

Impulsive fractional differential inclusions with state-dependent delay

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ABSTRACT. In this paper, we prove the existence of mild solution for impulsive fractional inclusions with state-dependent delay in Banach spaces. Our study is based on the nonlinear alternative of Leray-Schauder type for multivalued maps due to Martelli. An example is provided to illustrate the main result.

1. INTRODUCTION

Fractional differential equations have been proved to be one of the most effective tools in the modeling of many phenomena in various fields of physics, mechanics, chemistry, engineering, etc. For more details, see [1, 2, 31, 34, 41, 42, 48]. In order to describe various real-world problems in physical and engineering science subject to abrupt changes at certain instants during the evolution process, impulsive differential equations have been used to model the systems. The theory of impulsive differential equations is an important branch of differential equations, which has an extensive physical background [9, 12, 33].

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last few years, see for instance [4, 10, 23, 24, 25, 26, 27, 28, 44, 47] and the references therein. The literature devoted to this subject is concerned fundamentally with first-order functional differential equations for which the state belongs to some finite dimensional space, see among other works, [13, 15, 17, 22, 32, 36, 46]. The problem of the existence and uniqueness of

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solutions for fractional differential equations with delay was recently studied by Maraaba *et al.* in [37, 38]. In [5], the authors provide sufficient conditions for the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay, where as in [7, 40] we investigate the existence and controllability results for a class of impulsive fractional evolution equations with state-dependent delay.

El-Sayed and Ibrahim initiated the study of fractional differential inclusions in [19]. Recently several qualitative results for fractional differential inclusion several results were obtained in [14]. In [6] we establish sufficient conditions for the existence of mild solutions for fractional integro-differential inclusions with state-dependent delay in Banach spaces.

In this work we establish the existence of mild solutions for the class of impulsive fractional inclusions with state-dependent delay described by the form

$$(1) \quad \begin{aligned} D_t^\alpha x(t) &\in Ax(t) + F(t, x_{\rho(t, x_t)}, x(t)), \quad t \in J_k = (t_k, t_{k+1}], \quad k = 0, 1, \dots, m, \\ \Delta x(t_k) &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(t) &= \phi(t), \quad t \in (-\infty, 0], \end{aligned}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of an α -resolvent family $(S_\alpha(t))_{t \geq 0}$, $F : J \times \mathcal{B} \times E \rightarrow \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of the separable Banach space $(E, \|\cdot\|)$) and $J = [0, T]$, $T > 0$, and $\phi \in \mathcal{B}$ with $\phi(0) = 0$. Here, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k : E \rightarrow E$, $k = 1, 2, \dots, m$, are maps, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \rightarrow 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0} x(t_k - h)$ represent the right and the left limit of $x(t)$ at $t = t_k$, respectively. We denote by x_t the element of \mathcal{B} defined by $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$. Here x_t represents the history of the state from $-\infty$ up to the present time t . We assume that the histories x_t belongs to some abstract phase space \mathcal{B} , to be specified later, and $\phi \in \mathcal{B}$.

2. PRELIMINARIES

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$C = C(J, E)$ denotes the Banach space of continuous functions from J into E with the norm

$$\|y\|_C = \sup \{ \|y(t)\| : t \in J \}.$$

Let $L(E)$ be the Banach space of all linear and bounded operators on E .

Let $L^1(J, E)$ be the space of E -valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

Denote by $P_{cl}(E) = \{Y \in P(E) : Y \text{ closed}\}$, $P_b(E) = \{Y \in P(E) : Y \text{ bounded}\}$, $P_{cp}(E) = \{Y \in P(E) : Y \text{ compact}\}$, $P_{cp,c}(E) = \{Y \in P(E) : Y \text{ compact, convex}\}$, $P_{cl,c}(E) = \{Y \in P(E) : Y \text{ closed, convex}\}$.

A multivalued map $G : E \rightarrow P(E)$ is convex (closed) valued if $G(E)$ is convex (closed) for all $x \in E$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup_{x \in B} \{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on E if for each $x_0 \in E$ the set $G(x_0)$ is a nonempty, closed subset of E , and if for each open set U of E containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(E)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). For more details on multivalued maps see the books of Deimling [18], Górniewicz [20] and Hu and Papageorgiou [30].

Definition 2.1. The multivalued map $F : J \times \mathcal{B} \times E \rightarrow \mathcal{P}(E)$ is said to be Carathéodory if

- (i) $t \mapsto F(t, x, y)$ is measurable for each $(x, y) \in \mathcal{B} \times E$;
- (ii) $(x, y) \mapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in J$.

Definition 2.2. Let $\alpha > 0$ and $f \in L^1(J, E)$. The Riemann–Liouville integral is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [16].

Definition 2.3. [42]. The Caputo derivative of order α for a function $f : J \rightarrow E$ is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n-1 \leq \alpha < n.$$

If $0 \leq \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Obviously, The Caputo derivative of a constant is equal to zero.

In order to defined the mild solution of the problems (1) we recall the following definition.

Definition 2.4. A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi], M > 0$, such that the following two conditions are satisfied:

- (1) $\sum_{(\theta, \omega)} := \{\lambda \in C : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$ ($\rho(A)$ being the resolvent set of A).
- (2) $\|R(\lambda, A)\|_{L(E)} \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \sum_{(\theta, \omega)}$.

Sectorial operators are well studied in the literature. For details see [24].

Definition 2.5. [8] if A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of an α -resolvent

family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \rightarrow L(E)$ such that $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, \quad x \in E.$$

In this case, $S_\alpha(t)$ is called the α -resolvent family generated by A .

Definition 2.6. (see Definition 2.1 in [3]) if A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \rightarrow L(E)$ such that $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, \quad x \in E,$$

in this case, $S_\alpha(t)$ is called the solution operator generated by A . For more details see [36, 43].

In this paper, we will employ an axiomatic definition for the phase space \mathcal{B} which is similar to those introduced by Hale and Kato [21]. Specifically, \mathcal{B} will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1): If $x : (-\infty, T] \rightarrow E$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and

$$\|x(t)\| \leq C \|x_t\|_{\mathcal{B}},$$

where $C > 0$ is a constant.

(A2): There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s \in [0, t]} \|x(s)\| + C_2(t) \|x_0\|_{\mathcal{B}},$$

for $t \in [0, T]$ and x as in (A1).

(A3): The space \mathcal{B} is complete.

Example 2.1. The phase space $C_r \times L^p(g, X)$.

Let $r \geq 0, 1 \leq p < \infty$, and let $g : (-\infty, -r) \rightarrow \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [29]. Briefly, this means that g is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \rightarrow X$, such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|^p d\theta \right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms **(A1)**, **(A2)**, **(A3)**. Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X)$, $H = 1$, $M(t) = \Lambda(-t)^{\frac{1}{2}}$, $K(t) = 1 + \left(\int_{-r}^0 g(\tau) d\tau \right)^{\frac{1}{2}}$, for $t \geq 0$ (see [29], Theorem 1.3.8 for details).

Let $S_{F \circ x}$ be a set defined by

$$S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x_{\rho(t, x_t)}, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.1. [35]. *Let $F : J \times \mathcal{B} \times E \rightarrow P_{cp,c}(E)$ be an L^1 -Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(J, E)$ to $C(J, E)$, then the operator*

$$\begin{aligned} \Psi \circ S_F : C(J, E) &\rightarrow P_{cp,c}(C(J, E)), \\ x &\mapsto (\Psi \circ S_F)(x) := \Psi(S_{F \circ x}) \end{aligned}$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

The following is the multivalued version of the fixed-point theorem due to Martelli [39].

Lemma 2.2. *Let X be a Banach space, and $N : X \rightarrow P_{cp,cv}(X)$ be an upper semicontinuous and completely continuous multivalued map. If the set*

$$D = \{u \in X : \lambda u \in Nu \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. MAIN RESULTS

Before going further we need the following lemma ([45]).

Lemma 3.1. *Consider the Cauchy problem*

$$(2) \quad \begin{aligned} D_t^\alpha x(t) &= Ax(t) + F(t), \quad 0 < \alpha < 1, \\ x(0) &= x_0, \end{aligned}$$

if F satisfies the uniform Holder condition with exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem (2) is given by

$$x(t) = T_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)F(s)ds,$$

where

$$\begin{aligned} T_\alpha(t) &= \frac{1}{2\pi i} \int_{\hat{B}_r} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda, \\ S_\alpha(t) &= \frac{1}{2\pi i} \int_{\hat{B}_r} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda, \end{aligned}$$

\hat{B}_r denotes the Bromwich path. $S_\alpha(t)$ is called the α -resolvent family and $T_\alpha(t)$ is the solution operator, generated by A .

Theorem 3.1. [11, 45] *If $\alpha \in (0, 1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$, then for any $x \in E$ and $t > 0$, we have*

$$\|T_\alpha(t)\|_{L(E)} \leq Me^{\omega t} \text{ and } \|S_\alpha(t)\|_{L(E)} \leq Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0, \quad \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0 \leq t \leq T} \|T_\alpha(t)\|_{L(E)}, \quad \widetilde{M}_s = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{\alpha-1}),$$

so we have

$$\|T_\alpha(t)\|_{L(E)} \leq \widetilde{M}_T, \quad \|S_\alpha(t)\|_{L(E)} \leq t^{\alpha-1} \widetilde{M}_s.$$

Let us consider the set of functions

$$\mathcal{B}_1 = \left\{ x : (-\infty, T] \rightarrow E \text{ such that } x|_J \in C(J, E) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), x_0 = \phi, k = 1, 2, \dots, m \right\},$$

where $x|_J$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$.

From Lemma 3.1, we can define the mild solution of system (1) as follows:

Definition 3.1. A function $x : (-\infty, T] \rightarrow E$ is called a mild solution of (1) if the following holds: $x_0 = \phi \in \mathcal{B}$ on $(-\infty, 0]$ with $\phi(0) = 0$, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$, the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous and there exists $v(\cdot) \in L^1(J_k, E)$, such that $v(t) \in F(t, x_t, x(t))$ a.e. $t \in [0, T]$, and x satisfies the following integral equation:

$$(3) \quad x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T]. \end{cases}$$

Set

$$\mathcal{R}(\rho^-) = \{ \rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0 \}.$$

We always assume that $\rho : J \times \mathcal{B} \rightarrow (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_φ) The function $t \rightarrow \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into \mathcal{B} and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \rightarrow (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.1. The condition (H_φ), is frequently verified by continuous and bounded functions. For more details, see for instance [29].

Remark 3.2. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^* = \sup_{s \in J} C_1(s) \text{ and } C_2^* = \sup_{s \in J} C_2(s).$$

Lemma 3.2. [28] *If $x : (-\infty, T] \rightarrow X$ is a function such that $x_0 = \phi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Let us introduce the following hypotheses:

(H1) The semigroup $S(t)$ is compact for $t > 0$.

(H2) The multivalued map $F : J \times \mathcal{B} \times E \rightarrow \mathcal{P}(E)$ is Carathéodory with compact convex values.

(H3) There exist a function $\mu \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \rightarrow (0, +\infty)$ such that

$$\|F(t, x, y)\| \leq \mu(t)\psi(\|x\|_{\mathcal{B}} + \|y\|_E), \quad (t, x, y) \in J \times \mathcal{B} \times E,$$

with

$$(4) \quad \omega_2 \int_0^T \mu(s) ds < \int_{v(0)}^{+\infty} \frac{du}{\psi(u)},$$

where

$$\omega_2 = \frac{\beta_2 \widetilde{M}_s T^\alpha}{\alpha(1 - \widetilde{M}_T)},$$

$$v(0) = \omega_1 = \beta_1 + \frac{\beta_2 \widetilde{M}_T \Omega}{1 - \widetilde{M}_T},$$

and

$$\beta_1 = (C_2^* + L^\phi) \|\phi\|_{\mathcal{B}}, \quad \beta_2 = C_1^* + 1.$$

(H4) The functions $I_k : E \rightarrow E$ are continuous, and send bounded sets into bounded sets.

Set

$$\Omega = \max_{1 \leq k \leq m} \sup \{ \|I_k(x)\|, x \in B \},$$

where B is a bounded set of E .

Theorem 3.2. *Assume that assumptions (H_φ) , $(H1) - (H4)$ hold. If $\widetilde{M}_T < 1$, then the IVP (1) has a mild solution on $(-\infty, T]$.*

Proof. We transform the problem (1) into a fixed-point problem. Consider the multivalued operator $N : \mathcal{B}_1 \rightarrow \mathcal{P}(\mathcal{B}_1)$ defined by $N(h) = \{h \in \mathcal{B}_1\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T]. \end{cases}$$

It is clear that the fixed points of the operator N are mild solutions of the problem (1). Let us define $y(\cdot) : (-\infty, T] \rightarrow E$ as

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ 0, & t \in J. \end{cases}$$

Then $y_0 = \phi$. For each $z \in C(J, E)$ with $z(0) = 0$, we denote by \bar{z} the function defined by

$$\bar{z}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ z(t), & t \in J. \end{cases}$$

Let $x_t = y_t + \bar{z}_t, t \in J$. It is easy to see that $x(\cdot)$ satisfies (3) if and only if $z_0 = 0$ and for $t \in J$, we have

$$z(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1) [y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t-t_m) [y(t_m^-) + \bar{z}(t_m^-) + I_m(y(t_m^-) + \bar{z}(t_m^-))] \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T], \end{cases}$$

where $v(s) \in S_{F, y_\rho(s, y_s + \bar{z}_s) + \bar{z}_\rho(s, y_s + \bar{z}_s)}$. Let

$$\mathcal{B}_2 = \{z \in \mathcal{B}_1 \text{ such that } z_0 = 0\}.$$

For any $z \in \mathcal{B}_2$, we have

$$\begin{aligned} \|z\|_{\mathcal{B}_2} &= \sup_{t \in J} \|z(t)\| + \|z_0\|_{\mathcal{B}} \\ &= \sup_{t \in J} \|z(t)\|. \end{aligned}$$

Thus $(\mathcal{B}_2, \|\cdot\|_{\mathcal{B}_2})$ is a Banach space. We define the operator $P : \mathcal{B}_2 \rightarrow \mathcal{P}(\mathcal{B}_2)$ by: $P(z) = \{h \in \mathcal{B}_2\}$ with

$$h(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1) [y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t-t_m) [y(t_m^-) + \bar{z}(t_m^-) + I_m(y(t_m^-) + \bar{z}(t_m^-))] \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T], \end{cases}$$

where $v(s) \in S_{F, y_\rho(s, y_s + \bar{z}_s) + \bar{z}_\rho(s, y_s + \bar{z}_s)}$.

It is clear that the operator N has a fixed point if and only if P has a fixed point. So let us prove that P has a fixed point. We shall show that the operators P satisfy all conditions of Lemma 2.2. For better readability, we break the proof into a sequence of steps.

Step 1: P is convex for each $z \in \mathcal{B}_2$.

Indeed, if h_1 and h_2 belong to $P(z)$, then there exist $v_1, v_2 \in S_{F, y + \bar{z}}$ such that, for $t \in J$, we have

$$h_i(t) = \begin{cases} \int_0^t S_\alpha(t-s)v_i(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1) [y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)v_i(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t-t_m) [y(t_m^-) + \bar{z}(t_m^-) + I_m(y(t_m^-) + \bar{z}(t_m^-))] \\ + \int_{t_m}^t S_\alpha(t-s)v_i(s)ds, & t \in (t_m, T]. \end{cases} \quad (i = 1, 2)$$

Let $d \in [0, 1]$. Then for each $t \in [0, t_1]$, we get

$$dh_1(t) + (1 - d)h_2(t) = \int_0^t S_\alpha(t - s) [dv_1(s) + (1 - d)v_2(s)] ds.$$

Similarly, for any $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned} dh_1(t) + (1 - d)h_2(t) &= T_\alpha(t - t_i) [y(t_i^-) + \bar{z}(t_i^-) + I_i(y(t_i^-) + \bar{z}(t_i^-))] \\ &+ \int_{t_i}^t S_\alpha(t - s) [dv_1(s) + (1 - d)v_2(s)] ds \end{aligned}$$

Since $S_{F, y_\rho(s, y_s + \bar{z}_s) + \bar{z}_\rho(s, y_s + \bar{z}_s)}$ is convex (because F has convex values), we have

$$dh_1 + (1 - d)h_2 \in P(z).$$

Step 2: P maps bounded sets into bounded sets in \mathcal{B}_2 .

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant ℓ such that for each $z \in B_r = \{z \in \mathcal{B}_2 : \|z\|_{\mathcal{B}_2} \leq r\}$, we have $\|P(z)\|_{\mathcal{B}_2} \leq \ell$. Let $h \in P(z)$, and using (H2) we have for each $t \in [0, t_1]$,

$$\begin{aligned} \|h(t)\| &\leq \int_0^t \|S_\alpha(t - s)\| \|v(s)\| ds \\ &\leq \widetilde{M}_S \int_0^t (t - s)^{\alpha-1} \mu(\tau) \psi(\|y_\rho(s, y_s + \bar{z}_s) + \bar{z}_\rho(s, y_s + \bar{z}_s)\| + \|y(s) + \bar{z}(s)\|) ds \\ &\leq \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \int_0^t \mu(s) ds \\ &\leq \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \|\mu\|_{L^1}. \end{aligned}$$

Moreover, when $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have the estimate

$$\begin{aligned} \|h(t)\| &\leq \|T_\alpha(t - t_i) [z(t_i^-) + I_i(z(t_i^-))]\| + \int_{t_i}^t \|S_\alpha(t - s)\| \|v(s)\| ds \\ &\leq \widetilde{M}_T(r + \Omega) \\ &+ \widetilde{M}_S \int_{t_i}^t (t - s)^{\alpha-1} \mu(\tau) \psi(\|y_\rho(s, y_s + \bar{z}_s) + \bar{z}_\rho(s, y_s + \bar{z}_s)\| + \|y(s) + \bar{z}(s)\|) ds \\ &\leq \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi((C_2^* + L^\phi) L^\phi \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \|\mu\|_{L^1} \\ &\leq \ell. \end{aligned}$$

Hence $P(B_r)$ is bounded.

Step 3: P maps bounded sets into equicontinuous sets of \mathcal{B}_2 .

Let $\tau_1, \tau_2 \in [0, t_1]$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\leq \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\| \|v(s)\| ds \\ &+ \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\| \|v(s)\| ds \\ &\leq Q_1 + Q_2, \end{aligned}$$

where

$$\begin{aligned} Q_1 &= \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\| \|v(s)\| ds \\ &\leq \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \|\mu\|_{L^1} \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\| ds. \end{aligned}$$

Since $\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} \leq 2\widetilde{M}_s(t_1 - s)^{\alpha-1} \in L^1(J, \mathbb{R}_+)$ for $s \in [0, t_1]$ and $\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\| \rightarrow 0$ as $\tau_1 \rightarrow \tau_2$. This implies that

$$\lim_{\tau_1 \rightarrow \tau_2} Q_1 = 0.$$

$$\begin{aligned} Q_2 &= \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\| \|v(s)\| ds \\ &\leq \frac{\widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \psi((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \|\mu\|_{L^1}. \end{aligned}$$

Hence, we deduce that

$$\lim_{\tau_1 \rightarrow \tau_2} Q_2 = 0.$$

Similarly, for $\tau_1, \tau_2 \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\leq \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\| [\|z(t_i^-)\| + \|I_i(z(t_i^-))\|] + Q_1 + Q_2 \\ &\leq \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\| (r + \Omega) + Q_1 + Q_2. \end{aligned}$$

Since $T_\alpha(t)$ is also strongly continuous, so $\|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\| \rightarrow 0$ as $\tau_1 \rightarrow \tau_2$. Thus, from the above inequalities, we have

$$\lim_{\tau_1 \rightarrow \tau_2} \|h(\tau_2) - h(\tau_1)\| = 0.$$

So, $P(B_r)$ is equicontinuous.

Step 4: The set $(PB_r)(t)$ is relatively compact for each $t \in J$, where

$$(PB_r)(t) = \{h(t) : h \in P(B_r)\}.$$

For all $t \in [0, t_1]$, by the strong continuity of $S_\alpha(\cdot)$ and conditions (H1), (H2), that the set $\{S_\alpha(t - s)v(s), t, s \in [0, t_1]\}$ is relatively compact in E . Moreover, from the mean value theorem for the Bochner integral, we obtain

$$(PB_r)(t) \in \overline{t\text{conv}\{S_\alpha(t - s)v(s) : s \in [0, t_1]\}}.$$

On the other hand, for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, using the continuity of the operator $T_\alpha(\cdot)$, it follows that $(PB_r)(t)$ is relatively compact in E , for every $t \in [0, T]$. As a consequence of Step 2 to 3 together with Arzelá-Ascoli theorem we can conclude that P is completely continuous.

Step 5: P has a closed graph.

Suppose that $z_n \rightarrow z_*$, $h_n \in P(z_n)$ with $h_n \rightarrow h_*$. We claim that $h_* \in P(z_*)$. In fact, the assumption $h_n \in P(z_n)$ implies that there exists $v_n \in S_{F, y_n \rho(s, y_{ns} + \bar{z}_{ns}) + \bar{z}_n \rho(s, y_{ns} + \bar{z}_{ns})}$ such that, for each $t \in [0, t_1]$,

$$h_n(t) = \int_0^t S_\alpha(t - s)v_n(s) ds.$$

We will show that there exists $v_* \in S_{F, y_* \rho(s, y_{**} + \bar{z}_{**}) + \bar{z}_* \rho(s, y_{**} + \bar{z}_{**})}$ such that, for each $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds.$$

Consider the following linear continuous operator:

$$\begin{aligned} \Upsilon : L^1([0, t_1], E) &\longrightarrow C([0, t_1], E) \\ v &\longmapsto (\Upsilon v)(t) = \int_0^t S_\alpha(t-s)v(s)ds. \end{aligned}$$

By virtue of Lemma 2.1, we know that $\Upsilon \circ S_F$ is a closed graph operator. Moreover, for every $t \in [0, t_1]$, we get

$$h_n(t) \in \Upsilon(S_{F, y_n \rho(s, y_{ns} + \bar{z}_{ns}) + \bar{z}_n \rho(s, y_{ns} + \bar{z}_{ns})}).$$

Since $z_n \rightarrow z_*$ and $h_n \rightarrow h_*$, it follows, that for every $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds,$$

for some $v_* \in S_{F, y_* \rho(s, y_{**} + \bar{z}_{**}) + \bar{z}_* \rho(s, y_{**} + \bar{z}_{**})}$.

Similarly, for any $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned} h_n(t) &= T_\alpha(t-t_i) [y_n(t_i^-) + \bar{z}_n(t_i^-) + I_i(y_n(t_i^-) + \bar{z}_n(t_i^-))] \\ &+ \int_{t_i}^t S_\alpha(t-s)v_n(s)ds. \end{aligned}$$

We must prove that there exists $v_* \in S_{F, y_* \rho(s, y_{**} + \bar{z}_{**}) + \bar{z}_* \rho(s, y_{**} + \bar{z}_{**})}$ such that, for each $t \in (t_i, t_{i+1}]$,

$$\begin{aligned} h_*(t) &= T_\alpha(t-t_i) [y_*(t_i^-) + \bar{z}_*(t_i^-) + I_i(y_*(t_i^-) + \bar{z}_*(t_i^-))] \\ &+ \int_{t_i}^t S_\alpha(t-s)v_*(s)ds. \end{aligned}$$

Now, for every $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned} &\left\| \left(h_n(t) - T_\alpha(t-t_i) [y_n(t_i^-) + \bar{z}_n(t_i^-) + I_i(y_n(t_i^-) + \bar{z}_n(t_i^-))] \right) \right. \\ &\quad \left. - \left(h_*(t) - T_\alpha(t-t_i) [y_*(t_i^-) + \bar{z}_*(t_i^-) + I_i(y_*(t_i^-) + \bar{z}_*(t_i^-))] \right) \right\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consider the linear continuous operator:

$$\begin{aligned} \Upsilon : L^1((t_i, t_{i+1}], E) &\longrightarrow C((t_i, t_{i+1}], E) \\ v &\longmapsto (\Upsilon v)(t) = \int_{t_i}^t S_\alpha(t-s)v(s)ds. \end{aligned}$$

From Lemma 2.1, it follows that $\Upsilon \circ S_F$ is a closed graph operator. Also, from the definition of Υ , we have that, for every $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$,

$$\begin{aligned} &\left(h_n(t) - T_\alpha(t-t_i) [y_n(t_i^-) + \bar{z}_n(t_i^-) + I_i(y_n(t_i^-) + \bar{z}_n(t_i^-))] \right) \\ &\in \Upsilon(S_{F, y_n \rho(s, y_{ns} + \bar{z}_{ns}) + \bar{z}_n \rho(s, y_{ns} + \bar{z}_{ns})}). \end{aligned}$$

Since $z_n \rightarrow z_*$, for some $v_* \in S_{F, y_* \rho(s, y_{**} + \bar{z}_{**}) + \bar{z}_* \rho(s, y_{**} + \bar{z}_{**})}$ it follows that, for every $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} h_*(t) &= T_\alpha(t - t_i) [y_*(t_i^-) + \bar{z}_*(t_i^-) + I_i(y_*(t_i^-) + \bar{z}_*(t_i^-))] \\ &\quad + \int_{t_i}^t S_\alpha(t - s)v_*(s)ds. \end{aligned}$$

Hence the multivalued operator P is upper semi-continuous.

Step 6: A priori bounds.

Now it remains to show that the set

$$\mathcal{E} = \{z \in \mathcal{B}_2 : \lambda z \in Pz, \text{ for some } \lambda > 1\}$$

is bounded.

Let $z \in \mathcal{E}$ be any element, then there exists $v \in S_{F, y \rho(s, y_s + \bar{z}_s) + \bar{z} \rho(s, y_s + \bar{z}_s)}$ such that

$$z(t) = \lambda^{-1} \begin{cases} \int_0^t S_\alpha(t - s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t - t_1) [y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ \quad + \int_{t_1}^t S_\alpha(t - s)v(s)ds, & t \in (t_1, t_2]; \\ \vdots \\ T_\alpha(t - t_m) [y(t_m^-) + \bar{z}(t_m^-) + I_m(y(t_m^-) + \bar{z}(t_m^-))] \\ \quad + \int_{t_m}^t S_\alpha(t - s)v(s)ds, & t \in (t_m, T]. \end{cases}$$

Then for each $t \in [0, t_1]$ we have,

$$\begin{aligned} \|z(t)\| &\leq \int_0^t \|S_\alpha(t - s)\| \|v(s)\| ds \\ &\leq \widetilde{M}_s \int_0^t (t - s)^{\alpha-1} \|v(s)\| ds, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we get

$$\begin{aligned} \|z(t)\| &\leq \|T_\alpha(t - t_i)\| (\|z(t_i^-)\| + \|I_i(z(t_i^-))\|) + \int_{t_i}^t \|S_\alpha(t - s)\| \|v(s)\| ds \\ &\leq \widetilde{M}_T \|z(t_i^-)\| + \widetilde{M}_T \Omega + \widetilde{M}_s \int_0^t (t - s)^{\alpha-1} \|v(s)\| ds. \end{aligned}$$

Then for all $t \in [0, T]$, we have

$$\begin{aligned} \|z(t)\| &\leq \frac{\widetilde{M}_T \Omega}{1 - \widetilde{M}_T} + \frac{\widetilde{M}_s T^\alpha}{\alpha(1 - \widetilde{M}_T)} \int_0^t \mu(s) \psi((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) \|z(s)\|) ds \\ &\leq \frac{\widetilde{M}_T \Omega}{1 - \widetilde{M}_T} + \frac{\widetilde{M}_s T^\alpha}{\alpha(1 - \widetilde{M}_T)} \int_0^t \mu(s) \psi(\beta_1 + \beta_2 \|z(t)\|) ds. \end{aligned}$$

Then

$$\begin{aligned} \beta_1 + \beta_2 \|z(t)\| &\leq \beta_1 + \frac{\beta_2 \widetilde{M}_T \Omega}{1 - \widetilde{M}_T} + \frac{\beta_2 \widetilde{M}_s T^\alpha}{\alpha(1 - \widetilde{M}_T)} \int_0^t \mu(s) \psi(\beta_1 + \beta_2 \|z(s)\|) ds \\ &\leq \omega_1 + \omega_2 \int_0^t [\mu(s) \psi(\beta_1 + \beta_2 \|z(s)\|)] ds. \end{aligned}$$

Let

$$m(t) := \sup \{ \beta_1 + \beta_2 \|z(s)\| : 0 \leq s \leq t \}, \quad t \in J.$$

By the previous inequality, we have

$$m(t) \leq \omega_1 + \omega_2 \int_0^t [\mu(s)\psi(m(s))] ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$m(t) \leq v(t) \quad \text{for all } t \in J,$$

with

$$v(0) = \omega_1,$$

and

$$v'(t) = \omega_2 \mu(t) \psi(m(t)), \quad \text{a.e. } t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \omega_2 \mu(t) \psi(v(t)), \quad \text{a.e. } t \in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))} ds \leq \omega_2 \int_0^t \mu(s) ds.$$

By a change of variable we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s) ds.$$

Using the condition (4), this implies that for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s) ds \leq \omega_2 \int_0^T \mu(s) ds < \int_{v(0)}^{+\infty} \frac{du}{\psi(u)}.$$

Thus, for every $t \in J$, there exists a constant Λ such that $v(t) \leq \Lambda$ and hence $m(t) \leq \Lambda$. Since $\|z\|_{\mathcal{B}_2} \leq m(t)$, we have $\|z\|_{\mathcal{B}_2} \leq \Lambda$.

This shows that the set \mathcal{E} is bounded. As a consequence of the Lemma 2.2, we deduce that the operator P has a fixed point which gives rise to a mild solution of the problem (1). □

4. AN EXAMPLE

We consider the impulsive fractional integro-differential problem:

$$\begin{aligned} \frac{\partial_t^q}{\partial t^q} v(t, \zeta) &\in \frac{\partial^2}{\partial \zeta^2} v(t, \zeta) + \int_{-\infty}^s \eta(s, \tau - s, \zeta) G(\tau, v(\tau - \sigma(v(\tau, 0))), \zeta) d\tau ds, \\ (5) \quad v(t, 0) &= v(t, \pi) = 0 \\ v(t, \zeta) &= v_0(\theta, \zeta), \quad -\infty < \theta \leq 0 \\ \Delta v(t_k)(\zeta) &= \int_{-\infty}^{t_k} p_k(t_k - y) dy \cos(v(t_k)(\zeta)), \end{aligned}$$

where $0 < q < 1, t \in [0, T], \sigma \in C(\mathbb{R}, [0, T]), \zeta \in [0, \pi], p_k : \mathbb{R} \rightarrow \mathbb{R}, k = 1, 2, \dots, m$, and $G : [0, T] \times \mathcal{B} \rightarrow P(\mathbb{R})$ is an u.s.c. multivalued map with compact convex values.

Set $E = L^2([0, \pi])$ and define A by

$$D(A) = \{u \in E : u'' \in E, u(0) = u(\pi) = 0\},$$

$$Au = u''.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(S(t))_{t \geq 0}$ on E . For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } E \right\}$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta} |\phi(\theta)| : \theta \leq 0\}.$$

Notice that the phase space \mathcal{B}_γ satisfies axioms **(A1)** – **(A3)**.

For $t \in [0, T]$, $\zeta \in [0, \pi]$ and $\varphi \in \mathcal{B}_\gamma$ we set

$$\begin{aligned} x(t)(\zeta) &= v(t, \zeta), \\ \phi(\theta)(\zeta) &= v_0(\theta, \zeta), \\ F(t, \varphi, x(t))(\zeta) &= \int_{-\infty}^0 \eta(t, \theta, \zeta) G(t, \varphi(0, \zeta)) d\theta, \\ \rho(t, \varphi) &= t - \sigma(\varphi(0, 0)), \\ I_k(x(t_k^-))(\zeta) &= \int_{-\infty}^0 p_k(t_k - y) dy \cos(x(t_k)(\zeta)), \quad k = 1, 2, \dots, m. \end{aligned}$$

We can show that problem (5) is an abstract formulation of problem (1). The following result is a direct consequence of Theorem 3.2.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and let $t \rightarrow \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of (5).*

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