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New properties of the time-scale fractional operators with application to dynamic equations

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Abstract. We introduce new properties of Riemann–Liouville fractional integral and derivative on time scales. As well as sufficient conditions for existence and uniqueness of solution to an initial value problem for a class differential equations on time scales.

1. Introduction

Fractional calculus was introduced and developed by Leibniz, Liouville, Riemann, Letnikov, and Grünwald [15]. This branch was applied in physics, natural and social sciences. In recent years, there has been much research activity concerning the Fractional calculus of various dynamic equations. The theory of time scales was introduced by Stefan Hilger in his PhD thesis [22] in 1988, in order to unify and generalize continuous and discrete analysis. For more detailed discussions on the time scale calculus we refer to the books (Bohner and Peterson, 2001, 2003), see [17, 18].

In 2016, Benkhettou et al. [21], introduced a concept of fractional derivative of Riemann-Liouville on time scales. Several authors have obtained important results about different subjects on time scales. See for instance M. Rchid et al [19], A. Abdeljawad et al [25], T. Gülsen et al [26].

The main purpose of this paper is to be deduced some new properties of the Riemann-Liouville fractional operator. As applications, we investigate fon existence and uniqueness of solutions some classes fractional dynamic equations.

The paper is organized as follows. In the next sections, we give some definitions and facts of time scale calculus. In Section 3, we establish some new properties of the Riemann-Liouville fractional operator. In Section 4, we investigate some IVPs for some classes fractional dynamic equations. In Section 5, we illustrate our results with examples.

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2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. The graininess function μ, for a time scale, is defined by $\mu(t) = \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$, the function f^σ denotes $f \circ \sigma$. The Δ-derivative of $f : \mathbb{T} \to \mathbb{R}$ at a right dense point t is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

If t is right scattered, then the $\Delta-$derivative is defined by

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.
In what follows, with $\mathcal{C}([a,b]_\mathbb{T}, \mathbb{R})$ we denote the Banach space of all continuous functions from $[a,b]_\mathbb{T}$ into $\mathbb{R}$, where $[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}$, with the norm

$$\|x\|_\infty := \sup\{|x(t)| : t \in [a,b]_\mathbb{T}\}.$$

Definition 1. [17] Let $[a,b]_\mathbb{T}$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a,b]_\mathbb{T} \to \mathbb{R}$ is called a delta antiderivative function $f : [a,b]_\mathbb{T} \to \mathbb{R}$ provided F is continuous on $[a,b]_\mathbb{T}$, delta differentiable on $[a,b)$, and $F^\Delta(t) = f(t)$, for all $t \in [a,b)$. Then, we define the Δ-integral of f from a to b by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

Proposition 1. [1] *Let $[a,b]_\mathbb{T}$ denote a closed bounded interval in $\mathbb{T}$ and f is an integrable function on $[a,b]_\mathbb{T}$. Then*

$$\int_a^b f(t)\,\Delta t = \int_{[a,b]_\mathbb{T}} f(t)\,\mathrm{d}\,t + \sum_{t \in \mathcal{R} \cap [a,b)} \mu(t) f(t),$$

where $\mathcal{R} = \{t \in \mathbb{T} : \sigma(t) > t\}$ is at most countable.

Definition 2 (Fractional integral on time scales)**.** [21] Suppose $\mathbb{T}$ is a time scale, $[a,b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a,b]$. Let $0 < \alpha < 1$. Then the (left) fractional integral of order α of h is defined by

$${}_a^\mathbb{T}I_t^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\Delta s,$$

where Γ is the gamma function.

Definition 3 (Riemann–Liouville fractional derivative on time scales [21])**.** Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The (left) Riemann–Liouville fractional derivative of order α of h is defined by

$$ {}_a^{\mathbb{T}}D_t^{\alpha} h(t) = \left(\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} h(s)\,\Delta s \right)^{\Delta} = \left({}_a^{\mathbb{T}}I_t^{1-\alpha} h(t) \right)^{\Delta}. $$

Proposition 2. *If $\mathbb{T} = \mathbb{R}$, the Riemann–Liouville fractional integral satisfies*

(1) $$ {}_a^{\mathbb{T}}I_t^{\beta} \circ {}_a^{\mathbb{T}} I_t^{\alpha} = {}_a^{\mathbb{T}} I_t^{\beta+\alpha}, \quad \textit{for } \alpha > 0 \textit{ and } \beta > 0. $$

3. Main Results

In this section, we present some of the new properties of the time-scale fractional operators.

The following counter example, to prove that the equality (1) is not always satisfied on time scales.

Example 1 (Counter example)**.** We take $\mathbb{T} = \mathbb{N}$, $a = 1$ and $h : \mathbb{T} \to \mathbb{R}$, $h(t) = 1$. Let $\alpha > 0$ and $b > 0$, then by Definition 2, we have

$$ \begin{aligned} {}_1^{\mathbb{N}}I_t^{\alpha} h(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1}\,\Delta s = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{s=t-1} (t-s)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \left[(t-1)^{\alpha-1} + (t-2)^{\alpha-1} + \cdots + 1^{\alpha-1} \right] \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{s=t-1} s^{\alpha-1} := \varphi(t). \end{aligned} $$

By, the last equality, deduce

$$ {}_1^{\mathbb{N}}I_t^{\alpha+\beta} h(t) = \frac{1}{\Gamma(\alpha+\beta)} \sum_{s=1}^{s=t-1} s^{\alpha+\beta-1}. $$

On the other hand, we have

$$ \begin{aligned} \left({}_1^{\mathbb{T}}I_t^{\beta} \circ {}_1^{\mathbb{T}} I_t^{\alpha} \right) h(t) &= {}_1^{\mathbb{N}}I_t^{\beta} \varphi(t) = \frac{1}{\Gamma(\beta)} \int_1^t (t-s)^{\beta-1} \varphi(s)\,\Delta s \\ &= \frac{1}{\Gamma(\beta)} \sum_{s=1}^{s=t-1} (t-s)^{\beta-1} \varphi(s). \end{aligned} $$

Thus,

$$ \left({}_1^{\mathbb{T}}I_t^{\beta} \circ {}_1^{\mathbb{T}} I_t^{\alpha} \right) h(2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \quad \text{and} \quad {}_1^{\mathbb{N}}I_t^{\alpha+\beta} h(2) = \frac{1}{\Gamma(\alpha+\beta)} $$

and

$$ \left({}_1^{\mathbb{T}}I_t^{\beta} \circ {}_1^{\mathbb{T}} I_t^{\alpha} \right) h(2) \neq {}_1^{\mathbb{T}}I_t^{\alpha+\beta} h(2). $$

Remark 1. By counter example 1, we conclude that ${}_a^{\mathbb{T}}I_t^{\beta} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = {}_a^{\mathbb{T}}I_t^{\beta+\alpha}$, for $\alpha > 0$, $\beta > 0$ are not always correct on the time scales, which are proposition 16 in [21]. If you suggest a counterexample to ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$ and ${}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{\alpha} = Id$, $\alpha > 0$, you leave to provide exact calculations in Example 1. Example 1 is a for ${}_a^{\mathbb{T}}I_t^{\beta} \circ {}_a^{\mathbb{T}}I_t^{\alpha} \neq {}_a^{\mathbb{T}}I_t^{\beta+\alpha}$ only.

Before starting to introduce the properties of the time scale fractional operators, we present a new generalization for the Beta function on time scales.

Definition 4 (Beta function on time scales)**.** We will define the function $B_{a,b}^{\mathbb{T}}(\alpha,\beta)$ as follows

$$B_{a,b}^{\mathbb{T}}(\alpha,\beta) = \int_a^b (s-a)^{\beta-1}(b-s)^{\alpha-1}\,\Delta s, \quad \text{for } \alpha > 0,\ \beta > 0.$$

Remark 2. If $\mathbb{T} = \mathbb{R}$, $a = 0$ and $b = 1$, then Definition 4 takes the form

$$B_{0,1}^{\mathbb{R}}(\alpha,\beta) = B(\alpha,\beta), \quad \text{for } \alpha > 0 \quad \text{and } \beta > 0,$$

where B is the classical beta function.

Proposition 3. *The function $B_{a,b}^{\mathbb{T}}(\alpha,\beta)$ satisfies the following inequality*

$$B_{a,b}^{\mathbb{T}}(\alpha,\beta) \geq B(\alpha,\beta)(b-a)^{\alpha+\beta-1}, \quad \textit{for } \alpha > 0 \quad \textit{and } \beta > 0.$$

Proof. By Proposition 1, we have

$$B_{a,b}^{\mathbb{T}}(\alpha,\beta) \geq \int_a^b (s-a)^{\beta-1}(b-s)^{\alpha-1}\,ds = B_{a,b}^{\mathbb{R}}(\alpha,\beta).$$

By setting $s = a + r(b-a)$, $r \in [0,1]$, we obtain that

$$\begin{aligned} B_{a,b}^{\mathbb{T}}(\alpha,\beta) &\geq (b-a)^{\alpha+\beta-1}\int_a^b r^{\beta-1}(1-r)^{\alpha-1}\,\mathrm{d}\,s \\ &= (b-a)^{\alpha+\beta-1} B_{0,1}^{\mathbb{R}}(\alpha,\beta) \\ &= B(\alpha,\beta)(b-a)^{\alpha+\beta-1}. \end{aligned}$$

The proof is complete. □

Example 2. If $\mathbb{T} = \mathbb{N}$, let $a, b \in \mathbb{N}$, by Definition 4, we have

$$B_{a,b}^{\mathbb{N}}(\alpha,\beta) = \sum_{s=a}^{s=b-1}(s-a)^{\beta-1}(b-s)^{\alpha-1}, \quad \text{for } \alpha > 1, \beta > 1.$$

Definition 5. Let $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}$ we define the time scales $\lambda\mathbb{T}$ and $\mathbb{T}+a$ by:

$$\lambda\mathbb{T} := \{\lambda t : t \in \mathbb{T}\}, \qquad \mathbb{T}+a := \{t+a : t \in \mathbb{T}\}.$$

Definition 6. Let $\lambda \in \mathbb{R}$, $a \in \mathbb{R}$ and let be the function $v : \mathbb{T} \to \lambda(\mathbb{T}+a)$ defined by

$$v_{\lambda,a}(s) = \lambda(s+a), \qquad \text{for all } s \in \mathbb{T}.$$

Remark 3. Let $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}$, such as $\lambda \neq 0$, then function $v_{\lambda,a}$ is bijective and the inverse function $v_{\lambda,a}^{-1}$ given by

$$v_{\lambda,a}^{-1}(s) = \frac{s}{\lambda} - a = v_{\frac{1}{\lambda},-a\lambda}(s), \quad \text{for all } s \in \lambda(\mathbb{T}+a).$$

Notation 1. We define the time scales $\mathbb{T}_{a,b}$ by:

$$\mathbb{T}_{a,b} := \frac{1}{b-a}(\mathbb{T}-a).$$

Proposition 4. *The Beta function on time scales satisfies the following useful property:*

$$B_{a,b}^{\mathbb{T}}(\alpha,\beta) = (b-a)^{\beta+\alpha-1}\beta_{0,1}^{\mathbb{T}_{a,b}}(\beta,\alpha), \quad \textit{for } \alpha>0 \quad \textit{and} \quad \beta>0,$$

where

$$\beta_{0,1}^{\mathbb{T}_{a,b}}(\beta,\alpha) = \int_0^1 r^{\beta-1}(1-r)^{\alpha-1}\Delta_{\mathbb{T}_{a,b}}r.$$

Proof. The function $u_{a,b} = v_{\frac{1}{b-a},-a}$ is bijective and the inverse function $u_{a,b}^{-1}$ given by

$$u_{a,b}^{-1}(r) = a + r(b-a), \quad \text{for all } r \in \mathbb{T}_{a,b}.$$

By the chain rule [17], we see that

$$\begin{aligned} B_{a,b}^{\mathbb{T}}(\alpha,\beta) &= (b-a)\int_{u_{a,b}(a)}^{u_{a,b}(b)}\left[(s-a)^{\beta-1}(b-s)^{\alpha-1}\right]\circ u_{a,b}^{-1}(r)\,\Delta_{\mathbb{T}_{a,b}}r \\ &= (b-a)^{\beta+\alpha-1}\int_0^1 r^{\beta-1}(1-r)^{\alpha-1}\Delta_{\mathbb{T}_{a,b}}r \\ &= (b-a)^{\beta+\alpha-1}\beta_{0,1}^{\mathbb{T}_{a,b}}(\beta,\alpha). \end{aligned}$$

The proof is complete. □

Proposition 5. *For any function h integrable on $[a,b]_{\mathbb{T}}$, the Riemann–Liouville Δ-fractional integral satisfies*

$$\left({}_a^{\mathbb{T}}I_t^{\beta} \circ {}_a^{\mathbb{T}}I_t^{\alpha}\right)(h(t)) = \frac{1}{\Gamma(\alpha+\beta)}\int_a^t h(u)(t-u)^{\beta+\alpha-1}\frac{\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)}{B(\alpha,\beta)}\Delta u,$$

for $\alpha>0$ and $\beta>0$, where $\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)$ is defined as in Proposition 4.

Proof. By Definition 4, we get

$$\begin{aligned} \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta}\right)(h(t)) &= {}_a^{\mathbb{T}}I_t^{\alpha}\left({}_a^{\mathbb{T}}I_t^{\beta}h(t)\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left({}_a^{\mathbb{T}}I_t^{\beta}h(s)\right)\Delta s \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha-1}\left(\int_a^s (s-u)^{\beta-1}h(u)\Delta u\right)\Delta s. \end{aligned}$$

From Fubini's theorem, we interchange the order of integration to obtain

$$\begin{aligned}\left({}_a^{\mathbb{T}}I_t^{\alpha}\circ_a^{\mathbb{T}} I_t^{\beta}\right)(h(t)) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(u)\left(\int_u^t (s-u)^{\beta-1}(t-s)^{\alpha-1}\Delta s\right)\Delta u\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(u)\, B_{u,t}^{\mathbb{T}}(\alpha,\beta)\,\Delta u.\end{aligned}$$

By Proposition 4, we obtain that

$$\begin{aligned}\left({}_a^{\mathbb{T}}I_t^{\alpha}\circ_a^{\mathbb{T}} I_t^{\beta}\right)(h(t)) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(u)(t-u)^{\beta+\alpha-1}\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)\,\Delta u\\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t h(u)(t-u)^{\beta+\alpha-1}\frac{\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)}{B(\alpha,\beta)}\Delta u.\end{aligned}$$

The proof is complete. □

Remark 4. If $\mathbb{T}=\mathbb{R}$, we have $\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)=B(\alpha,\beta)$, then Proposition 5 gives the classic result.

Corollary 1. *For any function h integrable and positive on $[a,b]_{\mathbb{T}}$, the Riemann–Liouville Δ-fractional integral satisfies*

$$\left({}_a^{\mathbb{T}}I_t^{\beta}\circ_a^{\mathbb{T}} I_t^{\alpha}\right)(h(t)) \geq_a^{\mathbb{T}} I_t^{\beta+\alpha}(h(t)),$$

for $\alpha>0$ and $\beta>0$.

Proof. Form Proposition 5 and Proposition 3, we have

$$\begin{aligned}\left({}_a^{\mathbb{T}}I_t^{\beta}\circ_a^{\mathbb{T}} I_t^{\alpha}\right)(h(t)) &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t h(u)(t-u)^{\beta+\alpha-1}\frac{\beta_{0,1}^{\mathbb{T}_{u,t}}(\beta,\alpha)}{B(\alpha,\beta)}\Delta u\\ &\geq \frac{1}{\Gamma(\alpha+\beta)}\int_a^t h(u)(t-u)^{\beta+\alpha-1}\Delta u\\ &= {}_a^{\mathbb{T}}I_t^{\beta+\alpha}(h(t)).\end{aligned}$$

The proof is complete. □

Proposition 6. *Let $h:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be a function Δ is differentiable, such that h^{Δ} integrable on $[a,b]_{\mathbb{T}}$. Then*

$${}_a^{\mathbb{T}}I_t^{\alpha}h^{\Delta}(t) = \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}\eta(t,s)\,h^{\sigma}(s)\,\Delta s - \frac{h(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1},$$

for all $\alpha>1$, with $\eta(t,s)=\int_0^1\left[1-h\frac{\mu(s)}{t-s}\right]^{\alpha-2}dh$.

Proof. By Definition 2, we have

$$\begin{aligned}{}_a^{\mathbb{T}}I_t^{\alpha}h^{\Delta}(t) &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h^{\Delta}(s)\,\Delta s\\ &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[h(s)-h(t)]^{\Delta}\Delta s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \left[(t-s)^{\alpha-1} \left[h(s) - h(t) \right] \right]_a^t \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left[(t-s)^{\alpha-1} \right]^{\Delta_s} \left[h^\sigma(s) - h(t) \right] \Delta s \\
&= -\frac{1}{\Gamma(\alpha)} \int_a^t \left[(t-s)^{\alpha-1} \right]^{\Delta_s} \left[h^\sigma(s) - h(t) \right] \Delta s \\
&\quad - \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha-1} \left[h(a) - h(t) \right].
\end{aligned}
$$

By Pötzsche's chain rule, we have

$$
\begin{aligned}
-\frac{1}{\Gamma(\alpha)} \left[(t-s)^{\alpha-1} \right]^{\Delta_s} &= \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 \left[(1-h)(t-s) + h(t-\sigma(s)) \right]^{\alpha-2} \mathrm{d}\, h \\
&= \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 \left[1 - h + h\frac{t-\sigma(s)}{t-s} \right]^{\alpha-2} \mathrm{d}\, h \\
&= \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 \left[1 - h\frac{\mu(s)}{t-s} \right]^{\alpha-2} \mathrm{d}\, h \\
&= \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \eta(t,s).
\end{aligned}
$$

Then,

$$
\begin{aligned}
{}_a^{\mathbb{T}}I_t^\alpha h^\Delta(t) &= \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} \eta(t,s) h^\sigma(s) \Delta s \\
&\quad + \frac{h(t)}{\Gamma(\alpha)} \int_a^t \left[(t-s)^{\alpha-1} \right]^{\Delta_s} \Delta s - \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha-1} \left[h(a) - h(t) \right] \\
&= \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} \eta(t,s) h^\sigma(s) \Delta s - \frac{h(t)}{\Gamma(\alpha)} (t-a)^{\alpha-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha-1} \left[h(a) - h(t) \right] \\
&= \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} \eta(t,s) h^\sigma(s) \Delta s - \frac{h(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}.
\end{aligned}
$$

The proof is complete. □

Corollary 2. *Let $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function Δ is differentiable, such that h^Δ integrable on $[a,b]_{\mathbb{T}}$, then*

$$
\left| {}_a^{\mathbb{T}}I_t^\alpha h^\Delta(t) \right| \leq \left| {}_a^{\mathbb{T}}I_t^{\alpha-1} h(t) \right| + \frac{|h(a)|}{\Gamma(\alpha)} (t-a)^{\alpha-1},
$$

for all $\alpha > 1$.

Proof. Form Proposition 6, we have

$$\left|{}_{a}^{\mathbb{T}}I_{t}^{\alpha}h^{\Delta}(t)\right| \leq \frac{1}{\Gamma(\alpha-1)}\int_{a}^{t}(t-s)^{\alpha-2}\left|h^{\sigma}(s)\right|\Delta s+\frac{|h(a)|}{\Gamma(\alpha)}(t-a)^{\alpha-1}$$
$$=\left|{}_{a}^{\mathbb{T}}I_{t}^{\alpha-1}h(t)\right|+\frac{|h(a)|}{\Gamma(\alpha)}(t-a)^{\alpha-1}.$$

The proof is complete. □

We consider the following initial value problem:

$$\text{(2)}\qquad \begin{cases} \lambda x^{\Delta}(t)+{}_{a}^{\mathbb{T}}D_{t}^{\alpha}(p(t)x(t))=f(t,x(t)), \\ \lambda x^{\sigma}(a)+\left({}_{a}^{\mathbb{T}}I_{t}^{1-\alpha}\right)^{\sigma}(p(a)x(a))=0, \end{cases} \quad \text{for} \begin{cases} t\in[\sigma(a),b]_{\mathbb{T}}, \\ 0<\alpha<1, \end{cases}$$

where ${}_{a}^{\mathbb{T}}D_{t}^{\alpha}$ is the Riemann–Liouville fractional derivative operator of order α defined on $\mathbb{T}$. The problem (2) will be studied under the following assumptions $f\in\mathcal{C}([\sigma(a),b]_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, $p\in\mathcal{C}([\sigma(a),b]_{\mathbb{T}},\mathbb{R})$, and $\lambda\in\mathbb{R}-\{0\}$. Our main results give necessary and sufficient conditions for the existence and uniqueness of solution to the problem (2).

Lemma 1. *Let $\alpha\in(0,1)$ and $f:[\sigma(a),b]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$. Function $x\in\mathcal{C}([\sigma(a),b]_{\mathbb{T}},\mathbb{R})$ is a solution of the problem (2) if and only if it is a solution of the following integral equation:*

$$x(t)=\frac{1}{\lambda}\int_{\sigma(a)}^{t}f(s,x(s))\Delta s-\frac{1}{\lambda\Gamma(1-\alpha)}\int_{a}^{t}(t-s)^{-\alpha}p(s)x(s)\Delta s.$$

Proof. By Definition 2, we have

$$\lambda x^{\Delta}(t)=f(t,x(t))-\frac{1}{\Gamma(1-\alpha)}\left(\int_{a}^{t}(t-s)^{-\alpha}p(s)x(s)\Delta s\right)^{\Delta}$$
$$=f(t,x(t))-\left({}_{a}^{\mathbb{T}}I_{t}^{1-\alpha}p(t)x(t)\right)^{\Delta}$$
$$=f(t,x(t))-{}_{a}^{\mathbb{T}}D_{t}^{\alpha}(p(t)x(t)).$$

The proof is complete. □

Our first main result is based on the Banach fixed point theorem [2].

Theorem 1. *Let $\alpha\in(0,1)$ and $f\in\mathcal{C}([\sigma(a),b]_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, there exists a positive and continuous function $r:[\sigma(a),b]_{\mathbb{T}}\to\mathbb{R}$, such that*

$$\text{(3)}\qquad \begin{gathered}|f(t,x)-f(t,y)|\leq r(t)|x-y|.\\ \textit{for all}\quad x,y\in\mathbb{R}\quad\textit{and}\quad t\in[\sigma(a),b]_{\mathbb{T}}.\end{gathered}$$

If

$$\text{(4)}\qquad \left({}_{a}^{\mathbb{T}}I_{t}^{1}|p(t)|+{}_{a}^{\mathbb{T}}I_{t}^{1-\alpha}|p(t)|\right)<|\lambda|,\quad \textit{for all}\quad t\in[\sigma(a),b]_{\mathbb{T}},$$

then the problem (2) has a unique solution on $[\sigma(a),b]_{\mathbb{T}}$.

Proof. We transform the problem (2) into a fixed point problem. Consider the operator $L : \mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R}) \to \mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$ defined by

$$(5) \qquad Lx(t) = \frac{1}{\lambda} \int_{\sigma(a)}^{t} \left(f(s, x(s)) - \frac{1}{\Gamma(1-\alpha)} (t-s)^{-\alpha} p(s) x(s) \right) \Delta s.$$

We need to prove that L has a fixed point, which is a unique solution of (2) on $[\sigma(a), b]_{\mathbb{T}}$. For that, we show that F is a contraction. Let $x, y \in \mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$. For $t \in [\sigma(a), b]_{\mathbb{T}}$, we have

$$\begin{aligned} |Lx(t) - Ly(t)| &\leq \frac{1}{|\lambda|} \int_{\sigma(a)}^{t} |f(s, x(s)) - f(s, y(s))| \Delta s \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} |p(s)| |x(s) - y(s)| \Delta s \\ &\leq \frac{1}{|\lambda|} \left({}_{a}^{\mathbb{T}}I_{t}^{1} |p(t)| + {}_{a}^{\mathbb{T}}I_{t}^{1-\alpha} |p(t)| \right) \|x - y\| . \end{aligned}$$

By (4), L is a contraction and thus, by the contraction mapping theorem, we deduce that L has a unique fixed point. This fixed point is the unique solution of (2). □

Now, we give our second main result guarantees the existence of at least one solution of the problem (2). This result is based on the Schauder's fixed point theorem [2].

Theorem 2. *Let $\alpha \in (0,1)$ and $f \in \mathcal{C}([\sigma(a), b]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, there are two functions $r \in \mathcal{C}([\sigma(a), b]_{\mathbb{T}}, [0, \infty))$ and $\varphi \in \mathcal{C}(\mathbb{R}, [0, \infty))$, such that*

$$|f(t, x)| \leq r(t) \varphi(x), \quad \text{for all} \quad y \in \mathbb{R} \quad \text{and} \quad t \in [\sigma(a), b]_{\mathbb{T}} .$$

Then the problem (2) has a solution on $[\sigma(a), b]_{\mathbb{T}}$.

Proof. We use Schauder's fixed point theorem to prove that L defined by (5) has a fixed point. The proof is given in several steps.

Step 1: L is continuous. Let x_n be a sequence such that $x_n \to x$ in $\mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$. Then, for each $t \in [\sigma(a), b]_{\mathbb{T}}$,

$$\begin{aligned} |Lx_n(t) - Lx(t)| &\leq \frac{1}{|\lambda|} \int_{\sigma(a)}^{t} |f(s, x_n(s)) - f(s, x(s))| \Delta s \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} |p(s)| |x_n(s) - x(s)| \Delta s \\ &\leq \frac{b - \sigma(a)}{|\lambda|} \sup_{t \in [\sigma(a), b]_{\mathbb{T}}} |f(s, x_n(s)) - f(s, x(s))| \\ &\quad + \sup_{t \in [a, b]_{\mathbb{T}}} \left({}_{a}^{\mathbb{T}}I_{t}^{1-\alpha} |p(t)| \right) \|x_n - x\| \end{aligned}$$

$$= \frac{b-\sigma(a)}{|\lambda|} \left\| f(., x_n(.)) - f(s, x(.)) \right\| + \left\| {}_a^{\mathbb{T}}I_t^{1-\alpha} |p| \right\| \left\| x_n - x \right\| .$$

Since f is a continuous function, we have $Lx_n \to Lx$ in $\mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$.

Step 2: The map L maps bounded sets into bounded sets in $\mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$. Indeed, it is enough to show that for any ε there exists a positive constant δ such that, for each $x \in B(0, \varepsilon)$, we have $Lx \in B(0, \delta)$. By hypothesis, for each $t \in [\sigma(a), b]_{\mathbb{T}}$, we get

$$\begin{aligned} |Lx(t)| &\leq \frac{1}{|\lambda|} \int_{\sigma(a)}^{t} r(s)\varphi(x(s))\Delta s \\ &\quad + \frac{1}{|\lambda|\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} |p(s)| |x(s)| \Delta s \\ &\leq \frac{1}{|\lambda|} \max_{x\in[-\varepsilon,\varepsilon]} \varphi(x) \int_{\sigma(a)}^{b} r(s)\Delta s + \frac{\varepsilon}{|\lambda|} \left\| {}_a^{\mathbb{T}}I_t^{1-\alpha} |p| \right\| = \delta. \end{aligned}$$

Step 3: The map L maps bounded sets into equicontinuous sets of $\mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$. Let $t_1, t_2 \in [\sigma(a), b]_{\mathbb{T}}$, $t_1 < t_2$ and $B(0, \varepsilon)$ be a bounded set of $\mathcal{C}([\sigma(a), b]_{\mathbb{T}}, \mathbb{R})$. For all $x \in B(0, \varepsilon)$, we get

$$\begin{aligned} |Lx(t_2) - Lx(t_1)| &\leq \frac{1}{|\lambda|} \int_{t_1}^{t_2} r(s)\varphi(x(s))\Delta s \\ &\quad + \frac{1}{|\lambda|\Gamma(1-\alpha)} \int_a^{t_2} (t_2-s)^{-\alpha} |p(s)| |x(s)| \Delta s \\ &\quad - \frac{1}{|\lambda|\Gamma(1-\alpha)} \int_a^{t_1} (t_1-s)^{-\alpha} |p(s)| |x(s)| \Delta s \\ &\leq \frac{1}{|\lambda|} \max_{x\in[-\varepsilon,\varepsilon]} \varphi(x) \int_{t_1}^{t_2} r(s)\Delta s \\ &\quad + \frac{\varepsilon \|p\|}{|\lambda|\Gamma(1-\alpha)} \int_{t_1}^{t_2} (t_2-s)^{-\alpha} \Delta s \\ &\quad + \frac{\varepsilon \|p\|}{|\lambda|\Gamma(1-\alpha)} \int_a^{t_1} \left| (t_2-s)^{-\alpha} - (t_1-s)^{-\alpha} \right| \Delta s. \end{aligned}$$

On the other hand, by Proposition 1, we get

$$\begin{aligned} \int_{t_1}^{t_2} (t_2-s)^{-\alpha} \Delta s &\leq \int_{t_1}^{t_2} (t_2-s)^{-\alpha} \,\mathrm{d}\, s + \sum_{s\in[t_1,t_2)\cap\mathcal{R}} \mu(s)(t_2-s)^{-\alpha} \\ &\leq \frac{1}{1-\alpha} (t_2-t_1)^{1-\alpha} + \sum_{s\in[t_1,t_2)\cap\mathcal{R}} (t_2-s)^{1-\alpha} \end{aligned}$$

$$\leq \frac{1}{1-\alpha}(t_2-t_1)^{1-\alpha}+(t_2-t_1)^{2-\alpha}.$$

Similarly, we get

$$\int_a^{t_1}\left|(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}\right|\Delta s$$
$$\leq \frac{1}{1-\alpha}\Big[(t_2-t_1)^{1-\alpha}+(t_2-a)^{1-\alpha}$$
$$-(t_1-a)^{1-\alpha}\Big]+\sum_{s\in[a,t_1)\cap\mathcal{R}}(t_1-s)^{1-\alpha}$$
$$-\sum_{s\in[a,t_1)\cap\mathcal{R}}(t_1-s)(t_2-s)^{-\alpha}.$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we conclude that $L:\mathcal{C}([\sigma(a),b]_{\mathbb{T}},\mathbb{R})\to\mathcal{C}([\sigma(a),b]_{\mathbb{T}},\mathbb{R})$ is completely continuous. As a consequence of Schauder's fixed point theorem, we conclude that L has a fixed point, which is solution of the problem (2). □

4. Conclusion

If we take $\lambda=0$ in the problem (2), we get

$$\begin{cases} {}_a^{\mathbb{T}}D_t^{\alpha}(p(t)x(t))=f(t,x(t)), \text{ for } t\in[\sigma(a),b]_{\mathbb{T}},\ \alpha\in(0,1), \\ \left({}_a^{\mathbb{T}}I_t^{1-\alpha}\right)^{\sigma}(p(a)x(a))=0. \end{cases} \tag{6}$$

We consider the following integral equation

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}p(s)f(s,x(s))\Delta s, \text{ for } t\in[\sigma(a),b]_{\mathbb{T}}. \tag{7}$$

Since ${}_a^{\mathbb{T}}D_t^{\alpha}\circ{}_a^{\mathbb{T}}I_t^{\alpha}=Id$, and ${}_a^{\mathbb{T}}I_t^{\alpha}\circ{}_a^{\mathbb{T}}D_t^{\alpha}=Id$, for $\alpha>0$ are not always correct defined on the time scales. Then, if x is a solution to the problem (6), it has no permanent relationship the solution of integral equation (7).

5. Example

Remark 5. Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ and a is right-scattered. By Definition 2, we have

$$\left({}_a^{\mathbb{T}}I_t^{\alpha}h\right)^{\sigma}(a)=\int_a^{\sigma(a)}\frac{(\sigma(a)-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\Delta s=\frac{(\mu(a))^{\alpha}h(a)}{\Gamma(\alpha)}.$$

Example 3. Let $\mathbb{T}=h\mathbb{Z}$, $a=h$ and $b=mh$, where $h>1$ and $m\in\mathbb{N}$. Then $\sigma(t)=t+h$ and $\mu(t)=0$.

We consider the following initial value problem:

$$\begin{cases} \lambda x^{\Delta}(t)+{}_h^{h\mathbb{Z}}D_t^{\frac{1}{2}}x(t)=t^2x(t), \quad \text{for } t\in[h,mh]_{h\mathbb{Z}}, \\ \lambda x(2h)=-\dfrac{\sqrt{h}}{\pi}h(h). \end{cases} \tag{8}$$

Here, $\lambda > 0$, $\alpha = \frac{1}{2}$, $p(t) = 1$ and $f(t, x) = t^2 x$, for $t \in [h, mh]_{h\mathbb{Z}}$ and $x \in \mathbb{R}$. By Remark 5, we find that the problem (8) is a private case of the problem (2) in $\mathbb{T} = h\mathbb{Z}$. Then (3) holds, and

$$ {}_h^{h\mathbb{Z}}I_t^1 |p(t)| + {}_h^{h\mathbb{Z}}I_t^{\frac{1}{2}} |p(t)| \leq \frac{3m}{2}(h-1). $$

If $3m(h-1) < 2\lambda$, then (4) holds, Thus, the conditions of Theorem 1 are satisfied, and we conclude that there is a function $x \in \mathcal{C}([h, mh]_{h\mathbb{Z}}, \mathbb{R})$ the unique solution of (8).

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