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On relationships between q-products identities, R_α, R_β and R_m functions related to Jacobi's triple-product identity

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Abstract. The authors establish a set of two new relationships involving q-product identities, R_α, R_β, and R_m $(m = 1, 2, 3, \ldots)$ functions; and answer a open question of Srivastava *et al.* [18]. The present work is motivated and based upon recent findings of Chaudhary *et al.* [8].

1. Introduction

The q-Pochhammer symbol or q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) (see, for example, [3], [5, Chapter 3, Section 3.2.1], [19, Chapter 6] and [20, pp. 346 *et seq.*]), as

$$(a;q)_n := \begin{cases} 1, & (n=0), \\ \prod_{k=0}^{n-1}(1-aq^k), & (n \in \mathbb{N}), \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) = \prod_{k=1}^{\infty}(1-aq^{k-1}), \qquad (a, q \in \mathbb{C};\ |q| < 1). \tag{2}$$

When $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula,the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n\ (a_2; q)_n\ \cdots\ (a_m; q)_n \tag{3}$$

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and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty \, (a_2; q)_\infty \, \cdots \, (a_m; q)_\infty. \tag{4}$$

Ramanujan (see [12] and [13]) defined the general theta function $\mathfrak{f}(a, b)$ as follows (see, for details, [6, p. 31, Eq. (18.1)] and [15]):

$$\begin{aligned} \mathfrak{f}(a, b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} \, (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} \, b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b, a), \qquad (|ab| < 1), \end{aligned} \tag{5}$$

which is further expressed as

$$\mathfrak{f}(a, b) = a^{\frac{n(n+1)}{2}} \, b^{\frac{n(n-1)}{2}} \, \mathfrak{f}\big(a(ab)^n, b(ab)^{-n}\big) = \mathfrak{f}(b, a) \qquad (n \in \mathbb{Z}). \tag{6}$$

Equation (6) holds true as stated only if n is any integer. Moreover, in case n is not an integer, this result (6) is only approximately true (see, for details, [12, Vol. **2**, Chapter XVI, p. 193, Entry 18 (iv)]).

In fact, Ramanujan (see [12] and [13]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [6, p. 35, Entry 19]):

$$\mathfrak{f}(a, b) = (-a; ab)_\infty \, (-b; ab)_\infty \, (ab; ab)_\infty, \tag{7}$$

or, equivalently, by (see [10])

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2} \, z^n &= \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + zq^{2n-1}\right) \left(1 + \frac{1}{z} \, q^{2n-1}\right) \\ &= \left(q^2; q^2\right)_\infty \, \left(-zq; q^2\right)_\infty \, \left(-\frac{q}{z}; q^2\right)_\infty, \qquad (|q| < 1; \; z \neq 0). \end{aligned}$$

The q-series identity (7) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worthy of note here (see, for details, [6, pp. 36–37, Entry 22]):

$$\begin{aligned} \varphi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \\ &= \left\{(-q; q^2)_\infty\right\}^2 (q^2; q^2)_\infty = \frac{(-q; q^2)_\infty \, (q^2; q^2)_\infty}{(q; q^2)_\infty \, (-q^2; q^2)_\infty}; \end{aligned} \tag{8}$$

$$\psi(q) := \mathfrak{f}(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}; \tag{9}$$

and

$$\begin{aligned} f(-q) := \mathfrak{f}(-q,-q^2) &= \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n \, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \end{aligned} \tag{10}$$

Equation (10) is known as Euler's *Pentagonal Number Theorem.* Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{11}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [3] and [5]).

Theorem 1 (Euler's Pentagonal Number Theorem)**.** *The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$\begin{aligned} R(q) := q^{\frac{1}{5}} \frac{H(q)}{G(q)} &= q^{\frac{1}{5}} \frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty \, (q^4;q^5)_\infty}{(q^2;q^5)_\infty \, (q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+} \, \frac{q}{1+} \, \frac{q^2}{1+} \, \frac{q^3}{1+}, \quad (|q|<1). \end{aligned} \tag{12}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger-Ramanujan identities, are defined as follows:

$$\begin{aligned} G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)} = \frac{1}{(q;q^5)_\infty \, (q4;q^5)_\infty} = \\ &= \frac{(q^2;q^5)_\infty \, (q^3;q^5)_\infty \, (q^5;q^5)_\infty}{(q;q)_\infty} \end{aligned} \tag{13}$$

and

$$\begin{aligned} H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} &= \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} \\ = \frac{1}{(q^2;q^5)_\infty \, (q^3;q^5)_\infty} &= \frac{(q;q^5)_\infty \, (q^4;q^5)_\infty \, (q^5;q^5)_\infty}{(q;q)_\infty}, \end{aligned} \tag{14}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (5) and (10), respectively.

For a detailed historical account of (and for various related developments stemming from) the Rogers-Ramanujan continued fraction (12) as well as the Rogers-Ramanujan identities (13) and (14), the interested reader may refer to the monumental work [6, p. 77 *et seq.*] (see also [15] and [19]).

Theorem 2. *Suppose that* $|q|<1$. *Then*

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-\cdots}, \tag{15}$$

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+}\cdots, \tag{16}$$

and

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1 + \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+}\cdots. \tag{17}$$

In 2015, Andrews *et al.* [4] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities by introducing the general family $R(s,t,l,u,v,w)$ and also suggested its three special cases in term of multivaraite R-functions. Thereafter, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [15] to [17]). An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn *et al.* [9] is also worth mentioning in this connection. In this paper, our main objective is to establish a set of two new identities which depict the inter-relationships in terms of R_α, R_β and R_m functions along with q-product identities.

Recently, Srivastva *et al.* (see [18]) has introduced three notations:

$$\begin{gathered} R_\alpha = R(2,1,1,1,2,2); \qquad R_\beta = R(2,2,1,1,2,2); \\ R_m = R(m,m,1,1,1,2). \end{gathered} \tag{18}$$

where $m = 1,2,3,\ldots$

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [11]):

For

$$A = \frac{f(q)f(-q^2)}{q^{\frac{1}{4}}f(q^3)f(-q^6)}, \quad C_n = \frac{f(-q^n)f(-q^{2n})}{q^{\frac{n}{4}}f(-q^{3n})f(-q^{6n})}; \quad n = 1,2,3,\ldots$$

I. If

$$P = AC_1, \quad Q = \frac{A}{C_1},$$

then

$$\left(Q^8+\frac{1}{Q^8}\right) + 7\left(Q^4+\frac{1}{Q^4}\right) + \left(P^2-\frac{9^2}{P^2}\right)\left(Q^2-\frac{1}{Q^2}\right) - 24 = 0. \tag{19}$$

II. If

$$P = C_1C_4, \quad Q = \frac{C_1}{C_4},$$

then

$$\left(Q^{16}+\frac{1}{Q^{16}}\right)-200\left(Q^{12}+\frac{1}{Q^{12}}\right)-6848\left(Q^{8}+\frac{1}{Q^{8}}\right)-53255\left(Q^{4}+\frac{1}{Q^{4}}\right)$$
$$-\left\{9320\left(Q^{2}+\frac{1}{Q^{2}}\right)+2016\left(Q^{6}+\frac{1}{Q^{6}}\right)+103\left(Q^{10}+\frac{1}{Q^{10}}\right)\right\}\cdot\left(P^{2}+\frac{9^{2}}{P^{2}}\right)$$
$$-\left\{280\left(Q^{4}+\frac{1}{Q^{4}}\right)+16\left(Q^{8}+\frac{1}{Q^{8}}\right)+(29)^{2}\right\}\cdot\left(P^{4}+\frac{9^{4}}{P^{4}}\right)$$
$$-\left\{24\left(Q^{2}+\frac{1}{Q^{2}}\right)+\left(Q^{6}+\frac{1}{Q^{6}}\right)\right\}\cdot\left(P^{6}+\frac{9^{6}}{P^{6}}\right)-\left(P^{8}+\frac{9^{8}}{P^{8}}\right)-135032=0. \tag{20}$$

2. A Set of Main Results

In this section, we state and prove a set of two identities which depict inter-relationships among q-product identities; R_α, R_β, and R_m functions.

Theorem 3. *Each of the following relationships holds true:*

$$\left\{\frac{(-q;-q)_\infty(-q^3;-q^3)_\infty}{q^{\frac{1}{2}}R_1R_6(-q;-q)_\infty}\right\}^2\cdot\left\{\frac{(q,q^2,q^2,q^2,q^2,q^2;q^2)_\infty}{(-q^3;-q^3)_\infty(q^3,q^3,q^6,q^6,q^6;q^6)_\infty(q^6,q^6;q^{12})_\infty}\right\}^2$$
$$+\left\{\frac{9q^{\frac{1}{2}}R_1R_6}{(-q^3;-q^3)_\infty}\right\}^2\cdot\left\{\frac{(-q^3;-q^3)_\infty(q^3,q^3,q^6,q^6,q^6;q^6)_\infty(q^6,q^6;q^{12})_\infty}{(q,q^2,q^2,q^2,q^2,q^2;q^2)_\infty}\right\}^2$$
$$=\{R_1R_6R_{12}\}^8\cdot\left\{\frac{(-q;-q)_\infty(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(-q^3;-q^3)_\infty(q^2,q^2;q^2)_\infty(q^{24};q^{24})_\infty}\right\}^8$$
$$+\left\{\frac{(-q^3;-q^3)_\infty(q^2,q^2;q^2)_\infty}{R_1R_6(-q;-q)_\infty(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty}\right\}^8+7\{R_1R_6R_{12}\}^4$$
$$\cdot\left\{\frac{(-q;-q)_\infty(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(-q^3;-q^3)_\infty(q^2,q^2;q^2)_\infty(q^{24};q^{24})_\infty}\right\}^4+$$
$$+7\left\{\frac{(-q^3;-q^3)_\infty(q^2,q^2;q^2)_\infty}{R_1R_6(-q;-q)_\infty(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty}\right\}^4+\{R_1R_4R_6[R_{12}]^3\}^2$$
$$\cdot\left\{\frac{(-q,-q;-q)_\infty(q;q^2)_\infty(q^2;q^4)^3_\infty(q^3;q^6)_\infty(q^6;q^{12})^2_\infty(q^{12};q^{24})_\infty}{q^{\frac{1}{2}}(-q^3,-q^3;-q^3)_\infty(q^3,q^6,q^6,q^6;q^6)_\infty(q^{24};q^{24})_\infty(q^8,q^8;q^8)_\infty}\right\}^2$$
$$+\left\{\frac{9q^{\frac{1}{2}}(-q^3,-q^3;-q^3)_\infty(q^3,q^6,q^6,q^6;q^6)_\infty}{R_1R_6[R_{12}]^2}\right\}^2$$
$$\cdot\left\{\frac{(q^4,q^8;q^8)_\infty}{(-q,-q;-q)_\infty(q;q^2)_\infty(q^2,q^2,q^2;q^4)_\infty(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty}\right\}^2-24, \tag{21}$$

and

$$\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^{16}\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^8;q^{16})_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^8,q^{16},q^{16};q^{16})_\infty}\right\}^{16}$$
$$+\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^{16}$$
$$=200\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^{12}\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^{12}$$
$$+200\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^{12}+6848\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^8$$
$$\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^8+$$
$$+6848\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^8+53255\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^4$$
$$\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^4$$
$$+53255\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^4$$
$$+9320\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$
$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^2\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^2$$
$$+9320\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$
$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^2$$
$$+2016\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$
$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^6\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^6$$
$$+2016\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$
$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^6$$
$$+103\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$

$$\cdot\left\{\frac{q^{\frac{3}{4}}R_4R_8[R_{12}]^2(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^{10}$$

$$+103\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^2$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^{10}$$

$$+754920\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^2\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^2$$

$$+754920\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^2$$

$$+163296\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^6\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^6$$

$$+163296\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^6$$

$$+8343\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^{10}\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^{10}$$

$$+8343\left\{\frac{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^2$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^{10}$$

$$+280\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^4$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^4\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^4$$

$$+280\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^4$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^4$$

$$+16\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^4$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^8\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^8$$

$$+16\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^4$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^8$$

$$+841\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^4$$

$$+280\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^4$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^4\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^4$$

$$+280\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^4$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^4$$

$$+16\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^4$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^8\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^8$$

$$+16\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^4$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^8$$

$$+841\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^4$$

$$+24\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^6$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^2\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^2$$

$$+24\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^6$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^2$$

$$+\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^6\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^6$$

$$\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^6$$

$$+\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^6$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^6$$

$$+24\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^6$$

$$\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^2\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^2$$

$$+24\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^6$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^2$$

$$+\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^6\cdot\{q^{\frac{3}{4}}R_4R_8[R_{12}]^2\}^6$$

$$\cdot\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty(q^{12};q^{24})_\infty}{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24},q^{24};q^{24})_\infty(q^{16},q^{16};q^{16})_\infty}\right\}^6$$

$$+\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^6$$

$$\cdot\left\{\frac{(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{24};q^{24})_\infty(q^8,q^{16};q^{16})_\infty}{q^{\frac{3}{4}}R_4R_{12}(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4;q^8)_\infty}\right\}^6$$

$$+\left\{\frac{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}\right\}^8$$

$$+\left\{\frac{9q^{\frac{5}{4}}(q^3;q^6)_\infty(q^6,q^6;q^{12})_\infty(q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty}{(q;q^2)_\infty(q^2,q^2;q^4)_\infty(q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}\right\}^8+135032. \tag{22}$$

Proof. First of all, in order to prove our assertion (21), apply the identitity (10) into (19), we obtain:

$$P = \{AC_1\} = \left\{ \frac{(-q;-q)_\infty (q, q^2, q^2, q^2; q^2)_\infty}{q^{\frac{1}{2}} (-q^3; -q^3)_\infty (q^3, q^6, q^6, q^6; q^{12})_\infty} \right\}, \tag{23}$$

$$\begin{aligned} Q = \left\{ \frac{A}{C_1} \right\} &= \{R_1 R_6 R_{12}\} \cdot \\ &\cdot \left\{ \frac{(-q;-q)_\infty (q^3;q^6)_\infty (q^6;q^{12})^2_\infty (q^{12};q^{24})_\infty}{(-q^3;-q^3)_\infty (q^2,q^2;q^2)_\infty (q^{24};q^{24})_\infty} \right\}. \end{aligned} \tag{24}$$

combining (23) and (24), as precondition given in (19), by arranging the powers of suitable terms, we are led to the first assertion (21).

Finally. we attempt to prove our second identity (22), apply identity (10) into (20), we obtain:

$$\begin{aligned} P &= \{C_1 C_4\} = \\ &= \left\{ \frac{(q;q^2)_\infty (q^2,q^2;q^4)_\infty (q^4,q^4,q^4,q^8,q^8,q^8,q^8;q^8)_\infty}{q^{\frac{5}{4}} (q^3;q^6)_\infty (q^6,q^6;q^{12})_\infty (q^{12},q^{12},q^{12},q^{24},q^{24},q^{24},q^{24};q^{24})_\infty} \right\}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} Q = \left\{ \frac{C_1}{C_4} \right\} &= \{q^{\frac{3}{4}} R_4 R_8 [R_{12}]^2\} \cdot \\ &\cdot \left\{ \frac{(q;q^2)_\infty (q^2,q^2;q^4)_\infty (q^4,q^4;q^8)_\infty (q^8;q^{16})_\infty (q^{12};q^{24})_\infty}{(q^3;q^6)_\infty (q^6,q^6;q^{12})_\infty (q^{24},q^{24};q^{24})_\infty (q^8,q^{16},q^{16};q^{16})_\infty} \right\}. \end{aligned} \tag{26}$$

combining (25) and (26), as precondition given in (20), by arranging the powers of suitable terms, we are led to the second assertion (22).

We thus have completed our proof of the above Theorem. □

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4. Conflicts of Interest

Both the authors declare that they have no conflict of interest.

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