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On coupled systems of fractional impulsive differential equations by using a new Caputo-Fabrizio fractional derivative

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for coupled system of Caputo-Fabrizio fractional impulsive differential equations using the fixed point approach in generalized metric spaces. The compactness of solution sets of the system is also investigated. An example is provided to illustrate the developed theory.

1. Introduction

The fractional calculus is nowadays a very attractive subject to mathematicians, and many different forms of fractional differential operators were introduced. To increase the applicability of the fractional calculus, some authors proposed a new type of fractional derivatives possessing different kernels. The most used definitions proposed by Riemann-Liouville and the first Caputo version has the weakness that their kernel had singularity [9]. A recent new definition of fractional derivative without singular kernel has been provided by Caputo and Fabrizio [15] and its properties were discussed in [22]. A fractional order derivative is important for developing mathematical models in many scientific and engineering disciplines (see [7]). Several qualitative results for different classes of differential equations with different types of fractional integral and derivatives were obtained in [1, 3, 17, 23, 25, 27, 28].

Thereby, many evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses, which led to define a class of differential equations known as impulsive differential equations. We refer to [12, 21] for an introduction to the theory of impulsive differential equations. Recently, several papers have been devoted to study the solutions of differential equations with new types

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of fractional derivatives and their applications, see [8, 5, 10, 14] and the references therein.

Coupled systems of fractional-order differential equations appear in the mathematical formulation of several real areas like physics, engineering, chemistry, biology, viscoelasticity etc. [11, 24]. Recently, the study of existence and uniqueness of solutions of coupled systems of fractional order differential equations has also attracted some attention. Alsaedi et al. [5] examined the existence of solutions for a coupled system of time-fractional differential equations and inclusions by using the new Caputo-Fabrizio fractional derivative. Recently, Berrezoug et al. [13] studied the existence and uniqueness, continuous dependence on initial conditions and the boundedness of solutions for a system of impulsive differential equations using the fixed point approach in vector Banach spaces. Very recently, Chalishajar and Kumar [16] investigated the existence and uniqueness of the solutions to a fractional order nonlinear coupled system with integral boundary conditions. Furthermore, Ulam's type stability of the proposed coupled system is studied. For more results on the study of coupled systems of fractional differential equations, we refer to [2, 5, 4, 20, 13] and the references therein.

In this paper, we consider the following coupled system of fractional impulsive differential equations involving the Caputo-Fabrizio fractional derivative:

$$(1)\qquad \begin{cases} \left(D_t^{\alpha} x\right)(t) = f_1(t,x,y), & \\ & t\in[0,T],\ t\neq t_k,\ k=1,\dots,m,\\ \left(D_t^{\beta} y\right)(t) = f_2(t,x,y), & \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k),y(t_k)), & \\ & k=1,\dots,m,\\ \Delta y(t_k) = y(t_k^+) - y(t_k^-) = \overline{I}_k(x(t_k),y(t_k)), & \\ x(0)=x_0, \quad y(0)=y_0, & \end{cases}$$

where D_t^{α}, D_t^{β} are the Caputo-Fabrizio fractional derivative of order α and β, $0<\alpha,\beta<1$. Here $0=t_0\le t_1\le\cdots\le t_m\le t_{m+1}=T$, $\Delta x(t_k) = I_k(x(t_k^-)) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k+h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively. $x_0, y_0 \in \mathbb{R}$, $f_1, f_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and I_k , $\overline{I}_k \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are given functions.

The rest of this paper is organized as follows. In Section 2, we introduce all the background material used in this paper such as definition of Caputo-Fabrizio derivatives of fractional order and some properties of generalized Banach spaces and fixed point theory. In Sections 3 and 4, using Perov's and Schaefer fixed point type theorems in generalized Banach spaces, we prove some existence and compactness results for problem (1). An example is given to demonstrate the application of our main results in section 4. Finally, a conclusion is given in section 5.

2. Preliminaries

For completeness, in this section, we will give some notations, basic definitions and some fundamental facts of Caputo-Fabrizio derivatives of fractional order which can be found in [15] and [22].

Definition 1. The α order Caputo-Fabrizio time fractional differential derivative of the function u is written as

$$\left({}^{CF}D_t^\alpha u\right)(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] u'(s)ds, \quad t \geq 0, \tag{2}$$

where $M(\alpha)$ represents a normalization function such that $M(0) = M(1) = 1$, $0 < \alpha < 1$, and $u \in H^1(0, b)$, $b > 0$.

Note that, according to the definition 1, the new definition of fractional derivative is zero when u is constant, as in the usual Caputo fractional time derivative, but contrary to the usual Caputo fractional time derivative, the kernel does not have singularity for $t = s$ [15, 22].

The notion of Caputo-Fabrizio time-fractional integral is,

Definition 2. Let $0 < \alpha < 1$. the fractional integral of order α of a function f is defined by,

$$\left({}^{CF}I_t^\alpha f\right)(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s)ds, t \geq 0, \tag{3}$$

where $M(\alpha)$ represents a normalization function and $0 < \alpha < 1$.

Losada and Nieto [22] note that the fractional integral of Caputo-Fabrizio type of a function of order $0 < \alpha < 1$ is an average between function f and its integral of order one.

With imposing

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1,$$

we obtain an explicit formula for $M(\alpha)$,

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 < \alpha < 1.$$

By substituting $M(\alpha)$ in (1), we obtain the definition of the fractional Caputo-Fabrizio derivative of order $0 < \alpha < 1$ for a function u as follows:

Definition 3. Let $0 < \alpha < 1$. The fractional Caputo-Fabrizio derivative of order α of a function u is given by

$$\left({}^{CF}D_t^\alpha u\right)(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) u'(s)ds, \quad t \geq 0. \tag{4}$$

Lemma 1 ([22]). *Let $0 < \alpha < 1$ and u be a solution of the following fractional differential equation,*

$$\left({}^{CF}D_t^{\alpha}u\right)(t) = 0, \tag{5}$$

Then, u is a constant function. The converse, is also true.

Lemma 2 ([22]). *Suppose that $f \in L^1([0,T])$ and $0 < \alpha < 1$. Then, the unique solution of the following initial value problem*

$$\begin{gathered} \left({}^{CF}D_t^{\alpha}u\right)(t) = f(t), \\ u(0) = u_0 \in \mathbb{R} \end{gathered} \tag{6}$$

is given by

$$u(t) \quad = u_0 + A_\alpha(f(t) - f(0)) + B_\alpha \textstyle\int_0^t f(s)ds, \tag{7}$$

where $A_\alpha = \dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$ *and* $B_\alpha = \dfrac{2\alpha}{(2-\alpha)M(\alpha)}$.

Lemma 3. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0,x) = 0$, $\forall x \in \mathbb{R}$, and $0 < \alpha < 1$. Then, a function x is a solution of the following initial value problem*

$$\begin{cases} \left(D_t^{\alpha}x\right)(t) = f(t,x), t \in [0,T], & t \neq t_k, \\ \quad \Delta x(t) = I_k(x(t_k)), & k = 1, \dots, m, \\ \quad x(0) = x_0, \end{cases} \tag{8}$$

if and only if

$$x(t) = \begin{cases} x_0 + A_\alpha f(t,x) + B_\alpha \int_0^t f(s,x(s))ds, & t \in [0,t_1], \\ x_0 + A_\alpha f(t,x) + B_\alpha \int_0^t f(s,x(s))ds & +I_1(x(t_1)), \\ & \quad t \in (t_1,t_2], \\ \quad \vdots & \quad \vdots \\ x_0 + A_\alpha f(t,x) + B_\alpha \int_0^t f(s,x(s))ds & +\displaystyle\sum_{i=1}^{k} I_i(x(t_i)), \\ & \quad t \in (t_k,t_{k+1}]. \end{cases} \tag{9}$$

Proof. Assume x satisfies (8). If $t \in [0,t_1]$, then

$$\left(D_t^{\alpha}x\right)(t) = f(t,x).$$

Lemma (2) implies

$$x(t) = x_0 + A_\alpha f(t,x) + B_\alpha \int_0^t f(s,x(s))ds.$$

If $t \in (t_1,t_2]$ then Lemma (2) implies

$$x(t) = x(t_1^+) + A_\alpha\left(f(t,x) - f(t_1^+, x(t_1^+))\right) + B_\alpha \int_{t_1}^t f(s,x(s))ds$$

$$
\begin{aligned}
&= I_1(x(t_1)) + x(t_1) + A_\alpha \left(f(t,x) - f(t_1^+, x(t_1^+)) \right) \\
&\quad + B_\alpha \int_{t_1}^{t} f(s,x(s))ds \\
&= I_1(x(t_1)) + x_0 + A_\alpha (f(t_1, x(t_1)) - f(0,x_0)) \\
&\quad + B_\alpha \int_{0}^{t_1} f(s,x(s))ds \\
&\quad + A_\alpha \left(f(t,x) - f(t_1^+, x(t_1^+)) \right) + B_\alpha \int_{t_1}^{t} f(s,x(s))ds \\
&= I_1(x(t_1)) + x_0 + A_\alpha \left(f(t,x) - f(0,x_0)) \right) \\
&\quad + B_\alpha \int_{0}^{t} f(s,x(s))ds \\
&= I_1(x(t_1)) + x_0 + A_\alpha f(t,x) + B_\alpha \int_{0}^{t} f(s,x(s))ds.
\end{aligned}
$$

If $t \in (t_2, t_3]$, then from Lemma (2) we get

$$
\begin{aligned}
x(t) &= x(t_2^+) + A_\alpha \left(f(t,x) - f(t_2^+, x(t_2^+)) \right) + B_\alpha \int_{t_2}^{t} f(s,x(s))ds \\
&= I_2(x(t_2)) + x(t_2) + A_\alpha \left(f(t,x) - f(t_2^+, x(t_2^+)) \right) \\
&\quad + B_\alpha \int_{t_2}^{t} f(s,x(s))ds \\
&= I_2(x(t_2)) + I_1(x(t_1)) + x_0 + A_\alpha (f(t_2, x(t_2)) - f(0,x_0)) \\
&\quad + B_\alpha \int_{0}^{t_2} f(s,x(s))ds \\
&\quad + A_\alpha \left(f(t,x) - f(t_2^+, x(t_2^+)) \right) + B_\alpha \int_{t_2}^{t} f(s,x(s))ds \\
&= I_1(x(t_1)) + I_2(x(t_2)) + x_0 + A_\alpha \left(f(t,x) - f(0,x_0)) \right) \\
&\quad + B_\alpha \int_{0}^{t} f(s,x(s))ds \\
&= I_1(x(t_1)) + I_2(x(t_2)) + x_0 + A_\alpha f(t,x) + B_\alpha \int_{0}^{t} f(s,x(s))ds.
\end{aligned}
$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma (2) we get

$$
x(t) = x_0 + \sum_{i=1}^{k} I_i(x(t_i)) + A_\alpha f(t,x) + B_\alpha \int_{0}^{t} f(s,x(s))ds.
$$

Conversely, assume that x satisfies the impulsive fractional integral equation (9). If $t \in [0, t_1]$ then $x(0) = x_0$ and we can easily show that

$$
\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1 \ldots, m,
$$

and using the fact that $D^\alpha C = 0$, where C is a constant and the definition 1 we get

$$(D_t^\alpha x)(t) = f(t,x), t \in [0,t_1] \cup (t_k, t_{k+1}], \quad k = 1,\dots,m. \qquad \square$$

In the following, we introduce some notations and definitions of generalized metric space.

If $x, y \in \mathbb{R}^m$, with $x = (x_1, \cdots, x_m)$ and $y = (y_1, \dots, y_m)$, then by $x \le y$ we mean $x_i \le y_i$ for all $i = 1,\dots,m$. Also we set $|x| = (|x_1|,\dots,|x_m|)$, $\max(x,y) = (\max(x_1,y_1),\dots,\max(x_m,y_m))$ and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i > 0\}$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$ for each $i = 1,\dots,m$.

Definition 4. Let X be a nonempty set and consider space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a generalized metric in Perov's sense and (X,d) is called a generalized metric space.

Let (X,d) be a generalized metric space in Perov's sense with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_m(x,y) \end{pmatrix}, \quad (x,y) \in X \times X.$$

For $r = (r_1,\dots,r_m) \in \mathbb{R}^m_+$, we will denote by

$$\begin{aligned} B(x_0,r) &= \{x \in X : d(x_0,x) < r\} = \\ &= \{x \in X : d_i(x_0,x) < r_i, \quad i = 1,\dots,m\}, \end{aligned}$$

the open ball centered in x_0 with radius r and

$$\begin{aligned} B(x_0,r) &= \{x \in X : d(x_0,x) \le r\} = \\ &= \{x \in X : d_i(x_0,x) \le r_i, \quad i = 1,\dots,m\}, \end{aligned}$$

the closed ball centered in x_0 with radius r.

We mention that for a generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence, and completeness are similar to those in the usual metric spaces.

Definition 5. A square matrix A of real numbers is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

Lemma 4 (see [18]). *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

- *A is a matrix convergent to zero;*
- *The eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$; where I denote the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R}_+)$,*
- *The matrix $I - A$ is non-singular and $(I-A)^{-1} = I + A + \cdots + A^n + \cdots$;*

- *The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnegative elements;*
- *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.*

Some examples of matrices convergent to zero can be found in [18].

Definition 6. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix A such that

$$d(N(x), N(y)) \leq Ad(x, y), \quad \forall x, y \in X.$$

Theorem 1 (Perov's fixed point theorem, see [26])**.** *. Let (X, d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix A. Then N has a unique fixed point x^* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x^*) \leq A^k(I - A)^{-1} d(x_0, N(x_0)), \quad \forall k \in N.$$

Now, we state Schaefer fixed point theorem type in generalized Banach space.

Theorem 2 ([19])**.** *. Let X be a generalized Banach space and let $G : X \to X$ be completely continuous. Then, either*

(1) *the operator equation $x = Tx$ has a solution, or*
(2) *the set*

$$\mathcal{E} = \{x \in X : x = \lambda N(x), \lambda \in (0, 1)\}$$

is unbounded.

3. Existence and uniqueness of the solution

For a given $T > 0$, let $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. In order to define a solution for problem (1), consider the following space of picewise continuous functions:

$$\begin{aligned} PC(J, \mathbb{R}) = \Big\{ & y : [0, T] \to \mathbb{R},\ y_k \in C(J_k, \mathbb{R}) \text{ for } k = 0, \ldots m, \\ & \text{and there exist } y(t_k^-) and y(t_k^+) \\ & \text{with } y(t_k) = y(t_k^-),\ k = 1, \ldots, m \Big\}, \end{aligned}$$

endowed with the norm

$$\|y\| = \sup_{t \in [0, T]} (|y(t)|).$$

It is not difficult to check that $PC(J, \mathbb{R})$ is a Banach space with norm $\|.\|$. Now, we first define the solution to our problem.

Definition 7. A function $(x, y) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ is said to be a solution of (1) if and only if

$$\begin{cases} x(t) = x_0 + A_\alpha f_1(t, x, y) + B_\alpha \int_0^t f_1(s, x(s), y(s))ds \\ \qquad + \sum_{0 \le t_k \le t} I_k(x(t_k), y(t_k)), \\ y(t) = y_0 + A_\beta f_2(t, x, y) + B_\beta \int_0^t f_2(s, x(s), y(s))ds \\ \qquad + \sum_{0 \le t_k \le t} \overline{I}_k(x(t_k), y(t_k)). \end{cases} \tag{10}$$

First, we will list the following hypotheses which will be imposed in our main theorem.

(H_1) There exist constants $l_i > 0$, $i = 1, \dots, 4$, such that

$$\|f_1(t, x, y) - f_1(t, \overline{x}, \overline{y})\| \le l_1 \|x - \overline{x}\| + l_2 \|y - \overline{y}\|,$$

and

$$\|f_2(t, x, y) - f_2(t, \overline{x}, \overline{y})\| \le l_3 \|x - \overline{x}\| + l_4 \|y - \overline{y}\|,$$

for all $x, \overline{x}, y, \overline{y} \in \mathbb{R}$;

(H_2) There exist constants a_{1k}, a_{2k}, b_{1k}, $b_{2k} \ge 0$, $k = 1, \dots, m$, such that

$$\|I_k(x, y) - I_k(\overline{x}, \overline{y})\| \le a_{1k} \|x - \overline{x}\| + a_{2k} \|y - \overline{y}\|,$$

and

$$\|\overline{I}_k(x, y) - \overline{I}_k(\overline{x}, \overline{y})\| \le b_{1k} \|x - \overline{x}\| + b_{2k} \|y - \overline{y}\|,$$

for all $x, \overline{x}, y, \overline{y} \in \mathbb{R}$.

We will use the Perov fixed point theorem to prove the existence of a solution to the problem (1).

Theorem 3. *Assume that (H_1) - (H_2) are satisfied and the matrix*

$$M = \begin{pmatrix} A_\alpha l_1 + B_\alpha T l_1 + \sum_{k=1}^m a_{1k} & A_\alpha l_2 + B_\alpha T l_2 + \sum_{k=1}^m a_{2k} \\ A_\beta l_3 + B_\beta T l_3 + \sum_{k=1}^m b_{1k} & A_\beta l_4 + B_\beta T l_4 + \sum_{k=1}^m b_{2k} \end{pmatrix} \tag{11}$$

converges to zero and $f_1(\cdot, 0, 0) = f_2(\cdot, 0, 0) = I_k(0, 0) = \overline{I}_k(0, 0) = 0$. Then the problem (1) has unique solution.

Proof. Consider the operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x, y) = (N_1(x, y), N_2(x, y)),$$

where

$$N_1(x,y)(t) = x_0 + A_\alpha f_1(t,x,y) + B_\alpha \int_0^t f_1(s,x(s),y(s))ds + \sum_{0\le t_k \le t} I_k(x(t_k),y(t_k))$$

and

$$N_2(x,y)(t) = y_0 + A_\beta f_2(t,x,y) + B_\beta \int_0^t f_2(s,x(s),y(s))ds + \sum_{0\le t_k \le t} \overline{I}_k(x(t_k),y(t_k)).$$

We show that N was well defined. Given $(x,y) \in PC \times PC$, $t \in [0,T]$, we have

$$\begin{aligned}\|N_1(x,y)\| &\le \|x_0\| + A_\alpha \|f_1(t,x,y)\| + B_\alpha \int_0^t \|f_1(s,x(s),y(s))\|ds \\ &\quad + \sum_{0\le t_k\le t} \|I_k(x(t_k),y(t_k))\| \\ &\le \|x_0\| + A_\alpha(l_1\|x\| + l_2\|y\|) + B_\alpha T[l_1\|x\| + l_2\|y\|] \\ &\quad + \sum_{k=1}^m [\|a_{1k}\|x\| + a_{2k}\|y\|].\end{aligned}$$

Similarly we have

$$\begin{aligned}\|N_2(x,y)\| &\le \|y_0\| + A_\beta(l_3\|x\| + l_4\|y\|) + B_\beta T[l_3\|x\| + l_4\|y\|] \\ &\quad + \sum_{k=1}^m [\|b_{1k}\|x\| + b_{2k}\|y\|].\end{aligned}$$

Thus

$$\begin{pmatrix} \|N_1(x,y)\| \\ \|N_2(x,y)\| \end{pmatrix} \le \begin{pmatrix} \|x_0\| \\ \|y_0\| \end{pmatrix} + \\ + \begin{pmatrix} A_\alpha l_1 + B_\alpha T l_1 + \sum\limits_{k=1}^m a_{1k} & A_\alpha l_2 + B_\alpha T l_2 + \sum\limits_{k=1}^m a_{2k} \\ A_\beta l_3 + B_\beta T l_3 + \sum\limits_{k=1}^m b_{1k} & A_\beta l_4 + B_\beta T l_4 + \sum\limits_{k=1}^m b_{2k} \end{pmatrix} \cdot \begin{pmatrix} \|x\| \\ \|y\| \end{pmatrix}.$$

This implies that N is well defined.

Clearly, fixed points of N are solutions of problem (1). We show that N is a contraction. Let $(x,y),(x,y) \in PC \times PC$. Then $(H_1\ )$ and $(H_2\ )$ imply

$$\begin{aligned}\|N_1(x,y) - N_1(\overline{x},\overline{y})\| &\le A_\alpha \|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})\| \\ &\quad + B_\alpha \int_0^t \|f_1(s,x,y) - f_1(t,\overline{x},\overline{y})\|ds\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0\le t_k\le t} (\|I_k(x,y) - I_k(\overline{x},\overline{y})\| \\
&\le A_\alpha[l_1\|x-\overline{x}\| + l_2\|y-\overline{y}\|] \\
&\quad + B_\alpha T[l_1\|x-\overline{x}\| + l_2\|y-\overline{y}\|] \\
&\quad + \sum_{k=1}^{m}[a_{1k}\|x-\overline{x}\| + a_{2k}\|y-\overline{y}\|] \\
&\le (A_\alpha l_1 + B_\alpha T l_1 + \sum_{k=1}^{m} a_{1k})\|x-\overline{x}\| \\
&\quad + (A_\alpha l_2 + B_\alpha T l_2 + \sum_{k=1}^{m} a_{2k})\|y-\overline{y}\|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|N_2(x,y) - N_2(\overline{x},\overline{y})\| &\le (A_\beta l_3 + B_\beta T l_3 + \sum_{k=1}^{m} b_{1k})\|x-\overline{x}\| \\
&\quad + (A_\beta l_4 + B_\beta T l_4 + \sum_{k=1}^{m} b_{2k})\|y-\overline{y}\|.
\end{aligned}
$$

It follows that

$$
\|N(x,y) - N(\overline{x},\overline{y})\| \le M \begin{pmatrix} \|x-\overline{x}\| \\ \|y-\overline{y}\| \end{pmatrix}, \quad \text{for all } (x,y),(\overline{x},\overline{y}) \in PC\times PC.
$$

Hence, by Theorem 1, the operator N has a unique fixed point which is a solution of problem (1). □

4. EXISTENCE AND COMPACTNESS OF SOLUTION SETS

In this section we prove some existence and compactness results for problem (1) by application of Schaefer fixed point type theorem in generalized Banach spaces. We consider the following hypotheses

(H_3) There exist $c_1, c_2 \ge 0$ such that

$$
\|f_1(t,x,y)\| \le c_1\|x\| + c_2\|y\|, \quad \text{for all} \quad x, y, \in \mathbb{R}.
$$

(H_4) There exist $c_3, c_4 \ge 0$ such that

$$
\|f_2(t,x,y)\| \le c_3\|x\| + c_4\|y\|, \quad \text{for all} \quad x, y, \in \mathbb{R}.
$$

(H_5) There exist constants d_{1k}, d_{2k}, ≥ 0 , $k = 1,\dots,m$, such that

$$
\|I_k(x,y)\| \le d_{1k}\|x\| + d_{2k}\|y\|, \quad \text{for all } x, y, \in \mathbb{R}.
$$

(H_6) There exist constants e_{1k}, $e_{2k} \ge 0$, $k = 1,\dots,m$, such that

$$
\|\overline{I}_k(x,y)\| \le e_{1k}\|x\| + e_{2k}\|y\|, \quad \text{for all } x, y, \in \mathbb{R}.
$$

Now, we give prove of the existence result of problem (1) by using nonlinear alternative of Schaefer fixed point theorem type in generalized Banach space.

Theorem 4. *Assume that the hypotheses (H$_3$)-(H$_6$) hold. Then, the problem (1) has a solution defined on* $[0,T]$. *Moreover, the solution set*

$$S = \{(x,y) \in PC \times PC,\ (x,y) \quad \text{is solution of} \quad (1)\}, \tag{12}$$

is compact.

Proof. Let $N : PC(J,\mathbb{R})\times PC(J,\mathbb{R}) \to PC(J,\mathbb{R})\times PC(J,\mathbb{R})$ be an operator defined in the proof of Theorem 3.

In order to apply theorem 2, we first show that N is completely continuous. The proof will be given in several steps.

Step 1. $N(\cdot,\cdot)$ is continuous. Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ as $n \to \infty$. Then

$$\begin{aligned}\|N_1(x_n,y_n) - N_1(x,y)\| \le A_\alpha \|f_1(t,x_n,y_n) - f_1(t,x,y)\| \\ + B_\alpha \int_0^t \|f_1(s,x_n,y_n) - f_1(t,x,y)\| ds \\ + \sum_{0\le t_k \le t} (\|I_k(x_n,y_n) - I_k(x,y)\|.\end{aligned}$$

Since f_1 and I_k are a continuous functions. Thus

$$\|N_1(x_n,y_n) - N_1(x,y)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, we can get,

$$\|N_2(x_n,y_n) - N_2(x,y)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus N is continuous.

Step 2. N maps bounded sets into bounded sets in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) : \|x\| \le q, \|y\| \le q\}$, we have

$$N(x,y) \le l = (l_1, l_2).$$

Then for each $t \in [0,T]$, we get

$$\begin{aligned}\|N_1(x,y)\| &\le \|x_0\| + A_\alpha \|f_1(t,x,y)\| + B_\alpha \int_0^t \|f_1(s,x(s),y(s))\| ds \\ &\quad + \sum_{0\le t_k\le t} \|I_k(x(t_k), y(t_k))\| \\ &\le \|x_0\| + A_\alpha[c_1\|x\| + c_2\|y\|] + B_\alpha T[c_1\|x\| + c_2\|y\|] \\ &\quad + \sum_{k=1}^{m} [d_{1k}\|x\| + d_{2k}\|y\|]\end{aligned}$$

$$\leq \|x_0\| + q\Big(A_\alpha[c_1+c_2] + B_\alpha T[c_1+c_2] + \sum_{k=1}^{m}[d_{1k}+d_{2k}]\Big)$$
$$:= l_1.$$

Similarly, we have

$$\|N_2(x,y)\| \leq \|y_0\| + q\Big(A_\beta[c_3+c_4] + B_\beta T[c_3+c_4] + \sum_{k=1}^{m}[e_{1k}+e_{2k}]\Big)$$
$$:= l_2.$$

Step 3. N maps bounded sets into equicontinuous sets of $PC([0,T],\mathbb{R}) \times PC([0,T],\mathbb{R})$. Let B_q be a bounded set in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ as in Step 2. Let $r_1, r_2 \in J$, $r_1 < r_2$ and $u \in B_q$. Thus we have

$$\begin{aligned}
&\|N_1(x(r_2),y(r_2)) - N_1(x(r_1),y(r_1))\| \leq \\
&\leq A_\alpha\|f_1(r_2,x(r_2),y(r_2)) - f_1(r_1,x(r_1),y(r_1))\| + \\
&\quad + B_\alpha\int_{r_1}^{r_2}\|f_1(s,x(s),y(s)))\|ds + \sum_{r_1\leq t_k\leq r_2}(\|I_k(x,y)\|.
\end{aligned}$$

This implies that $\|N_1(x(r_2),y(r_2)) - N_1(x(r_1),y(r_1))\| \to 0$ whenever $r_2 \to r_1$. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli, we conclude that N maps B_q into a precompact set in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

Similarly, we have

$$\begin{aligned}
&\|N_2(x(r_2),y(r_2)) - N_2(x(r_1),y(r_1))\| \leq \\
&\leq A_\beta\|f_2(r_2,x(r_2),y(r_2)) - f_2(r_1,x(r_1),y(r_1))\| + \\
&+ \quad B_\beta\int_{r_1}^{r_2}\|f_2(s,x(s),y(s)))\|ds + \sum_{r_1\leq t_k\leq r_2}\|\overline{I}_k(x,y)\|.
\end{aligned}$$

Again, by utilizing the Arzelà-Ascoli theorem we observe that N_2 is completely continuous. Therefore, we get $\|N_2(x(r_2),y(r_2)) - N_2(x(r_1),y(r_1))\| \to 0$ whenever r_2 tends to r_2. Thus, N is completely continuous operator.

Step 4. It remains to show that

$$\mathcal{A} = \Big\{(x(\cdot),y(\cdot)) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) :$$
$$(x(\cdot),y(\cdot)) = \lambda N(x(\cdot),y(\cdot)), \lambda \in (0,1)\Big\}$$

is bounded.

Let $(x,y) \in \mathcal{A}$. Then $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $t \in [0,b]$, we have

$$\begin{aligned}|x(t)| &\le |x_0| + A_\alpha |f_1(t,x(t),y(t))| + B_\alpha \int_0^t |f_1(s,x(s),y(s))|ds \\ &\quad + \sum_{0\le t_k \le t} |I_k(x(t_k),y(t_k))| \\ &\le \|x_0\| + A_\alpha[c_1\|x\| + c_2\|y\|] + B_\alpha T[c_1\|x\| + c_2\|y\|] \\ &\quad + \sum_{k=1}^{m}[d_{1k}\|x\| + d_{2k}\|y\|]\end{aligned}$$

and

$$\begin{aligned}|y(t)| &\le |y_0| + A_\beta |f_2(t,x(t),y(t))| + B_\beta \int_0^t |f_2(s,x(s),y(s))|ds \\ &\quad + \sum_{0\le t_k \le t} |\overline{I}_k(x(t_k),y(t_k))| \\ &\le \|y_0\| + A_\beta[c_3\|x\| + c_4\|y\|] + B_\beta T[c_3\|x\| + c_4\|y\|] \\ &\quad + \sum_{k=1}^{m}[e_{1k}\|x\| + e_{2k}\|y\|].\end{aligned}$$

Therefore

$$\begin{aligned}|x(t)| + |y(t)| &\le C + \Big(A_\alpha c_1 + B_\alpha T c_1 + \sum_{k=1}^{m} d_{1k} + A_\alpha c_3 \\ &\qquad + B_\alpha T c_3 + \sum_{k=1}^{m} e_{1k}\Big)\|x\| \\ &\qquad + \Big(A_\beta c_2 + B_\beta T c_2 + \sum_{k=1}^{m} d_{2k} + A_\beta c_4 \\ &\qquad + B_\beta T c_4 + \sum_{k=1}^{m} e_{2k}\Big)\|y\| \\ &\le C + \max(\gamma_1,\gamma_2)(\|x\| + \|y\|), \quad \text{for all } t\in[0,T],\end{aligned}$$

where

$$C = \|x_0\| + \|y_0\|$$

and

$$\begin{aligned}\gamma_1 &= A_\alpha c_1 + B_\alpha T c_1 + \sum_{k=1}^{m} d_{1k} + A_\alpha c_3 + B_\alpha T c_3 + \sum_{k=1}^{m} e_{1k}, \\ \gamma_2 &= A_\beta c_2 + B_\beta T c_2 + \sum_{k=1}^{m} d_{2k} + A_\beta c_4 + B_\beta T c_4 + \sum_{k=1}^{m} e_{2k}.\end{aligned}$$

Hence

$$\|x\| + \|y\| \leq C + \max(\gamma_1, \gamma_2)(\|x\| + \|y\|).$$

This implies that

$$\|x\| + \|y\| \leq \frac{C}{1 - \max(\gamma_1, \gamma_2)} := K.$$

Consequently

$$\|x\| \leq K \quad \text{and} \quad \|y\| \leq K.$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 2 we deduce that N has a fixed point $(x(\cdot), y(\cdot))$ which is a solution to the problem (1).

Step 5. Compactness of the solution sets. We will show that the set

$$S = \{(x, y) \in PC \times PC,\ (x, y) \text{ is solution of (1)}\}$$

is compact.

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in $S(x_0, y_0)$. For every $n \in \mathbb{N}$, we get

$$x_n(t) = x_0 + A_\alpha f_1(t, x_n, y_n) + B_\alpha \int_0^t f_1(s, x_n(s), y_n(s))ds + \sum_{0\leq t_k \leq t} I_k(x_n(t_k), y_n(t_k)),$$

and

$$y_n(t) = y_0 + A_\beta f_2(t, x_n, y_n) + B_\beta \int_0^t f_2(s, x_n(s), y_n(s))ds + \sum_{0\leq t_k \leq t} \overline{I}_k(x_n(t_k), y_n(t_k)),$$

and set $B = \{(x_n, y_n) : n \in \mathbb{N}\} \subset PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$. From earlier parts of the proof of this theorem, we see that B is bounded and equicontinuous. Then, from the Ascoli-Arzelà theorem, we can conclude that B is compact. Hence, $(x_n, y_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k}, y_{n_k})_{n_k\in\mathbb{N}} \subseteq S$ that converges to some $(x(\cdot), y(\cdot)) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$. Let

$$z_x(t) = x_0 + A_\alpha f_1(t, x, y) + B_\alpha \int_0^t f_1(s, x(s), y(s))ds + \sum_{0\leq t_k \leq t} I_k(x(t_k), y(t_k)),$$

and

$$z_y(t) = y_0 + A_\beta f_2(t, x, y) + B_\beta \int_0^t f_2(s, x(s), y(s))ds + \sum_{0\leq t_k \leq t} \overline{I}_k(x(t_k), y(t_k)).$$

Then

$$
\begin{aligned}
|x_{n_k} - z_x(t)| \leq A_\alpha |f_1(t, x_{n_k}, y_{n_k}) - f_1(t, x, y)| \\
+ B_\alpha \int_0^t |f_1(s, x_{n_k}(s), y_{n_k}(s)) - f_1(s, x(s), y(s))| ds \\
+ \sum_{0 \leq t_k \leq t} |I_k(x_{n_k}(t_k), y_{n_k}(t_k)) - I_k(x(t_k), y(t_k))|,
\end{aligned}
$$

and

$$
\begin{aligned}
|y_{n_k} - z_y(t)| \leq A_\alpha |f_1(t, x_{n_k}, y_{n_k}) - f_2(t, x, y)| \\
+ B_\alpha \int_0^t |f_2(s, x_{n_k}(s), y_{n_k}(s)) - f_1(s, x(s), y(s))| ds \\
+ \sum_{0 \leq t_k \leq t} |\overline{I}_k(x_{n_k}(t_k), y_{n_k}(t_k)) - \overline{I}_k(x(t_k), y(t_k))|.
\end{aligned}
$$

Since $f_1(.,.,.)$, $f_2(.,.,.)$, $I_k(\cdot)$ and $\overline{I}_k(\cdot)$ are continuous functions, then as $n_k \to \infty$, $x_{n_k} \to z_x(t)$ and $y_{n_k} \to z_y(t)$, so

$$
\begin{aligned}
x(t) = x_0 + A_\alpha f_1(t, x, y) + B_\alpha \int_0^t f_1(s, x(s), y(s)) ds \\
+ \sum_{0 \leq t_k \leq t} I_k(x(t_k), y(t_k)),
\end{aligned}
$$

and

$$
\begin{aligned}
y(t) = y_0 + A_\beta f_2(t, x, y) + B_\beta \int_0^t f_2(s, x(s), y(s)) ds \\
+ \sum_{0 \leq t_k \leq t} \overline{I}_k(x(t_k), y(t_k)).
\end{aligned}
$$

Hence, S is compact. □

5. An example

In this section we present an example to illustrate the usefulness and applicability of our results. Consider the following differential equation system

$$(13)\qquad \begin{cases} D^{\frac{1}{2}}x(t) = \dfrac{\sin(x(t)+y(t))}{16(\ln(t+1)+1)} + 1, \quad t \in [0,1],\ t \neq \frac{3}{4}, \\ D^{\frac{1}{2}}y(t) = \dfrac{\sin x(t) + \sin y(t)}{18\ln(t\exp(t^2)+1)}, \quad t \in [0,1],\ t \neq \frac{3}{4}, \\ \Delta x(\frac{3}{4}) = \dfrac{1}{5}\sin(x(\frac{3}{4}) + y(\frac{3}{4})), \\ \Delta y(\frac{3}{4}) = \dfrac{1}{7}\left[\cos(x(\frac{3}{4})) + \cos(y(\frac{3}{4}))\right], \\ x(0) = \sqrt{3}, \\ y(0) = \sqrt{2}. \end{cases}$$

Here

$$f_1(t,x,y) = \frac{\sin(x(t)+y(t))}{16(\ln(t+1)+1)} + 1,$$

$$f_2(t,x,y) = \frac{\sin x(t) + \sin y(t)}{18\ln(t\exp(t^2)+1)}.$$

Clearly, the map $t \mapsto f_1(t,x,y)$ is jointly continuous for all $x, y \in \mathbb{R}$. The same for the map f_2. Also the maps $x \mapsto f_1(t,x,y)$ and $y \mapsto f_2(t,x,y)$ are continuous for all $t \in J$. Firstly, we show that f_1, f_2,I_1 and $\widetilde{I}_1$ are Lipschitz functions. Indeed, let $x, y \in \mathbb{R}$, then

$$\begin{aligned} |f_1(t,x,y) - f_1(t,\widetilde{x},\widetilde{y})| &= \left| \frac{\sin(x(t)+y(t))}{16(\ln(t+1)+1)} - \frac{\sin(\widetilde{x}(t)+\widetilde{y}(t))}{16(\ln(t+1)+1)} \right| \\ &\leq \frac{1}{16}|x-\widetilde{x}| + \frac{1}{16}|y-\widetilde{y}|. \end{aligned}$$

Then

$$|f_1(t,x,y) - f_1(t,\widetilde{x},\widetilde{y})| \leq \frac{1}{16}|x-\widetilde{x}| + \frac{1}{16}|y-\widetilde{y}|.$$

Analogously for the function f_2, we get

$$|f_2(t,x,y) - f_2(t,\widetilde{x},\widetilde{y})| \leq \frac{1}{18}|x-\widetilde{x}| + \frac{1}{18}|y-\widetilde{y}|,$$

$$\left| I_1\left(x\left(\frac{3}{4}\right), y\left(\frac{3}{4}\right)\right) - I_1\left(\widetilde{x}\left(\frac{3}{4}\right), \widetilde{y}\left(\frac{3}{4}\right)\right) \right| \leq \frac{1}{5}|x-\widetilde{x}| + \frac{1}{5}|y-\widetilde{y}|,$$

$$\left| \widetilde{I}_1\left(x\left(\frac{3}{4}\right), y\left(\frac{3}{4}\right)\right) - \widetilde{I}_1\left(\widetilde{x}\left(\frac{3}{4}\right), \widetilde{y}\left(\frac{3}{4}\right)\right) \right| \leq \frac{1}{7}|x-\widetilde{x}| + \frac{1}{7}|y-\widetilde{y}|.$$

Therefore the matrix

$$M = \begin{pmatrix} 0.26 & 0.26 \\ 0.20 & 0.20 \end{pmatrix}$$

converges to zero, since its eigenvalues are $\lambda_1 = 0,46 < 1$, $\lambda_2 = 0 < 1$. From Theorem 3, the problem (13) has a unique solution.

6. Conclusion

In this paper, we used the Perov and Schaefer fixed point theorems type in generalized Banach space to achieve the necessary criteria for the existence and uniqueness of the solution of considered coupled system of Caputo-Fabrizio fractional impulsive differential equations. Similarly, under particular assumptions and conditions, we have established the compactness of the solution sets of the system.

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