Mathematica Moravica
Vol. **24**, No. **2** (2020), 117–131
doi: 10.5937/MatMor2002117B

New inequalities for F-convex functions pertaining generalized fractional integrals

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Abstract. In this paper, the authors, utilizing F-convex functions which are defined by B. Samet, establish some new Hermite-Hadamard type inequalities via generalized fractional integrals. Some special cases of our main results recaptured the well-known earlier works.

1. Introduction

Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [17]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities in (1) hold in the reversed direction if f is concave.

Over the last decade, this classical double inequality has been improved and generalized in a number of ways, see [5, 7, 8, 13, 18], [23]–[25] and the references therein. Also, many types of convexities have been defined, such as quasi–convex in [6], pseudo–convex in [14], strongly convex in [20], ε–convex in [11], s–convex in [10], h–convex in [28], etc. Recently, Samet in [21], has defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfying the following axioms:

2010 *Mathematics Subject Classification.* Primary: 26D07; Secondary: 26D10, 26D15, 26A33.

Key words and phrases. Convex function, trigonometrically ρ-convex functions.

Full paper. Received 9 December 2019, revised 21 July 2020, accepted 28 July 2020, available online 16 September 2020.

(A1) If $e_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda)\mathrm{d}t = F\left(\int_0^1 e_1(t)\mathrm{d}t, \int_0^1 e_2(t)\mathrm{d}t, \int_0^1 e_3(t)\mathrm{d}t, \lambda\right);$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)\mathrm{d}t = T_{F,w}\left(\int_0^1 w(t)u(t)\mathrm{d}t, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable;

(A3) For any $(w, e_1, e_2, e_3) \in \mathbb{R}^4$, $e_4 \in [0,1]$, we have

$$wF(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

Definition 1. Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F-convex function), if and only if:

$$F(f(tx + (1-t)y), f(x), f(y), t) \le 0, \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Remark 1. 1) Let $\varepsilon \ge 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, see [11], that is

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon, \ \ (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1-e_4)e_3 - \varepsilon \tag{2}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 tw(t)\mathrm{d}t\right) e_2 - \left(\int_0^1 (1-t)w(t)\mathrm{d}t\right) e_3 - \varepsilon. \tag{3}$$

For

$$L_w = (1-w)\varepsilon, \tag{4}$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \le 0,$$

that is f is an F–convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an F–convex function with respect to F defined above.

2) Let $h : J \to [0, +\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0,1) \subseteq J$. Let $f : [a,b] \to [0,+\infty)$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an h-convex function, see [28], that is

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y), \ \ (x,y,t) \in [a,b] \times [a,b] \times [0,1]\,.$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3 \tag{5}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 h(t)w(t)\mathrm{d}t \right) e_2 - \left(\int_0^1 h(1-t)w(t)\mathrm{d}t \right) e_3. \tag{6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y) - h(t)f(x) - h(1-t)f(y) \le 0,$$

that is, f is an F–convex function.

Samet in [21], established the following Hermite–Hadamard type inequalities using the new convexity concept:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a,b]$. Then*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x, \ \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x, \frac{1}{2} \right) \le 0,$$

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x, \ f(a), \ f(b) \right) \le 0.$$

Definition 2. Let $f \in L^1[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

Definition 3. Let $f \in L^1[a,b]$. Then k–fractional integrals of order $\alpha, k > 0$ are defined by

$$I^{\alpha}_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\mathrm{d}t, \quad x > a,$$

and

$$I_{b^-,k}^{\alpha}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f(t)\mathrm{d}t,\quad b>x, \tag{7}$$

where $\Gamma_k(\cdot)$ stands for the k-gamma function. For $k=1$, the k–fractional integrals yield Riemann–Liouville integrals. For $\alpha=k=1$, the k–fractional integrals yield classical integrals. For more details, see [9, 12, 15, 19].

It is remarkable that Sarikaya et al. in [26], first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}, \tag{8}$$

with $\alpha>0$.

Budak et al. in [1], prove the following Hermite-Hadaamrd type inequalities for F-convex functions via fractional integrals:

Theorem 3. *Let $I\subseteq R$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on I°, $a,b\in I^{\circ}$, $a<b$. If f is F-convex on $[a,b]$ for some $F\in\mathcal{F}$, then we have*

$$\begin{aligned}&F\left(f\left(\frac{a+b}{2}\right),\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a+}^{\alpha}f(b),\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a),\frac{1}{2}\right)\\&\quad+\int_0^1L_{w(t)}\mathrm{d}t\le0,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right],f(a)+f(b),f(a)+f(b)\right)\\&\quad+\int_0^1L_{w(t)}\mathrm{d}t\le0,\end{aligned} \tag{10}$$

where $w(t)=\alpha t^{\alpha-1}$.

For other papers involving F-convex functions, see [1]-[4], [16, 27].

Now we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [22].

Let's define a function $\varphi:[0,+\infty)\to[0,+\infty)$ satisfying the following conditions:

$$\int_0^1\frac{\varphi(t)}{t}\mathrm{d}t<+\infty, \tag{11}$$

$$\frac{1}{A_1} \leq \frac{\varphi(v)}{\phi(u)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2, \tag{12}$$

$$\frac{\varphi(u)}{u^2} \leq A_2 \frac{\varphi(v)}{v^2} \quad \text{for} \quad v \leq u, \tag{13}$$

$$\left| \frac{\varphi(u)}{u^2} - \frac{\varphi(v)}{v^2} \right| \leq A_3 |u-v| \frac{\varphi(u)}{u^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2, \tag{14}$$

where $A_1, A_2, A_3 > 0$ are independent of $u, v > 0$. If $\varphi(u)u^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(u)}{u^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies the above conditions.

The following left-sided and right–sided generalized fractional integral operators are defined respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) \mathrm{d}t, \quad x > a, \tag{15}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) \mathrm{d}t, \quad x < b. \tag{16}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, k–Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarikaya and Ertuğral in [22], establish the following Hermite–Hadamard inequality and lemmas for the generalized fractional integral operators:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)} \left[_{a+}I_\varphi f(b) + _{b-} I_\varphi f(a) \right] \leq \frac{f(a)+f(b)}{2}, \tag{17}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\varphi((b-a)t)}{t} \mathrm{d}t.$$

Budak et al. prove the following Hermite Hadamard type inequalities for F-convex functions.

Theorem 5 ([4])**.** *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^\circ$, $a < b$. If f is F-convex on $[a,b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Psi(1)}\ _{a+}I_\varphi f(b), \frac{1}{\Psi(1)}\ _{b-}I_\varphi f(a), \frac{1}{2} \right) + \int_0^1 L_{w(t)} \mathrm{d}t \leq 0,$$

and

$$T_{F,w}\left(\frac{1}{\Psi(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right],f(a)+f(b),f(a)+f(b)\right)$$
$$+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

where $w(t)=\frac{\varphi((b-a)t)}{t\Psi(1)}$.

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities via generalized fractional integrals utilizing F–convex functions. Some special cases of our main results recaptured the well–known earlier works. At the end, a briefly conclusion will be given as well.

2. Main results

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via F–convex functions.

Theorem 6. *Let $I\subseteq\mathbb{R}$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on I°, $a,b\in I^{\circ}$, $a<b$ and let F be linear with respect to the first three variables. If f is F-convex on $[a,b]$ for some $F\in\mathcal{F}$, then we have*

$$\begin{aligned}F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^{+}}I_{\varphi}f\left(b\right),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^{-}}I_{\varphi}f\left(a\right),\frac{1}{2}\right)\\+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,\end{aligned}\tag{18}$$

and

$$\begin{aligned}T_{F,w}\Bigg(\frac{1}{\Lambda(1)}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}I_{\varphi}f\left(b\right)+{}_{\left(\frac{a+b}{2}\right)^{-}}I_{\varphi}f\left(a\right)\right],\\ f(a)+f(b),f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,\end{aligned}\tag{19}$$

where $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ *and the function* $\Lambda:[0,1]\to\mathbb{R}$ *is defined by*

$$\Lambda(x)=\int_0^x\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}\mathrm{d}t.$$

Kanıt. Since f is F–convex, we have

$$F\left(f\left(\frac{x+y}{2}\right),f(x),f(y),\frac{1}{2}\right)\le 0,\quad\forall x,y\in[a,b].$$

For

$$x=\frac{t}{2}a+\left(\frac{2-t}{2}\right)\ b \text{ and } \ y=\left(\frac{2-t}{2}\right)a+\frac{t}{2}b,$$

we have

$$F\left(f\left(\frac{a+b}{2}\right),f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right),f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\frac{1}{2}\right)\leq 0.$$

for all $t\in[0,1]$. Multiplying this inequality by $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ and using axiom (A3), we get

$$F\left(\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{a+b}{2}\right),\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right),\right.$$
$$\left.\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\frac{1}{2}\right)+L_{w(t)}\leq 0$$

for all $t\in(0,1)$. Integrating over $(0,1)$ with respect to the variable t and using axiom (A1), we obtain

$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}\mathrm{d}t,\frac{1}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t,\right.$$
$$\left.\frac{1}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\mathrm{d}t,\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0.$$

Using the facts that

$$\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t$$
$$=\int_{\frac{a+b}{2}}^b\frac{\varphi(b-x)}{b-x}f(x)\mathrm{d}x={}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b)$$

and

$$\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\mathrm{d}t$$
$$=\int_a^{\frac{a+b}{2}}\frac{\varphi(x-a)}{x-a}f(x)\mathrm{d}x={}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a),$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\ {}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b),\frac{1}{\Lambda(1)}\ {}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a),\frac{1}{2}\right)$$
$$+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

which gives (18).

On the other hand, since f is F–convex, we have

$$F\left(f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0,1],$$

and

$$F\left(f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0,1].$$

Using the linearity of F, we get

$$\begin{aligned}F\Bigg(&f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\\ &f(a)+f(b), f(a)+f(b), t\Bigg) \leq 0,\end{aligned}$$

for all $t \in [0,1]$. Applying the axiom (A3) for $w(t)=\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$\begin{aligned}F\Bigg(&\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\right],\\ &\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], t\Bigg)+L_{w(t)} \leq 0,\end{aligned}$$

for all $t \in (0,1)$. Integrating over $(0,1)$ and using axiom (A2), we have

$$\begin{aligned}T_{F,w}\Bigg(&\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\right]\mathrm{d}t,\\ &f(a)+f(b), f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0,\end{aligned}$$

that is

$$\begin{aligned}T_{F,w}&\left(\frac{1}{\Lambda(1)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_{\varphi}f(a)\right], f(a)+f(b), f(a)+f(b)\right)\\ &+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0.\end{aligned}$$

The proof of Theorem 6 is completed. □

Remark 2. If we choose $\varphi(t) = t$ in Theorem 6, then we have the following inequalities

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(t)\mathrm{d}t, \frac{2}{b-a}\int_{a}^{\frac{a+b}{2}} f(t)\mathrm{d}t, \frac{1}{2}\right) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0, \tag{20}$$

and

$$T_{F,w}\left(\frac{2}{b-a}\int_a^b f(t)\mathrm{d}t, f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0, \tag{21}$$

where $w(t) = 1$.

Remark 3. If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 6, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha} J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b), \frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha} J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

and

$$T_{F,w}\left(\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

Corollary 1. *If we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 6, then we have the following inequalities for k–Riemann–Liouville fractional integrals*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^\alpha_{\left(\frac{a+b}{2}\right)+,\ k} f(b), \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^\alpha_{\left(\frac{a+b}{2}\right)-,\ k} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

and

$$T_{F,w}\left(\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^\alpha_{\left(\frac{a+b}{2}\right)+,\ k} f(b) + I^\alpha_{\left(\frac{a+b}{2}\right)-,\ k} f(a)\right],\right.$$

$$f(a)+f(b),f(a)+f(b)\Bigg)+\int\limits_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

where $w(t)=\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}$.

Theorem 7. *Let $I\subseteq\mathbb{R}$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on I°, $a,b\in I^{\circ}$, $a<b$ and let F be linear with respect to the first three variables. If f is F-convex on $[a,b]$ for some $F\in\mathcal{F}$, then we have*

$$\begin{aligned}F\Bigg(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\ {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right),\\ \frac{1}{\Lambda(1)}\ {}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right),\frac{1}{2}\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,\end{aligned}\tag{22}$$

and

$$\begin{aligned}T_{F,w}\Bigg(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+\ {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right],\\ f(a)+f(b),f(a)+f(b)\Bigg)+\int\limits_0^1 L_{w(t)}\mathrm{d}t\leq 0,\end{aligned}\tag{23}$$

where $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$.

Kanıt. Since f is F-convex, we have

$$F\left(f\left(\frac{x+y}{2}\right),f(x),f(y),\frac{1}{2}\right)\leq 0,\quad\forall\, x,y\in[a,b].$$

For

$$x=\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\ \text{ and }\ y=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b,$$

we have

$$\begin{aligned}F\Bigg(f\left(\frac{a+b}{2}\right),f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right),\\ f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right),\frac{1}{2}\Bigg)\leq 0,\end{aligned}$$

for all $t\in[0,1]$. Multiplying this inequality by $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ and using axiom (A3), we get

$$\begin{aligned}F\Bigg(\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{a+b}{2}\right),\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right),\\ \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right),\frac{1}{2}\Bigg)+L_{w(t)}\leq 0,\end{aligned}$$

for all $t \in (0,1)$. Integrating over $(0,1)$ with respect to the variable t and using axiom (A1), we obtain

$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}\mathrm{d}t,\right.$$
$$\frac{1}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\mathrm{d}t,$$
$$\left.\frac{1}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\mathrm{d}t,\frac{1}{2}\right)$$
$$+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0.$$

Using the facts that

$$\begin{aligned}
&\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\mathrm{d}t \\
&= \int_{\frac{a+b}{2}}^b \frac{\varphi\left(x-\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}f(x)\mathrm{d}x \\
&= {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\mathrm{d}t \\
&= \int_a^{\frac{a+b}{2}} \frac{\varphi\left(\frac{a+b}{2}-x\right)}{\frac{a+b}{2}-x}f(x)\mathrm{d}x \\
&= {}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right),
\end{aligned}$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\,{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\,{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right),\frac{1}{2}\right)$$
$$+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

which gives (22).

On the other hand, since f is F–convex, we have

$$F\left(f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right),f(a),f(b),t\right) \leq 0, \quad \forall t \in [0,1],$$

and

$$F\left(f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right), f(a), f(b), t\right) \le 0, \quad \forall t \in [0,1].$$

Using the linearity of F, we get

$$F\Bigg(f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right), \\ f(a)+f(b), f(a)+f(b), t\Bigg) \le 0, \quad \forall t \in [0,1].$$

Applying the axiom (A3) for $w(t)=\dfrac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$F\Bigg(\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+ \\ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\Bigg], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], \\ \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], t\Bigg) + L_{w(t)} \le 0,$$

for all $t \in (0,1)$. Integrating over $(0,1)$ and using axiom (A2), we have

$$T_{F,w}\Bigg(\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+ \\ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\Bigg]\mathrm{d}t, \\ f(a)+f(b), f(a)+f(b)\Bigg) + \int_0^1 L_{w(t)}\mathrm{d}t \le 0,$$

that is

$$T_{F,w}\Bigg(\frac{1}{\Lambda(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)+\ {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right], \\ f(a)+f(b), f(a)+f(b)\Bigg) + \int_0^1 L_{w(t)}\mathrm{d}t \le 0.$$

The proof of Theorem 7 is completed. □

Remark 4. If we take $\varphi(t)=t$ in Theorem 7, then the inequalities (22) and (23) reduce to the inequalities (20) and (21)

Remark 5. If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 7, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F\Bigg(f\left(\frac{a+b}{2}\right),\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right),$$
$$\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right),\frac{1}{2}\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

and

$$T_{F,w}\Bigg(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)\right],$$
$$f(a)+f(b),f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

where $w(t)=\alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

Corollary 2. *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 7, then we have the following inequalities for k–Riemann–Liouville fractional integrals:*

$$F\left(f\left(\frac{a+b}{2}\right),\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I^{\alpha}_{b-,\ k}f\left(\frac{a+b}{2}\right),\right.$$
$$\left.\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I^{\alpha}_{a+,\ k}f\left(\frac{a+b}{2}\right),\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

and

$$T_{F,w}\left(\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b-,\ k}f\left(\frac{a+b}{2}\right)\right],\right.$$
$$f(a)+f(b),f(a)+f(b))+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,$$

where $w(t)=\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}$.

Remark 6. One can obtain several results for convexity, ε-convexity, h-convexity, etc by special choice of the function F in Theorems 6 and 7.

3. Conclusion

In the development of this work, using the definition of F-convex functions some new Hermite-Hadamard type inequalities via generalized fractional integrals have been deduced. We also give several results capturing Riemann-Liouville fractional integrals and k-Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area.

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