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Derivations satisfying certain algebraic identities on Lie ideals

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Abstract. Let d be a derivation of a semiprime ring R and L a nonzero Lie ideal of R. In this note, it is proved that every noncentral square-closed Lie ideal of R contains a nonzero ideal of R. Further, we use this result to characterize the conditions: $d(xy) = d(x)d(y)$, $d(xy) = d(y)d(x)$ on L. With this, a theorem of Ali et al. [14] can be deduced.

1. Introduction

This article deals with the derivations acting as homomorphisms or anti-homomorphisms on Lie ideals of semiprime rings, directly motivated by a work of Ali et al. [14]. These types of studies were initiated by Bell and Kappe [11]. Throughout this paper, R will denote an associative ring with at least two elements and $Z(R)$ denotes the center of R. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. For any positive integer n, ring R is said to be n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $a, b \in R$, if $aRb = (0)$ implies either $a = 0$ or $b = 0$ then R is said to be a prime ring and if $aRa = (0)$ implies $a = 0$ then R is called a semiprime ring. An additive subgroup L of R is called a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L of R is said to be square-closed if $x^2 \in L$ for all $x \in L$. It is well-known that if L is square-closed, then $2xy \in L$ for all $x, y \in L$. Recall that an additive map $d : R \to R$ is said to be a derivation if $d(r_1r_2) = d(r_1)r_2 + r_1d(r_2)$ for all $r_1, r_2 \in R$. A familiar example of a derivation is an inner derivation, which is a mapping $\phi_\alpha : R \to R$ given by $\phi_\alpha(r) = [\alpha, r]$ for all $r \in R$ and α be a fixed element of R. Let K be a nonempty subset of R and d a derivation of R. The derivation d is said to be a derivation acting as homomorphism (resp. anti-homomorphism) on K if $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$) for all $x, y \in K$. Further, if $[d(x), x] \in Z(R)$ (resp. $[d(x), x] = 0$) for all $x \in K$ then d is called a centralizing derivation (resp. a commuting derivation) on K. By $C_R(K)$ we shall mean the centralizer of K, defined by $C_R(K) = \{x \in R : xk = kx \ \forall\ k \in K\}$.

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In [9], Posner gave a remarkable and pioneering result on centralizing derivations of prime rings, which is stated as: *If a prime ring R admits a centralizing derivation, then $d = 0$ or R is commutative.* After that a number of generalizations of this result took place (see [1], [4], [5] and references therein). In [10], Awtar proved that: *Let R be a prime ring of characteristic different from 2 and 3. Let d be a nonzero derivation of R, and U a Lie ideal of R with $[u, d(u)] \in Z(R)$ for all $u \in U$. Then $U \subseteq Z(R)$.* Lee and Lee [5] improved this result by excluding the condition of 3-torsion freeness of R.

In the literature, there are many papers investigating the derivations acting as homomorphism or anti-homomorphism on prime rings, but very few on semiprime rings. In 1989, Bell and Kappe [11] proved that: *If d is a derivation of a prime ring R which acts as homomorphism or as anti-homomorphism on a nonzero right ideal I of R, then $d = 0$ on R.* Yenigul and Argaç [12], Ashraf et al. [13] generalized this result by proving it for (σ, τ)-derivations of prime rings. In [14], Ali et al. extended this result to Lie ideals of prime rings. Precisely, they proved the following theorem:

Theorem 1.1. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If d is a derivation of R which acts as homomorphism or anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$.*

The following example demonstrates that one can not expect the above result for semiprime rings.

Example 1.1. Let R^1 be any noncommutative semiprime ring and S^1 be any commutative integral domain. Clearly, $R = S^1 \times R^1$ is a semiprime ring and $L = S^1 \times \{0\}$ is a nonzero Lie ideal of R. Let $\delta : R^1 \to R^1$ be a derivation of R^1. We define a mapping $d : R \to R$ as $(s, r) \mapsto (0, \delta(r))$. Note that, d is a derivation of R that acts as homomorphism and as anti-homomorphism on L, but neither $d = 0$ nor $L \subseteq Z(R)$.

2. Preliminary Results

The commutator identities: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$ and the following results are extensively used in the main section:

Lemma 2.1 ([2], Corollary 2.1)**.** *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \not\subseteq Z(R)$ and let $a, b \in L$. (i) If $aLa = (0)$, then $a = 0$. (ii) If $aL = (0)$ (or $La = (0)$), then $a = 0$. (iii) If L is square-closed and $aLb = (0)$, then $ab = 0$ and $ba = 0$.*

Lemma 2.2 ([3], Lemma 2.4)**.** *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \not\subseteq Z(R)$ and let $a \in L$. If $aLa = (0)$, then $a^2 = 0$ and there exists a nonzero ideal $M = R[L, L]R$ of R generated by $[L, L]$ such that $[M, R] \subseteq L$ and $Ma = aM = 0$.*

Lemma 2.3. *Let R be a 2-torsion free semiprime ring and L be a noncentral square-closed Lie ideal of R. Then there exists a nonzero ideal $M = R[L, L]R$ of R such that $2M \subseteq L$.*

Proof. For the existence of such an ideal one must check Lemma 2.2. For any $x, y \in L$ and $r \in R$, $[x, yr] \in L$. And so we have $y[x, r] + [x, y]r \in L$. Since L is an additive subgroup of R so $2y[x, r] + 2[x, y]r \in L$. As L is square-closed so $2y[x, r] \in L$. The above expression yields that $2[x, y]r \in L$. For any $s \in R$, $2[x, y]rs - 2s[x, y]r \in L$. Therefore, $2s[x, y]r \in L$ for all $x, y \in L$ and $r, s \in R$. Hence, $2R[L, L]R \subseteq L$, i.e., $2M \subseteq L$. If $M = R[L, L]R = (0)$, it implies that $(R[L, L])^2 = (0)$. Since R contains no non-zero nilpotent left-ideal, it gives $R[L, L] = (0)$ and so $[L, L] = (0)$. With the aid of Lemma 1 of [7], $L \subseteq Z(R)$, which is a contradiction. □

Remark 2.1. In [7], authors proved the following: *Let R be a 2-torsion free semiprime ring, d be a derivation of R. If an element $a \in R$ satisfies $ad(L) = (0)$, then $ad(M) = (0)$ where $M = R[L, L]R$.* Note that, above Lemma makes the proof of this result insignificant. Moreover, if $d(L)a = 0$ then $d(M)a = 0$.

Lemma 2.4 ([6], Remark 2.1)**.** *Let R be a ring, L a square-closed Lie ideal of R. Then $2R[L, L] \subseteq L$ and $2[L, L]R \subseteq L$.*

Lemma 2.5 ([8], Corollary 1.4)**.** *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose $a \in R$ such that $ax[x, y] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aM = (0)$, where $M = R[L, L]R$.*

Lemma 2.6. *Let R be a 2-torsion free semiprime ring and L be a nonzero Lie ideal of R. Then $C_R(L) = Z(R)$.*

Proof. Clearly, $Z(R) \subseteq C_R(L)$. It is easy to see that $C_R(L)$ is both a Lie ideal and a subring of R. Since $C_R(L)$ can not contain a nonzero ideal of R, in the light of Herstein [[15], Lemma 1.3] $C_R(L) \subseteq Z(R)$. Hence, $C_R(L) = Z(R)$. □

3. Main Results

Now onwards R will denote a 2-torsion free semiprime ring and L a non-central square-closed Lie ideal of R (unless otherwise mentioned).

Theorem 3.1. *Let d is a derivation of R. If d is centralizing on L, then d maps R into $Z(R)$.*

Proof. First we show that d is commuting on L. By hypothesis, we have $[d(x), x] \in Z(R)$ for all $x \in L$. Since L is square-closed, we may find $[d(x^2), x^2] \in Z(R)$. That means

$$[d(x)x + xd(x), x^2] = [[d(x), x], x^2] + 2[xd(x), x^2]$$

$$
\begin{aligned}
&= 2[xd(x), x^2] \\
&= 2x[xd(x), x] + 2[xd(x), x]x \\
&= 4x^2[d(x), x] \in Z(R).
\end{aligned}
$$

It implies that $[d(x), x^2[d(x), x]] = [d(x), x^2][d(x), x] = 0$ for all $x \in L$. Again by using our hypothesis, we obtain $2x[d(x), x]^2 = 0$. That is, $[d(x), x]^3 = 0$ for any $x \in L$. But the center of a semiprime ring does not contain nonzero nilpotent elements, so we must have $[d(x), x] = 0$ on L.

By Lemma 2.3, $2M \subseteq L$, we find that d is commuting on $M = R[L, L]R$. Therefore, R contains a central ideal generated by the set $d(R)M$ (see the proof of Theorem 3 in [4]). That means, $\langle d(R)M\rangle \subseteq Z(R)$. Thus for any $r, s, p, q \in R$ and $x, y \in L$, we have $[d(r)s[x, y]p, q] = 0$. Replacing p by pr_1, we get $d(r)s[x, y]p[r_1, q] = 0$. In particular, we have $d(r)s[x, y]Rd(r)s[x, y] = (0)$ for all $x, y \in L$ and $r, s \in R$. That yields, $d(r)R[x, y] = (0)$.

Now, we choose a family $\{P_\alpha : \alpha \in \Lambda\}$ of prime ideals of R such that $\bigcap P_\alpha = (0)$. Let P_α be a typical member of that family, so we have $\overline{R} = \frac{R}{P_\alpha}$ is a prime ring. Therefore, our last expression gives $\overline{d(R)}\overline{R}[\overline{L}, \overline{L}] = (\overline{0})$. The fact that $\overline{R}$ is a prime ring implies that either $[\overline{L}, \overline{L}] = (\overline{0})$ or $\overline{d(R)} = (\overline{0})$. If $[\overline{L}, \overline{L}] = (\overline{0})$, then $\overline{L} \subseteq Z(\overline{R})$ by [[7], Lemma 1], that means $[L, R] \subseteq P_\alpha$. Therefore, we have either $d(R) \subseteq P_\alpha$ or $[L, R] \subseteq P_\alpha$.

Together with these both cases, we obtain $d(R)[L, R] \subseteq P_\alpha$ for any prime ideal P_α of R. It yields $d(R)[L, R] \subseteq \bigcap P_\alpha$, i.e., $d(R)[L, R] = (0)$.

Now for any $r, s \in R$ and $x \in L$, we have $d(r)[x, s] = 0$. For some $p \in R$, replace s by sp in the last relation, we find $d(r)s[x, p] = 0$, where $r, s, p \in R$ and $x \in L$. In particular, we obtain $[d(r), x]R[d(r), x] = (0)$ for all $x \in L$ and $r \in R$. Hence, we obtain $[d(R), L] = (0)$.

In the latter case, if $L \subseteq Z(R)$, we clearly have the $d(R)[L, R] = (0)$ and hence $[d(R), L] = (0)$. In each case we have $d(R) \subseteq C_R(L)$. In light of Lemma 2.6, we get $d(R) \subseteq Z(R)$. □

Immediately we have the following consequences of Theorem 3.1:

Corollary 3.1 ([5], Theorem 5)**.** *Let $d \neq 0$ is a derivation of a 2-torsion free prime ring R. If d is centralizing on L, then $L \subseteq Z(R)$. Further, d maps R into $Z(R)$.*

Corollary 3.2. *If d and g be derivations of R such that $d(x)y = xg(y)$ for all $x, y \in L$, then d and g both maps R into $Z(R)$.*

Proof. Let us assume that, $L \nsubseteq Z(R)$. For any $x, y \in L$, we consider $d(x)y = xg(y)$. Replacing x by $2xz$, where $z \in L$, we get $2d(x)zy + 2xd(z)y = 2xzg(y)$. Our hypothesis forces that,

$$d(x)zy = 0, \tag{1}$$

where $x, y, z \in L$. Substitute $[d(x), y]$ for y in (1), we find

$$d(x)z[d(x), y] = 0. \tag{2}$$

Replacing z by $2yz$ in (2), we obtain

$$d(x)yz[d(x),y]=0. \tag{3}$$

Multiplying (2) by y from the left hand side and subtracting from (3), we get $[d(x),y]z[d(x),y]=0$ for all $x,y,z\in L$. In light of Lemma 2.1, we have $[d(x),y]=0$ for any $x,y\in L$. In particular, we have $[d(x),x]=0$ for all $x\in L$. Analogously, we can obtain $[g(x),x]=0$ for all $x\in L$. Hence by Theorem 3.1, we get the conclusions. □

Now, we are well occupied to prove our main result:

Theorem 3.2. (i) *Every derivation d of R that acts as homomorphism on L maps R into $Z(R)$.*

(ii) *Every derivation d of R that acts as anti-homomorphism on L maps R into $Z(R)$.*

Proof. (i) By hypothesis, we have

$$d(xy)=d(x)d(y) \text{ for all } x,y\in L. \tag{4}$$

Replacing x by $2wx$ in (4), where $w\in L$, we get $2d(w)xy+2wd(xy)=2d(w)xd(y)+2wd(x)d(y)$. Since R is 2-torsion free, (4) yields

$$d(w)x(y-d(y))=0. \tag{5}$$

Replacing y by $2yz$ in (5), where $z\in L$, we get $d(w)x(2yz-d(2yz))=0$. Using the condition of 2-torsion free and expanding it, we get $d(w)x(y-d(y))z-d(w)xyd(z)=0$ for all $x,y,w,z\in L$. By using (5), we obtain

$$d(w)xyd(z)=0. \tag{6}$$

Interchanging the role of x and y in (6), we find

$$d(w)yxd(z)=0. \tag{7}$$

On subtracting (7) from (6), we obtain

$$d(w)[x,y]d(z)=0 \tag{8}$$

where $x,y,w,z\in L$. Replace w by $2tw$ in (8), where $t\in L$, we have $2d(t)w[x,y]d(z)+2td(w)[x,y]d(z)=d(t)w2[x,y]d(z)=0$. In particular, we have $(2[x,y]d(z))L(2[x,y]d(z))=(0)$. By Lemma 2.1 and Lemma 2.4, we have $2[x,y]d(z)=0$, and so $[x,y]d(z)=0$ for all $x,y,z\in L$. Analogously, we have $d(x)[y,z]=0$ for any $x,y,z\in L$. Now, using Lemma 2.3, we replace y and z by $2m$ and $2m_1$ in order to obtain, $d(x)[m,m_1]=0$ for all $x\in L$ and $m,m_1\in M=R[L,L]R$. Substituting $m_1d(x)$ for m_1 and expanding, we get $d(x)[m,m_1]d(x)+d(x)m_1[m,d(x)]=0$. It reduces to $d(x)m_1[m,d(x)]=0$ for all $x\in L$ and $m,m_1\in M$. It implies that $[d(x),m]M[d(x),m]=(0)$ for all $x\in L$ and $m\in M$. We know that every nonzero ideal of a semiprime ring is a semiprime ring in itself. Therefore, we obtain $[d(x),m]=0$ for all $x\in L$ and $m\in M$. Now, as $R[L,L]\subseteq M$ so we put $m=r[y,z]$ in the last expression, where $r\in R$ and $y,z\in L$, we find $[d(x),r[y,z]]=0$.

Expanding last expression and using the fact that $[L,L] \subseteq M$ we obtain $[d(x),r][y,z] = 0$. Since L is square closed, substituting y^2 for y, we get $[d(x),r]y[y,z] = 0$ for each $x,y,z \in L$ and $r \in R$. Now, by Lemma 2.5, we get

$$[d(x),r][y,s] = 0 \text{ for all } x,y \in L \text{ and } r,s \in R. \tag{9}$$

For any $p \in R$, replacing s by sp in (9), we get $[d(x),r]s[y,p] = 0$. In particular, we have $[d(x),x]R[d(x),x] = (0)$ for all $x \in L$. Since R is semiprime ring, we find that $[d(x),x] = 0$ for all $x \in L$. Hence, Theorem 3.1 completes the proof.

(ii) By hypothesis, we have

$$d(xy) = d(y)d(x) \text{ for all } x,y \in L. \tag{10}$$

Replacing x by $2xy$ in (10), we get $d(xy^2) = d(y)d(xy)$. By expanding it, we get $d(xy)y + xyd(y) = d(y)d(x)y + d(y)xd(y)$ for all $x,y \in L$. Our hypothesis reduces it to

$$xyd(y) = d(y)xd(y) \tag{11}$$

For any $z \in L$, we replace x by $2zx$ in (11) in order to get

$$zxyd(y) = d(y)zxd(y) \tag{12}$$

Multiplying (11) by z from the left hand side and we have

$$zxyd(y) = zd(y)xd(y) \tag{13}$$

Combining (12) and (13) and we find $[d(y),z]xd(y) = 0$. By easy substitutions, we obtain $[d(y),z]x[d(y),z] = 0$ for any $x,y,z \in L$. That is, $[d(y),z]L[d(y),z] = (0)$ where $y,z \in L$. By Lemma 2.1, $[d(y),z] = 0$ for all $y,z \in L$. In particular, for $y = z$, we have $[d(y),y] = 0$ for all $y \in L$. Again by Theorem 3.1, we obtain the desired results. □

In the following example, we show that the hypothesis of semiprimeness in our Theorem 3.2 is essential.

Example 3.1. Let $\mathbb{Z}$ be a ring of integers and

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}, \qquad L = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

It is easy to verify that L is a noncentral Lie ideal of R and R not a semiprime ring. Let us define a mapping $d : R \to R$ such that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

We see that d is a derivation of R that satisfies $F(XY) = F(X)F(Y)$ and $F(XY) = F(Y)F(X)$ for all $X,Y \in L$. But $d(R) \nsubseteq Z(R)$.

Corollary 3.3. *Every derivation d of R that acts as homomorphism or anti-homomorphism on L, is a commutativity preserving mapping.*

Corollary 3.4. *If d be a derivation of R that acts as homomorphism or anti-homomorphism on R, then there exists $\alpha \in C$ and an additive mapping $\psi : R \to C$ such that $d(x) = \alpha x + \psi(x)$ for all $x \in R$.*

Proof. By Theorem 3.2, we get $[d(R), R] = (0)$, i.e., d is commuting on R. In the view of Brešar [[16], Corollary 4.2], we get the desired conclusion. □

Corollary 3.5 ([14], Theorem 3.1). *Let R be a 2-torsion free prime ring and L be a square-closed Lie ideal of R. If d is a derivation of R, which acts as homomorphism or anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$.*

Proof. Suppose that $L \nsubseteq Z(R)$. By Theorem 3.2, we obtain $d(R)[L, R] = (0)$, i.e., $d(r)[x, s] = 0$ for any $r, s \in R$ and $x \in L$. Replacing r by $r_1 r$, where $r_1 \in R$, we get $d(r_1)R[x, r] = (0)$. By primeness of R we have either $d(r_1) = 0$ or $[x, r] = 0$. In view of our assumption, we get $d = 0$. □

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