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Initial Coefficient estimates for a classes of m-fold symmetric bi-univalent functions involving Mittag-Leffler function

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Abstract. The main object of the present paper is to use Mittag-Leffler function to introduce and study two new classes

$$\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha) \quad \text{and} \quad \mathcal{R}^*_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \beta)$$

of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions defined in the open unit disk U. Also, we determine the estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new classes. Furthermore, we indicate certain special cases for our results.

1. Introduction

Let $\mathcal{A}$ stands for the class of functions f that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, are normalized by the conditions $f(0) = f'(0) - 1 = 0$, and have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Let S be the subclass of $\mathcal{A}$ consisting of functions of the form (1) which are also univalent in U. The Koebe one-quarter theorem (see [5]) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U

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satisfying (1). In fact, Srivastava et al. [18] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Murugusundaramoorthy et al. [11], Caglar et al. [4], Adegani et al. [1] and others (see, for example [8, 13, 14, 17, 25]).

For each function $f \in S$, the function $h(z) = (f(z^m))^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [7]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \tag{3}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (3). In fact, the functions in the class S are one-fold symmetric.

In [20] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= w - a_{m+1} w^{m+1} + \left[(m+1) a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &- \left[\tfrac{1}{2}(m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots, \end{aligned} \tag{4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}, \quad \left[-\log(1-z^m)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [2, 6, 12, 15, 16, 19, 22, 23, 24, 26, 27]).

The Mittag-Leffler function $E_\lambda(z)$, $(z \in \mathbb{C})$ (see [9, 10]) is defined by

$$E_\lambda(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + 1)}, \quad (\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > 0).$$

Recently, Srivastava and Tomovski [21] introduced the function $E_{\lambda,\eta}^{\delta,\tau}(z)$, $(z \in \mathbb{C})$ in the form:

$$E_{\lambda,\eta}^{\delta,\tau}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_{k\tau}\, z^k}{\Gamma(\lambda k + \eta) k!}, \quad (\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0).$$

where $\lambda, \eta, \delta \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(\tau) - 1\}$, $\mathrm{Re}(\tau) > 0$ and $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & \text{for } k = 0; \\ x(x+1)\cdots(x+n-1), & \text{for } k \in \mathbb{N}. \end{cases}$$

In 2016, Attiya [3] introduced and investigated a linear operator $\mathcal{H}_{\lambda,\eta}^{\delta,\tau} : \mathcal{A} \longrightarrow \mathcal{A}$ by using the Hadamard product (or convolution) and defined as follows

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = Q_{\lambda,\eta}^{\delta,\tau}(z) * f(z), \quad (z \in U),$$

where "$*$" indicate the Hadamard product (or convolution) of two series and

$$Q_{\lambda,\eta}^{\delta,\tau}(z) = \frac{\Gamma(\lambda+\eta)}{(\delta)_\tau} \left(E_{\lambda,\eta}^{\delta,\tau}(z) - \frac{1}{\Gamma(\eta)} \right),$$

$\lambda, \eta, \delta \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(\tau) - 1\}$, $\mathrm{Re}(\tau) > 0$.
By some easy computations, we conclude that

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\delta + k\tau)\Gamma(\lambda+\eta)}{\Gamma(\delta+\tau)\Gamma(\lambda k + \eta)\Gamma(k+1)} a_k z^k.$$

It is easily verified that if $f \in S_m$, then we have

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\delta + (mk+1)\tau)\Gamma(\lambda+\eta)}{\Gamma(\delta+\tau)\Gamma(\lambda(mk+1) + \eta)\Gamma(mk+2)} a_{mk+1} z^{mk+1}.$$

We require the following lemma to prove our main results.

Lemma 1. [5] *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which*

$$\mathrm{Re}\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2. Coefficient estimates for the function class $\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$

Definition 1. For $0 < \alpha \leq 1$, $0 \leq \gamma \leq 1$ and $m \in \mathbb{N}$, a function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$, if it satisfies the following conditions:

$$\left| \arg \left(\gamma z \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) \right)'' + (2\gamma + 1) \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) \right)' - 2\gamma \right) \right| < \frac{\alpha\pi}{2}, \tag{5}$$

and

$$\left|\arg\left(\gamma w\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)'' + (2\gamma+1)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)' - 2\gamma\right)\right| < \frac{\alpha\pi}{2}, \tag{6}$$

where the function $g = f^{-1}$ is given by (4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class

$$\mathcal{R}_{\Sigma_1}(\gamma,\lambda,\eta,\delta,\tau;\alpha) \equiv \mathcal{R}_{\Sigma}(\gamma,\lambda,\eta,\delta,\tau;\alpha).$$

Remark 1. If we put $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$:

(1) The class $\mathcal{R}_{\Sigma_m}(\gamma,\lambda,\eta,\delta,\tau;\alpha)$ reduces to the class $\mathcal{H}_{\Sigma,m}^{\alpha}$ which was given by Srivastava et al. [20];

(2) The class $\mathcal{R}_{\Sigma}(\gamma,\lambda,\eta,\delta,\tau;\alpha)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ which was given by Srivastava et al. [18].

Theorem 1. *Let* $f \in \mathcal{R}_{\Sigma_m}(\gamma,\lambda,\eta,\delta,\tau;\alpha)$ $(0 < \alpha \leq 1,\ 0 \leq \gamma \leq 1,\ m \in \mathbb{N})$ *be given by (3). Then*

$$|a_{m+1}| \leq \frac{2\alpha\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\sqrt{\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}}{\sqrt{\left|\alpha\Omega_1(\gamma,\lambda,\eta,\delta,\tau,m) + (1-\alpha)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)\right|}} \tag{7}$$

and

$$\begin{aligned} |a_{2m+1}| \leq{}& \frac{2\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}{\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)} \\ &+ \frac{2\alpha\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \Omega_1(\gamma,\lambda,\eta,\delta,\tau,m) = {}& (m+1)\Gamma(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\times \\ & \times\Gamma^2(m+2)\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\times \\ & \times\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right], \end{aligned} \tag{9}$$

$$\begin{aligned} \Omega_2(\gamma,\lambda,\eta,\delta,\tau,m) = {}& \Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\times \\ & \times\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]. \end{aligned} \tag{10}$$

Proof. It follows from conditions (5) and (6) that

$$\gamma z\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}f(z)\right)'' + (2\gamma+1)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}f(z)\right)' - 2\gamma = [p(z)]^{\alpha} \tag{11}$$

and

$$\gamma w\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)'' + (2\gamma+1)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)' - 2\gamma = [q(w)]^{\alpha}, \tag{12}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m}z^{2m} + p_{3m}z^{3m} + \cdots \tag{13}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{14}$$

Comparing the corresponding coefficients of (11) and (12) yields

$$\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)} a_{m+1} = \alpha p_m, \tag{15}$$

$$\begin{aligned} &\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))} a_{2m+1} \\ &= \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \end{aligned} \tag{16}$$

$$-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)} a_{m+1} = \alpha q_m \tag{17}$$

and

$$\begin{aligned} &\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))} \times \\ &\times \left((m+1)a_{m+1}^2 - a_{2m+1}\right) = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \end{aligned} \tag{18}$$

In view of (15) and (17), we find that

$$p_m = -q_m \tag{19}$$

and

$$\begin{aligned} &\frac{2\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]^2}{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)} a_{m+1}^2 \\ &= \alpha^2(p_m^2 + q_m^2). \end{aligned} \tag{20}$$

Also, from (16), (18) and (20), we obtain

$$\begin{aligned} &\frac{(m+1)\,\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))} a_{m+1}^2 \\ &= \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right) = \alpha(p_{2m}+q_{2m}) + \\ &\quad + \frac{(\alpha-1)\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]^2}{\alpha\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)} a_{m+1}^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &a_{m+1}^2 = \\ &\frac{\alpha^2\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(p_{2m}+q_{2m})}{\alpha\Omega_1(\gamma,\lambda,\eta,\delta,\tau,m)+(1-\alpha)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)}, \end{aligned} \tag{21}$$

where $\Omega_1(\gamma,\lambda,\eta,\delta,\tau,m)$ and $\Omega_2(\gamma,\lambda,\eta,\delta,\tau,m)$ are given by (9) and (10), respectively.

Now, taking the absolute value of (21) and applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we deduce that

$$|a_{m+1}| \leq \frac{2\alpha\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)\sqrt{\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}}{\sqrt{\left|\alpha\Omega_1(\gamma,\lambda,\eta,\delta,\tau,m)+(1-\alpha)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (7).
In order to find the bound on $|a_{2m+1}|$, by subtracting (18) from (16), we get

$$\begin{aligned}&\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\times\\ &\times\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right).\end{aligned} \tag{22}$$

It follows from (19), (20) and (22) that

$$\begin{aligned}a_{2m+1}=&\frac{\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\left(p_m^2+q_m^2\right)}{4\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)}\\ &+\frac{\alpha\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))\left(p_{2m}-q_{2m}\right)}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}.\end{aligned} \tag{23}$$

Taking the absolute value of (23) and applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$\begin{aligned}|a_{2m+1}|\leq&\frac{2\alpha^2(m+1)\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}{\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)}\\ &+\frac{2\alpha\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]},\end{aligned}$$

which completes the proof of Theorem 1. □

Remark 2. In Theorem 1, if we choose $\gamma=\lambda=0$ and $\eta=\delta=\tau=1$, then we obtain the results which was given by Srivastava et al. [20, Theorem 2].

For one-fold symmetric bi-univalent functions, Theorem 1 reduces to the following corollary:

Corollary 1. *Let* $f\in\mathcal{R}_\Sigma(\gamma,\lambda,\eta,\delta,\tau;\alpha)$ $(0<\alpha\leq 1,\ 0\leq\gamma\leq 1)$ *be given by (1). Then*

$$|a_2|\leq\frac{4\alpha\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)\sqrt{6\Gamma(3\lambda+\eta)}}{\sqrt{\left|\alpha\Omega_3(\gamma,\lambda,\eta,\delta,\tau)+24(1-\alpha)\Gamma(3\lambda+\eta)\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)\left(3\gamma+1\right)^2\right|}}$$

and

$$|a_3|\leq\frac{4\alpha^2\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}{\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)\left(3\gamma+1\right)^2}+\frac{4\alpha\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)(4\gamma+1)},$$

where

$$\Omega_3(\gamma,\lambda,\eta,\delta,\tau)=24\Gamma(\delta+\tau)\Gamma^2(2\lambda+\eta)\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)(4\gamma+1).$$

Remark 3. In Corollary 1, if we choose $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$, then we obtain the results which was given by Srivastava et al. [18, Theorem 1].

3. Coefficient estimates for the functions class $\mathcal{R}^*_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \beta)$

Definition 2. For $0 \leq \beta < 1$, $0 \leq \gamma \leq 1$ and $m \in \mathbb{N}$, a function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{R}^*_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \beta)$ if it satisfies the following conditions:

$$\operatorname{Re}\left\{\gamma z \left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z)\right)'' + (2\gamma+1)\left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z)\right)' - 2\gamma\right\} > \beta \tag{24}$$

and

$$\operatorname{Re}\left\{\gamma w \left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} g(w)\right)'' + (2\gamma+1)\left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} g(w)\right)' - 2\gamma\right\} > \beta, \tag{25}$$

where the function $g = f^{-1}$ is given by (4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class

$$\mathcal{R}^*_{\Sigma_1}(\gamma, \lambda, \eta, \delta, \tau; \beta) \equiv \mathcal{R}^*_{\Sigma}(\gamma, \lambda, \eta, \delta, \tau; \beta).$$

Remark 4. If we put $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$:

(1) The class $\mathcal{R}^*_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \beta)$ reduces to the class $\mathcal{H}_{\Sigma,m}(\beta)$ which was given by Srivastava et al. [20];
(2) The class $\mathcal{R}^*_{\Sigma}(\gamma, \lambda, \eta, \delta, \tau; \beta)$ reduces to the class $\mathcal{H}_{\Sigma}(\beta)$ which was given by Srivastava et al. [18].

Theorem 2. *Let* $f \in \mathcal{R}^*_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \beta)$ $(0 \leq \beta < 1,\ 0 \leq \gamma \leq 1,\ m \in \mathbb{N})$ *be given by (3). Then*

$$\begin{aligned}&|a_{m+1}| \\ &\leq 2\sqrt{\frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)}{(m+1)\,\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}}\end{aligned} \tag{26}$$

and

$$\begin{aligned}|a_{2m+1}| \leq\ & \frac{2\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\left(1-\beta\right)^2(m+1)}{\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ &+ \frac{2\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]},\end{aligned} \tag{27}$$

where $\Omega_2(\gamma, \lambda, \eta, \delta, \tau, m)$ *is given by (10).*

Proof. It follows from conditions (24) and (25) that there exist $p, q \in \mathcal{P}$ such that

$$\gamma z \left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z)\right)'' + (2\gamma+1)\left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z)\right)' - 2\gamma = \beta + (1-\beta)p(z) \tag{28}$$

and

$$\gamma w\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)''+(2\gamma+1)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)'-2\gamma=\beta+(1-\beta)q(w), \tag{29}$$

where $p(z)$ and $q(w)$ have the forms (13) and (14), respectively. Equating coefficients (28) and (29) yields

$$\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)}a_{m+1} = (1-\beta)p_m, \tag{30}$$

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}a_{2m+1} = (1-\beta)p_{2m}, \tag{31}$$

$$-\frac{\Gamma(\delta+(m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(m+1)+\eta)\Gamma(m+2)}a_{m+1} = (1-\beta)q_m \tag{32}$$

and

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\times \left((m+1)a_{m+1}^2-a_{2m+1}\right)=(1-\beta)q_{2m}. \tag{33}$$

From (30) and (32), we get

$$p_m=-q_m \tag{34}$$

and

$$\frac{2\Gamma^2(\delta+(m+1)\tau)\Gamma^2(\lambda+\eta)\left[\gamma\left((m+1)^2+1\right)+m(\gamma+1)+1\right]^2}{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)}a_{m+1}^2 = (1-\beta)^2\left(p_m^2+q_m^2\right). \tag{35}$$

Adding (31) and (33), we obtain

$$\frac{(m+1)\,\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\times a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}).$$

Therefore, we have

$$a_{m+1}^2=\frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)(p_{2m}+q_{2m})}{(m+1)\,\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}.$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq 2\sqrt{\frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)}{(m+1)\,\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (26).
In order to find the bound on $|a_{2m+1}|$, by subtracting (33) from (31), we get

$$\frac{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))}\times$$
$$\times\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)=(1-\beta)\left(p_{2m}-q_{2m}\right),$$

or equivalently

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2 + \frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)\left(p_{2m}-q_{2m}\right)}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]}.$$

Upon substituting the value of a_{m+1}^2 from (35), it follows that

$$a_{2m+1}=\frac{\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\left(1-\beta\right)^2(m+1)(p_m^2+q_m^2)}{4\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)} + \frac{\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)\left(p_{2m}-q_{2m}\right)}{2\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]},$$

where $\Omega_2(\gamma,\lambda,\eta,\delta,\tau,m)$ is given by (10).
Applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{2\Gamma^2(\delta+\tau)\Gamma^2(\lambda(m+1)+\eta)\Gamma^2(m+2)\left(1-\beta\right)^2(m+1)}{\Omega_2^2(\gamma,\lambda,\eta,\delta,\tau,m)} + \frac{2\Gamma(\delta+\tau)\Gamma(\lambda(2m+1)+\eta)\Gamma(2(m+1))(1-\beta)}{\Gamma(\delta+(2m+1)\tau)\Gamma(\lambda+\eta)\left[\gamma\left(4m(m+1)+2\right)+2m(\gamma+1)+1\right]},$$

which completes the proof of Theorem 2. □

Remark 5. In Theorem 2, if we choose $\gamma=\lambda=0$ and $\eta=\delta=\tau=1$, then we obtain the results which was given by Srivastava et al. [20, Theorem 3].

For one-fold symmetric bi-univalent functions, Theorem 2 reduces to the following corollary:

Corollary 2. *Let* $f\in\mathcal{R}_\Sigma^*(\gamma,\lambda,\eta,\delta,\tau;\beta)$ $(0\leq\beta<1,\ 0\leq\gamma\leq1)$ *be given by (1). Then*

$$|a_2|\leq2\sqrt{\frac{\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)(1-\beta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)(4\gamma+1)}}$$

and

$$|a_3|\leq\frac{8\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)\left(1-\beta\right)^2}{\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)\left(3\gamma+1\right)^2}+\frac{4\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)(1-\beta)}{\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)(4\gamma+1)}.$$

Remark 6. In Corollary 2, if we choose $\gamma=\lambda=0$ and $\eta=\delta=\tau=1$, then we obtain the results which was given by Srivastava et al. [18, Theorem 2].

4. Conclusion

In the present work, we have introduced two new classes of analytic and m-fold symmetric bi-univalent functions in the open unit disk U associated with Mittag-Leffler function. We have then derived the initial coefficient estimations for functions belonging to these new classes. Further by specializing the parameters, several consequences of these new classes are mentioned.

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