i = 0, 1, 2$. Moreover, the rate of convergence of a solution that converges to the zero equilibrium of the system is discussed. Finally, some numerical examples are given to demonstrate the effectiveness of the results obtained.

1. INTRODUCTION

Recently, the analysis of equilibrium solutions of various systems of non-linear difference equations has been one of the main topics in the theory of dynamical systems (see [2, 4, 7, 8, 9, 10, 11, 13, 14, 17, 18, 23, 24, 26, 27] and the references cited therein). Difference equations and systems of difference equations play an important role in the analysis of mathematical models of many areas such as ecology, population dynamics, statistical problems, number theory, geometry, genetics in biology, economics, psychology, sociology, physics, engineering, economics (see [1, 3, 5, 12, 15, 16, 21, 22]).

In [6], El-Owaidy et al. studied the third order non-linear rational difference equation

$$(1) \quad x_{n+1} = \frac{bx_{n-1}^2}{A + Bx_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters A, B, b and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary non-negative real numbers.

Motivated by the aforementioned study, our goal in this paper is to investigate the equilibrium points, the local asymptotic stability of these points,

2020 *Mathematics Subject Classification*. Primary: 39A20.

Key words and phrases. System of difference equations, equilibrium, positive solutions, invariant subsets.

Full paper. Received 5 June 2020, revised 25 January 2021, accepted 7 February 2021, available online 16 March 2021.

the global behavior of positive solutions, the existence of unbounded solutions and the existence of the prime two-periodic solutions of the following system

$$(2) \quad u_{n+1} = \frac{\alpha u_{n-1}^2}{\beta + \gamma v_{n-2}}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}^2}{\beta_1 + \gamma_1 u_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial values $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0$ are positive real numbers. This paper is natural extension of the paper [6].

Note that the system (2) can be reduced to the following system of difference equations

$$(3) \quad x_{n+1} = \frac{rx_{n-1}^2}{1 + y_{n-2}}, \quad y_{n+1} = \frac{sy_{n-1}^2}{1 + x_{n-2}}, \quad n = 0, 1, \dots,$$

by the change of variables $u_n = \left(\frac{\beta_1}{\gamma_1}\right) x_n$ and $v_n = \left(\frac{\beta}{\gamma}\right) y_n$ with $r = \frac{\alpha\beta_1}{\beta\gamma_1}$ and $s = \frac{\alpha_1\beta}{\beta_1\gamma}$.

We shall study the behavior of the solutions of the system (3) rather than system (2). Note that if

$$(4) \quad r = s \text{ and } x_{-i} = y_{-i} \text{ for } i = 0, 1, 2,$$

in system (3), then the system (3) reduces into Eq.(1) which has been studied by El-Owaidy in [6]. In this paper, we exclude the condition (4).

Using Mathematical induction on n , we see that if x_{-i}, y_{-i} are positive real numbers for $i \in \{0, 1, 2\}$ in system (3), then

$$\min\{x_n, y_n\} > 0, \quad n \geq -2.$$

To the best of our knowledge, the difference equations system (2) was not dealt with. Therefore, it is meaningful to study their deep results.

2. PRELIMINARIES

For the completeness in this paper, we find useful to remind some basic concepts of the difference equations theory as follows:

Let $f_1 : I_1^3 \times I_2^3 \rightarrow I_1$ and $f_2 : I_1^3 \times I_2^3 \rightarrow I_2$ are continuously differentiable functions where I_1, I_2 are intervals of real numbers. For any initial values $(x_{-i}, y_{-i}) \in I_1 \times I_2$ for $i \in \{0, 1, 2\}$, the six-dimensional discrete dynamical system

$$(5) \quad \left. \begin{aligned} x_{n+1} &= f_1(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \\ y_{n+1} &= f_2(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \end{aligned} \right\}$$

has a unique solution $\{(x_n, y_n)\}_{n=-2}^\infty$.

Definition 1. An equilibrium point of system (5) is a point (\bar{x}, \bar{y}) that satisfies

$$\bar{x} = f_1(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}),$$

$$\bar{y} = f_2(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}).$$

Together with the system (5), if we consider the associated vector map $F = (f_1, x_n, x_{n-1}, f_2, y_n, y_{n-1})$, then the point (\bar{x}, \bar{y}) is also called a fixed point of the vector map F .

Definition 2. Let (\bar{x}, \bar{y}) be an equilibrium point of the map F where f_1 and f_2 are continuously differentiable functions at (\bar{x}, \bar{y}) . The linearized system of (5) about the equilibrium point (\bar{x}, \bar{y}) is

$$(6) \quad X_{n+1} = F(X_n) = BX_n, \quad n = 0, 1, \dots,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and B is a Jacobian matrix of the system (5) about the equilibrium point (\bar{x}, \bar{y}) .

Definition 3. Let \bar{X} be a fixed point of the system of difference equations (6). If no eigenvalues of the Jacobian matrix B about \bar{X} has absolute value equal to one, then \bar{X} is called hyperbolic. If there exists an eigenvalue of the Jacobian matrix J_F about \bar{X} with absolute value equal to one, then \bar{X} is called non-hyperbolic.

Theorem 1. Let \bar{X} be a fixed point of the system of difference equations (6). If all eigenvalues of the Jacobian matrix J_F about \bar{X} lie inside the open unit disk $|\lambda| < 1$, then \bar{X} is locally asymptotically stable. If one of them has a modulus greater than one, then \bar{X} is unstable.

Theorem 2 (Routh-Hurwitz criterion). Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \dots$, is a system of difference equations and \bar{X} is a fixed point of F , the characteristic polynomial of this system about the equilibrium point \bar{X} is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if $\Delta_k > 0$ for $k = 1, 2, \dots, n$ where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

For other basic knowledge about difference equations and their systems, the reader is referred to [19, 20, 21].

3. STABILITY NATURE OF EQUILIBRIUM POINTS

In this section, we shall state the equilibrium points of system (3) and we will investigate the stability character of these points. One can easily see that the values of the equilibrium points depends on r and s .

Lemma 1. We have the following:

- $(\bar{x}_0, \bar{y}_0) = (0, 0)$ is always an equilibrium point of system (3) for all values of r and s ;
- if $rs > 1$, then system (3) has the unique positive equilibrium point $(\bar{x}_1, \bar{y}_1) = \left(\frac{s+1}{rs-1}, \frac{r+1}{rs-1} \right)$.

Before we investigate local asymptotic stability of the aforementioned equilibrium points, we shall build the corresponding linearized form of the system (3) and consider the following transformation

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \rightarrow (f, f_1, f_2, g, g_1, g_2),$$

where

$$f = \frac{rx_{n-1}^2}{1+y_{n-2}}, \quad f_1 = x_n, \quad f_2 = x_{n-1}, \quad g = \frac{sy_{n-1}^2}{1+x_{n-2}}, \quad g_1 = y_n, \quad g_2 = y_{n-1}.$$

The Jacobian matrix about the fixed point (\bar{x}, \bar{y}) under this transformation is as follows

$$J_F(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{2r\bar{x}}{1+\bar{y}} & 0 & 0 & 0 & -\frac{r\bar{x}^2}{(1+\bar{y})^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{s\bar{y}^2}{(1+\bar{x})^2} & 0 & \frac{2s\bar{y}}{1+\bar{x}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where $r, s \in (0, \infty)$.

Theorem 3. *The zero equilibrium point (\bar{x}_0, \bar{y}_0) of system (3) is locally asymptotically stable.*

Proof. The linearized system of (3) about the equilibrium point (\bar{x}_0, \bar{y}_0) is given by

$$X_{n+1} = J_F(\bar{x}, \bar{y})X_n,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$J_F(\bar{x}_0, \bar{y}_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation associated with $J_F(\bar{x}_0, \bar{y}_0)$ is

$$P(\lambda) = \lambda^6 = 0.$$

All roots of $P(\lambda)$ are equal to zero. Since all eigenvalues of the Jacobian matrix J_F about $(0, 0)$ lie inside the open unit disk $|\lambda| < 1$, the zero equilibrium point is locally asymptotically stable and this completes the proof. \square

Theorem 4. *If $rs > 1$, then the positive equilibrium point (\bar{x}_1, \bar{y}_1) is unstable, namely saddle point.*

Proof. The linearized system of (3) about the equilibrium point $(\bar{x}_1, \bar{y}_1) = (\frac{s+1}{rs-1}, \frac{r+1}{rs-1})$ is given by

$$X_{n+1} = J_F(\bar{x}_1, \bar{y}_1)X_n, \quad n = 0, 1, \dots,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$J_F(\bar{x}_1, \bar{y}_1) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & -\frac{1}{r} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{s} & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $J_F(\bar{x}_0, \bar{y}_0)(\bar{x}_1, \bar{y}_1)$ is as follows:

$$\lambda^6 - 4\lambda^4 + 4\lambda^2 - \frac{1}{rs} = 0.$$

From this characteristic equation, we obtain

$$\Delta_{6 \times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 4 & -\frac{1}{rs} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 4 & -\frac{1}{rs} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 4 & -\frac{1}{rs} \end{pmatrix}.$$

It is clear that not all of $\Delta_{6 \times 6} > 0$. Therefore, by Theorem 2, the unique positive equilibrium point (\bar{x}_1, \bar{y}_1) is unstable. But as the product of roots equals $-\frac{1}{rs}$, we conclude that there is at least one root with modulus less than one. Therefore, (\bar{x}_1, \bar{y}_1) is a Saddle point.

This completes the proof. \square

4. GLOBAL STABILITY OF THE ZERO EQUILIBRIUM POINT

Consider the subsets I_j , $j \in \{1, 2, 3, 4\}$ in \mathbb{R} , where

$$I_1 =]0, \bar{x}_1[\times]\bar{y}_1, \infty[;$$

$$I_2 =]\bar{x}_1, \infty[\times]0, \bar{y}_1[;$$

$$I_3 =]0, \bar{x}_1[\times]0, \bar{y}_1[;$$

$$I_4 =]\bar{x}_1, \infty[\times]\bar{y}_1, \infty[.$$

Theorem 5. *If $rs > 1$, then the subsets $I_1 =]0, \bar{x}_1[\times]\bar{y}_1, \infty[$ and $I_2 =]\bar{x}_1, \infty[\times]0, \bar{y}_1[$ are invariant subsets of \mathbb{R}^2 for system (3).*

Proof. We show that I_1 is an invariant subset of \mathbb{R}^2 for system (3). The proof is by induction on n .

Let $(x_{-i}, y_{-i}) \in I_1$ for $i \in \{0, 1, 2\}$. Then

$$x_1 = \frac{rx_{-1}^2}{1+y_{-2}} < \frac{r\bar{x}_1^2}{1+\bar{y}_1} = \bar{x}_1$$

and

$$y_1 = \frac{sy_{-1}^2}{1+x_{-2}} > \frac{s\bar{y}_1^2}{1+\bar{x}_1} = \bar{y}_1.$$

That is $(x_1, y_1) \in I_1$. Suppose that for a fixed $n \in \mathbb{N}$ we have $(x_{n-i}, y_{n-i}) \in I_1$, for $i \in \{0, 1, 2\}$. Then

$$x_{n+1} = \frac{rx_{n-1}^2}{1+y_{n-2}} < \frac{r\bar{x}_1^2}{1+\bar{y}_1} = \bar{x}_1$$

and

$$y_{n+1} = \frac{sy_{n-1}^2}{1+x_{n-2}} > \frac{s\bar{y}_1^2}{1+\bar{x}_1} = \bar{y}_1.$$

That is $(x_{n+1}, y_{n+1}) \in I_1$. Therefore, I_1 is an invariant subset of \mathbb{R}^2 for system (3).

Similarly, we can show that I_2 is an invariant subset of \mathbb{R}^2 for system (3). This completes the proof. \square

We have shown that the zero equilibrium point is locally asymptotically stable. In the following results, we shall show that the zero equilibrium point is a global attractor with basin as well as it is globally asymptotically stable.

Theorem 6. *Assume that $r < 1$ and $s < 1$ in system (3). Then the equilibrium point (\bar{x}_0, \bar{y}_0) is a global attractor with basin $]0, 1[^2$.*

Proof. Suppose that the initial points $(x_{-i}, y_{-i}) \in]0, 1[^2$, $i \in \{0, 1, 2\}$. It follows that

$$x_{n+1} = \frac{rx_{n-1}^2}{1+y_{n-2}} < rx_{n-1}^2 < rx_{n-1}.$$

Then the subsequences $\{x_{2n+i}\}_{n=-1}^{\infty}$, $i = 1, 2$ converges to zero and so the $x_n \rightarrow 0$ as $n \rightarrow \infty$.

Similarly $y_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\{(x_n, y_n)\}_{n=-2}^{\infty}$ converges to the zero equilibrium point $(0, 0)$ as $n \rightarrow \infty$. This completes the proof. \square

We put the following result without proof.

Theorem 7. *Consider system (3) and let x_{-i}, y_{-i} , $i \in \{0, 1, 2\}$ be nonnegative real numbers. Then for all $n \geq 1$, we have*

$$x_n \leq \begin{cases} r^{2\frac{n-1}{2}} x_{-1}^{2\frac{n+1}{2}}, & n = 1, 3, \dots \\ r^{2(\frac{n}{2})-1} x_0^{2\frac{n}{2}}, & n = 2, 4, \dots \end{cases}$$

and

$$y_n \leq \begin{cases} s^{2\frac{n-1}{2}} y_{-1}^{2\frac{n+1}{2}}, & n = 1, 3, \dots \\ s^{2(\frac{n}{2})-1} y_0^{2\frac{n}{2}}, & n = 2, 4, \dots \end{cases}$$

Theorem 8. *Assume that $r < 1$ and $s < 1$. Then the equilibrium point (\bar{x}_0, \bar{y}_0) of system (3) is globally asymptotically stable.*

Proof. The proof is a direct consequence of Theorem 3 and Theorem 7. \square

In the following result, we shall show that there exist unbounded solutions of system (3).

Theorem 9. *Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a solution of system (3). If $rs > 1$, then we have the following:*

- (1) *If $(x_{-i}, y_{-i}) \in I_3$, $i \in \{0, 2\}$ and $(x_{-1}, y_{-1}) \in I_4$, then the subsequence $\{(x_{2n}, y_{2n})\}_{n=0}^{\infty}$ converges to the point $(0, 0)$ and the subsequence $\{(x_{2n+1}, y_{2n+1})\}_{n=0}^{\infty}$ diverges to (∞, ∞) .*
- (2) *If $(x_{-i}, y_{-i}) \in I_4$, $i \in \{0, 2\}$ and $(x_{-1}, y_{-1}) \in I_3$, then the subsequence $\{(x_{2n+1}, y_{2n+1})\}_{n=0}^{\infty}$ converges to the point $(0, 0)$ and the subsequence $\{(x_{2n}, y_{2n})\}_{n=0}^{\infty}$ diverges to (∞, ∞) .*

Proof. We shall prove (1). The proof of (2) is similar and will be omitted.

The condition $(x_{-i}, y_{-i}) \in I_3$, $i \in \{0, 2\}$ and $(x_{-1}, y_{-1}) \in I_4$ is equivalent to

$$x_{-2}, x_0 < \bar{x}_1 < x_{-1} \text{ and } y_{-2}, y_0 < \bar{y}_1 < y_{-1}.$$

Then we have

$$x_1 = \frac{rx_{-1}}{1+y_{-2}} x_{-1} > \frac{r\bar{x}_1}{1+\bar{y}_1} x_{-1} = x_{-1} > \bar{x}_1$$

and

$$x_3 = \frac{rx_1}{1+y_0} x_1 > \frac{r\bar{x}_1}{1+\bar{y}_1} x_1 = x_1 > x_{-1} > \bar{x}_1.$$

Using induction on n , we can obtain

$$\bar{x}_1 < x_{-1} < \dots < x_{2n-1} < x_{2n+1} < \dots$$

Similarly,

$$\bar{y}_1 < y_{-1} < \dots < y_{2n-1} < y_{2n+1} < \dots$$

That is the subsequence $\{(x_{2n+1}, y_{2n+1})\}_{n=0}^{\infty}$ diverges to (∞, ∞) .

By the same argument, we can obtain

$$\bar{x}_1 > x_0 > \dots > x_{2n-2} > x_{2n} > \dots,$$

and

$$\bar{y}_1 > y_0 > \dots > y_{2n-2} > y_{2n} > \dots$$

That is the subsequence $\{(x_{2n}, y_{2n})\}_{n=0}^{\infty}$ converges to the point $(0, 0)$. This completes the proof. \square

5. PERIODICITY

In the following result, we show that under certain conditions, system (3) possesses period-2 solutions.

Theorem 10. *Assume that $rs > 1$ in system (3). Then we have the following:*

- (1) *If $x_{-2} = x_0 = 0$, $x_{-1} = \frac{1}{r}$ ($x_{-2} = x_0 = \frac{1}{r}$, $x_{-1} = 0$) and $y_{-2} = y_{-1} = y_0 = 0$, then system (3) possesses the period-2 solution*

$$\{\dots, (0, 0), (\frac{1}{r}, 0), (0, 0), (\frac{1}{r}, 0), \dots\}.$$

- (2) *If $x_{-2} = x_{-1} = x_0 = 0$ and $y_{-2} = y_0 = 0$, $y_{-1} = \frac{1}{s}$ ($y_{-2} = y_0 = \frac{1}{s}$, $y_{-1} = 0$), then system (3) possesses the period-2 solution*

$$\{\dots, (0, 0), (0, \frac{1}{s}), (0, 0), (0, \frac{1}{s}), \dots\}.$$

- (3) *If $x_{-2} = x_0 = 0$, $x_{-1} = \bar{x}_1$ ($x_{-2} = x_0 = \bar{x}_1$, $x_{-1} = 0$) and $y_{-2} = y_0 = \bar{y}_1$, $y_{-1} = 0$ ($y_{-2} = y_0 = 0$, $y_{-1} = \bar{y}_1$), then system (3) possesses the period-2 solution*

$$\{\dots, (0, \bar{y}_1), (\bar{x}_1, 0), (0, \bar{y}_1), (\bar{x}_1, 0), \dots\}.$$

- (4) *If $x_{-2} = x_0 = 0$, $x_{-1} = \frac{1}{r}$ ($x_{-2} = x_0 = \frac{1}{r}$, $x_{-1} = 0$) and $y_{-2} = y_0 = 0$, $y_{-1} = \frac{1}{s}$ ($y_{-2} = y_0 = \frac{1}{s}$, $y_{-1} = 0$), then system (3) possesses the period-2 solution*

$$\{\dots, (0, 0), (\frac{1}{r}, \frac{1}{s}), (0, 0), (\frac{1}{r}, \frac{1}{s}), \dots\}.$$

- (5) *If $x_{-2} = x_0 = 0$, $x_{-1} = \bar{x}_1$ ($x_{-2} = x_0 = \bar{x}_1$, $x_{-1} = 0$) and $y_{-2} = y_0 = \bar{y}_1$, $y_{-1} = \frac{1}{s}$ ($y_{-2} = y_0 = \frac{1}{s}$, $y_{-1} = \bar{y}_1$), then system (3) possesses the period-2 solution*

$$\{\dots, (0, \bar{y}_1), (\bar{x}_1, \frac{1}{s}), (0, \bar{y}_1), (\bar{x}_1, \frac{1}{s}), \dots\}.$$

- (6) *If $x_{-2} = x_0 = \frac{1}{r}$, $x_{-1} = \bar{x}_1$ ($x_{-2} = x_0 = \bar{x}_1$, $x_{-1} = \frac{1}{r}$) and $y_{-2} = y_0 = \bar{y}_1$, $y_{-1} = 0$ ($y_{-2} = y_0 = 0$, $y_{-1} = \bar{y}_1$), then system (3) possesses the period-2 solution*

$$\{\dots, (\frac{1}{r}, \bar{y}_1), (\bar{x}_1, 0), (\frac{1}{r}, \bar{y}_1), (\bar{x}_1, 0), \dots\}.$$

Proof. Suppose that

$$(7) \quad \{(p_1, q_1), (p_2, q_2), (p_1, q_1), (p_2, q_2), \dots\}$$

is a period-2 solution of system (3), where at least $p_1 \neq p_2$ or $q_1 \neq q_2$. Then we have

$$(8) \quad \begin{aligned} p_2 &= \frac{rp_2^2}{1+q_1}, & p_1 &= \frac{rp_1^2}{1+q_2}, \\ q_2 &= \frac{sq_2^2}{1+p_1}, & q_1 &= \frac{sq_1^2}{1+p_2}. \end{aligned}$$

If p_i, q_i are positive real numbers for $i = 1, 2$, then after some calculations using (7) and (2) we can obtain $p_1 = p_2 = \bar{x}_1$ and $q_1 = q_2 = \bar{y}_1$ which is a contradiction of our assumption. This means that, period-2 solution of the form (7) with positive real numbers $p_i, q_i, i = 1, 2$ does not exist.

(1) Suppose that $y_{-2} = y_{-1} = y_0 = 0$. Then from system (3), we have

$$x_{n+1} = rx_{n-1}^2, \quad y_n = 0, \quad \text{for all } n \geq 0.$$

Using (7) and (8), we get

$$q_1 = q_2 = 0, \quad p_1 = rp_1^2 \quad \text{and} \quad p_2 = rp_2^2.$$

Taking into account that $p_1 \neq p_2$, we get

$$p_1 = 0, \quad p_2 = \frac{1}{r} \quad \text{or} \quad p_2 = 0, \quad p_1 = \frac{1}{r}.$$

Therefore, if $x_{-2} = x_0 = p_1 = 0, x_{-1} = p_2 = \frac{1}{r}$ ($x_{-2} = x_0 = p_2 = \frac{1}{r}, x_{-1} = p_1 = 0$), we get the period-2 solution

$$\{\dots, (0, 0), (\frac{1}{r}, 0), (0, 0), (\frac{1}{r}, 0), \dots\}.$$

(2) The proof is similar to (1) and will be omitted.

(3) Suppose that $p_1 = q_2 = 0$. Using (7) and (8), one of the nonzero solutions for q_1 and p_2 is $q_1 = \bar{y}_1$ and $p_2 = \bar{x}_1$. Therefore, if $x_{-2} = x_0 = 0, x_{-1} = p_2 = \bar{x}_1$ ($x_{-2} = x_0 = p_2 = \bar{x}_1, x_{-1} = 0$) and $y_{-2} = y_0 = q_1 = \bar{y}_1, y_{-1} = 0$ ($y_{-2} = y_0 = 0, y_{-1} = q_1 = \bar{y}_1$), then system (3) possesses the period-2 solution $\{\dots, (0, \bar{y}_1), (\bar{x}_1, 0), (0, \bar{y}_1), (\bar{x}_1, 0), \dots\}$.

The proof of (4), (5) and (6) is similar and will be omitted.

This completes the proof. \square

6. RATE OF CONVERGENCE

In this section, we shall study the rate of convergence of a solution that converges to the equilibrium point $(0, 0)$ of the system (3). The following result gives the rate of convergence of the solution of a system of difference equations:

$$(9) \quad X_{n+1} = [A + B(n)]X_n,$$

where X_n is a six-dimensional vector, $A \in C^{6 \times 6}$ is a constant matrix and $B: \mathbb{Z}^+ \rightarrow C^{6 \times 6}$ is a matrix function satisfying

$$(10) \quad \|B(n)\| \rightarrow 0, \quad \text{when } n \rightarrow \infty,$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm. Also $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^2 given by

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Theorem 11 ([25]). Assume that condition (10) holds, if X_n is a solution of (9), then either $X_n = 0$ for all large n or

$$\theta = \lim_{n \rightarrow \infty} \sqrt[n]{\|X_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A .

Theorem 12 ([25]). Assume that condition (10) holds, if X_n is a solution of (9), then either $X_n = 0$ for all large n or

$$\theta = \lim_{n \rightarrow \infty} \frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A .

Assume that $\lim_{n \rightarrow \infty} x_n = \bar{x}$ and $\lim_{n \rightarrow \infty} y_n = \bar{y}$ and consider the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ x_{n-2} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ y_{n-2} - \bar{y} \end{pmatrix}.$$

We shall find a system satisfied by the error terms. The error terms satisfy the relations

$$(11) \quad \begin{aligned} x_{n+1} - \bar{x} &= \sum_{i=0}^2 A_{in}(x_{n-i} - \bar{x}) + \sum_{i=0}^2 B_{in}(y_{n-i} - \bar{y}), \\ y_{n+1} - \bar{y} &= \sum_{i=0}^2 C_{in}(x_{n-i} - \bar{x}) + \sum_{i=0}^2 D_{in}(y_{n-i} - \bar{y}). \end{aligned}$$

where

$$\begin{aligned} A_{0n} &= 0, & A_{1n} &= \frac{r(x_{n-1} + \bar{x})}{1 + \bar{y}}, & A_{2n} &= 0, \\ B_{0n} &= 0, & B_{1n} &= 0, & B_{2n} &= -\frac{rx_{n-1}^2}{(1 + y_{n-2})(1 + \bar{y})}, \\ C_{0n} &= 0, & C_{1n} &= 0, & C_{2n} &= -\frac{sy_{n-1}^2}{(1 + x_{n-2})(1 + \bar{x})}, \\ D_{0n} &= 0, & D_{1n} &= \frac{s(y_{n-1} + \bar{y})}{1 + \bar{x}}, & D_{2n} &= 0. \end{aligned}$$

We can write system (11) as

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^2 A_{in}e_{n-i}^1 + \sum_{i=0}^2 B_{in}e_{n-i}^2, \\ e_{n+1}^2 &= \sum_{i=0}^2 C_{in}e_{n-i}^1 + \sum_{i=0}^2 D_{in}e_{n-i}^2. \end{aligned}$$

Taking the limits of $A_{in}, B_{in}, C_{in}, D_{in}$, $i = 0, 1, 2$ we obtain

$$\lim_{n \rightarrow \infty} A_{in} = 0 \text{ for } i \in \{0, 2\} \text{ and } \lim_{n \rightarrow \infty} A_{1n} = \frac{2r\bar{x}}{1 + \bar{y}},$$

$$\begin{aligned} \lim_{n \rightarrow \infty} B_{in} &= 0 \text{ for } i \in \{0, 1\} \text{ and } \lim_{n \rightarrow \infty} B_{2n} = -\frac{r\bar{x}^2}{(1+\bar{y})^2}, \\ \lim_{n \rightarrow \infty} C_{in} &= 0 \text{ for } i \in \{0, 1\} \text{ and } \lim_{n \rightarrow \infty} C_{2n} = -\frac{s\bar{y}^2}{(1+\bar{x})^2}, \\ \lim_{n \rightarrow \infty} D_{in} &= 0 \text{ for } i \in \{0, 2\} \text{ and } \lim_{n \rightarrow \infty} D_{1n} = \frac{2s\bar{y}}{1+\bar{x}}. \end{aligned}$$

That is we can write

$$\begin{aligned} A_{1n} &= \frac{2r\bar{x}}{1+\bar{y}} + a_n, \quad B_{2n} = -\frac{r\bar{x}^2}{(1+\bar{y})^2} + b_n, \\ C_{2n} &= -\frac{s\bar{y}^2}{(1+\bar{x})^2} + c_n, \quad D_{1n} = \frac{2s\bar{y}}{1+\bar{x}} + d_n, \end{aligned}$$

where $a_n \rightarrow 0$, $b_n \rightarrow 0$, $c_n \rightarrow 0$ and $d_n \rightarrow 0$ as $n \rightarrow \infty$.

Then we can write

$$E_{n+1} = [K + B_n]E_n,$$

where

$$K = \begin{pmatrix} 0 & \frac{2r\bar{x}}{1+\bar{y}} & 0 & 0 & 0 & -\frac{r\bar{x}^2}{(1+\bar{y})^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{s\bar{y}^2}{(1+\bar{x})^2} & 0 & \frac{2s\bar{y}}{1+\bar{x}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$B_n = \begin{pmatrix} 0 & a_n & 0 & 0 & 0 & b_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_n & 0 & d_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $\|B_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Thus, the limitting system of error terms about an equilibrium point (\bar{x}, \bar{y}) can be written

$$\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n-1}^1 \\ e_{n+1}^2 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2r\bar{x}}{1+\bar{y}} & 0 & 0 & 0 & -\frac{r\bar{x}^2}{(1+\bar{y})^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{s\bar{y}^2}{(1+\bar{x})^2} & 0 & \frac{2s\bar{y}}{1+\bar{x}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}.$$

System (6) is similar to the linearized system of system (3) about an equilibrium point (\bar{x}, \bar{y}) .

System (6) about the equilibrium point $(0, 0)$ can be written as

$$\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n-1}^1 \\ e_{n+1}^2 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}.$$

The matrix K is the same as the Jacobian matrix evaluated at the zero equilibrium point. Using Theorems 11 and 12, we have the following result.

Corollary 1. *Let $\{(x_n, y_n)\}_{n=-2}^\infty$ be a solution of the system (3) such that*

$$\lim_{n \rightarrow \infty} x_n = 0 \text{ and } \lim_{n \rightarrow \infty} y_n = 0.$$

Then the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix} = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

of the solution $\{(x_n, y_n)\}_{n=-2}^\infty$ of the system (3) satisfies both of the following asymptotic relations:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\|E_n\|} = \sqrt{\sum_{i=0}^2 x_{n-i}^2 + \sum_{i=0}^2 y_{n-i}^2} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|} = \frac{\sqrt{\sum_{i=0}^2 x_{n+1-i}^2 + \sum_{i=0}^2 y_{n+1-i}^2}}{\sqrt{\sum_{i=0}^2 x_{n-i}^2 + \sum_{i=0}^2 y_{n-i}^2}} = 0,$$

where the eigenvalues of the Jacobian matrix evaluated at the equilibrium point $(0, 0)$ are all equal to zero.

7. NUMERICAL SIMULATIONS

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions of the system (3). All plots in this section are drawn with Mathematica.

Example 1. Figure 1. shows that if $r = 0.7$ and $s = 0.5$ ($r < 1$ and $s < 1$), then a solution $\{(x_n, y_n)\}_{n=-2}^\infty$ of system (3) with initial conditions $x_{-2} = 2.3$, $x_{-1} = 1.3$, $x_0 = 1.5$, $y_{-2} = 1.2$, $y_{-1} = 0.5$ and $y_0 = 2$ converges to the zero equilibrium point $(0, 0)$.

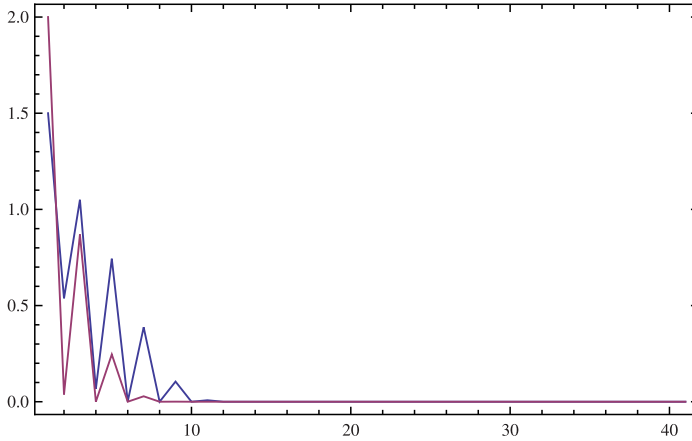


FIGURE 1. $x_{n+1} = \frac{0.7x_{n-1}^2}{1+y_{n-2}}, y_{n+1} = \frac{0.5y_{n-1}^2}{1+x_{n-2}}$

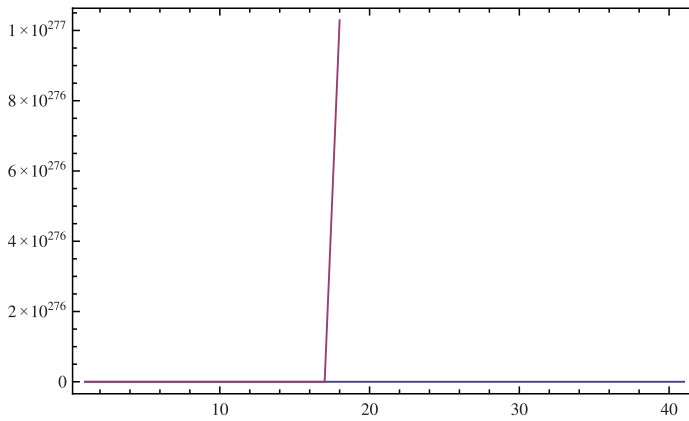


FIGURE 2. $x_{n+1} = \frac{2x_{n-1}^2}{1+y_{n-2}}, y_{n+1} = \frac{3y_{n-1}^2}{1+x_{n-2}}$

Example 2. Figure 2. shows that if $r = 2$ and $s = 3$ ($rs > 1$ and $(\bar{x}, \bar{y})=(0.8, 0.6)$), then a solution $\{(x_n, y_n)\}_{n=-2}^\infty$ of system (3) with initial conditions $x_{-2} = 0.3, x_{-1} = 0.3, x_0 = 0.5, y_{-2} = 1.2, y_{-1} = 1.5$ and $y_0 = 2$ ($(x_{-i}, y_{-i}) \in I_1 =]0, 0.8] \times]0.6, \infty[$, where $i \in \{0, 1, 2\}$) stay in the same subset I_1 .

Example 3. Figure 3. shows that if $r = 0.8$ and $s = 1.3$ ($rs = 1.04 > 1$), then a solution $\{(x_n, y_n)\}_{n=-2}^\infty$ of system (3) with initial conditions $x_{-2} = x_0 = \frac{1}{0.8} = 1.25, x_{-1} = 0, y_{-2} = y_0 = 0$ and $y_{-1} = 0$ is periodic with prime period-2. The solution is of the form

$$\{(1.25, 0), (0, 0), (1.25, 0), (0, 0), \dots\}.$$

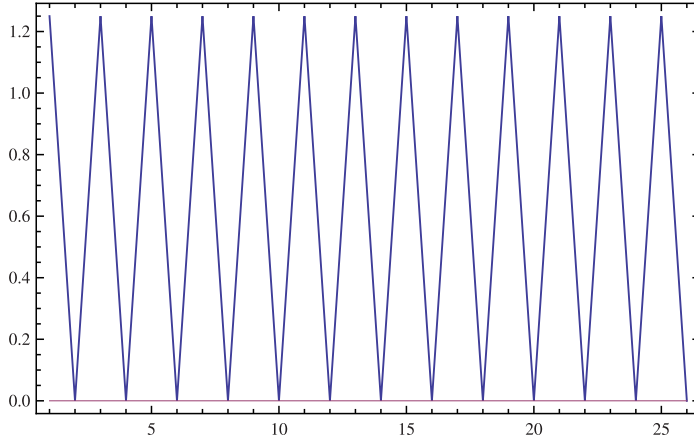


FIGURE 3. $x_{n+1} = \frac{0.8x_{n-1}^2}{1+y_{n-2}}, y_{n+1} = \frac{1.3y_{n-1}^2}{1+x_{n-2}}$

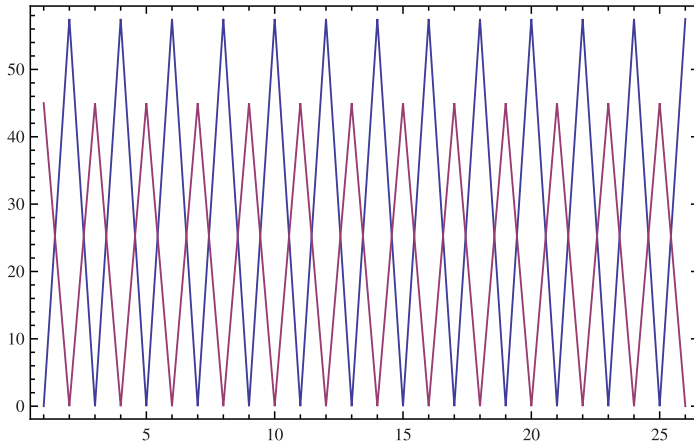


FIGURE 4. $x_{n+1} = \frac{0.8x_{n-1}^2}{1+y_{n-2}}, y_{n+1} = \frac{1.3y_{n-1}^2}{1+x_{n-2}}$

Example 4. Figure 4. shows that if $r = 0.8$ and $s = 1.3$ ($rs = 1.04 > 1$), then a solution $\{(x_n, y_n)\}_{n=-2}^\infty$ of system (3) with initial conditions $x_{-2} = x_0 = 0, x_{-1} = 57.5, y_{-2} = y_0 = 45$ and $y_{-1} = 0$ is periodic with prime period-2. In fact the solution is

$$\{(0, 45), (57.5, 0), (0, 45), (57.5, 0), \dots\}.$$

8. CONCLUSION AND SOME OPEN PROBLEMS

In this paper, some properties of a higher dimensional difference system of equations were studied. Namely, we investigated the equilibria of the system (3) in details. Also, we investigated the stability character of these points

using the linearization method . The main goal of dynamical systems theory is to approach the global behavior and the rate of convergence. Therefore, here we studied the global asymptotic stability and the rate of convergence of the zero equilibrium point of the system. Also, the existence of unbounded solutions and the periodicity of solutions of the system were studied. Even if it will be possible to obtain analytical results, it would be quite difficult to deal with them. So, numerical simulations were used to verify the correctness of analytical results.

Finally, we conjecture that our study can be extended to a system with higher order. We shall give some interesting open problems for difference systems of equations.

We conjecture that the obtained results are satisfied to the system

$$x_{n+1} = \frac{\alpha x_{n-2k-1}^2}{1 + y_{n-2l}}, \quad y_{n+1} = \frac{\beta y_{n-2k-1}^2}{1 + x_{n-2l}}, \quad n = 0, 1, \dots,$$

where the parameters α, β are non-negative real numbers, the initial conditions x_{-i}, y_{-i} are non-negative real numbers, $i = 0, 1, \dots, \max\{2l, 2k + 1\}$ and with natural numbers $l \geq 1, k \geq 1$.

Open Problem 1. Investigate the global behavior of the system of difference equations

$$x_{n+1} = \frac{\alpha x_{n-1}^p}{1 + y_{n-2}}, \quad y_{n+1} = \frac{\beta y_{n-1}^p}{1 + x_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters α, β are non-negative real numbers, the initial conditions x_{-i}, y_{-i} are non-negative real numbers, $i = 0, 1, 2$ and p is a positive integer.

Open Problem 2. Investigate the global behavior of the system of difference equations

$$x_{n+1} = \frac{\alpha_n x_{n-1}^2}{1 + y_{n-2}}, \quad y_{n+1} = \frac{\beta_n y_{n-1}^2}{1 + x_{n-2}}, \quad n = 0, 1, \dots,$$

where α_n, β_n are sequences (these sequences can be chosen as convergent, periodic or bounded), the initial conditions x_{-i}, y_{-i} are non-negative real numbers for $i = 0, 1, 2$.

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