

## Existence of positive periodic solutions for third-order nonlinear delay differential equations with variable coefficients

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ABSTRACT. In this paper, the following third-order nonlinear delay differential equation with periodic coefficients

$$\begin{aligned} & x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) \\ & = f(t, x(t), x(t - \tau(t))) + \frac{d}{dt}g(t, x(t - \tau(t))), \end{aligned}$$

is considered. By employing Green's function and Krasnoselskii's fixed point theorem, we state and prove the existence of positive periodic solutions to the third-order nonlinear delay differential equation.

### 1. INTRODUCTION

Third order differential equations arise from in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [24, 29].

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monograph [12, 26] and the papers [1]-[23], [25] [27]-[29], [31]-[34] and the references therein.

The second order nonlinear delay differential equation with periodic coefficients

$$\begin{aligned} & x''(t) + p(t)x'(t) + q(t)x(t) = \\ & = f(t, x(t), x(t - \tau(t))) + \frac{d}{dt}g(t, x(t - \tau(t))), \end{aligned}$$

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2010 *Mathematics Subject Classification*. Primary: 34K13, 34A34; Secondary: Secondary 34K30, 34L30.

*Key words and phrases*. Fixed point, positive periodic solutions, third-order delay differential equations.

*Full paper*. Received 28 February 2019, accepted 6 September 2019, available online 2 October 2019.

has been investigated in [9]. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Ardjouni and Djoudi obtained existence and uniqueness of periodic solutions.

In [29], Ren, Siegmund and Chen discussed the existence of positive periodic solutions for the third-order differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = g(t, x(t)).$$

By employing the fixed point index, the authors obtained existence results for positive periodic solutions.

Inspired and motivated by the works mentioned above and the papers [1]-[23], [25], [27]-[29], [31]-[34] and the references therein, we concentrate on the existence of positive periodic solutions for the third-order nonlinear delay differential equation

$$(1) \quad \begin{aligned} & x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) \\ & = f(t, x(t), x(t - \tau(t))) + \frac{d}{dt}g(t, x(t - \tau(t))), \end{aligned}$$

where  $p, q, r, \tau$  are continuous real-valued functions. The functions  $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous in their respective arguments. To show the existence of positive periodic solutions, we transform (1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact.

The organization of this paper is as follows. In section 2, we introduce some notations and lemmas, and state some preliminary results needed in later section, then we give the Green's function of (1), which plays an important role in this paper. In section 3, we present our main results on existence of positive periodic solutions of (1).

We state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [30].

**Theorem 1.1** (Krasnoselskii). *Let  $\mathbb{M}$  be a closed convex nonempty subset of a Banach space  $(\mathbb{B}, \|\cdot\|)$ . Suppose that  $H_1$  and  $H_2$  map  $\mathbb{M}$  into  $\mathbb{B}$  such that*

- (i)  $x, y \in \mathbb{M}$ , implies  $H_1x + H_2y \in \mathbb{M}$ ,
- (ii)  $H_1$  is compact and continuous,
- (iii)  $H_2$  is a contraction mapping.

*Then there exists  $z \in \mathbb{M}$  with  $z = H_1z + H_2z$ .*

In this paper, we give the assumptions as follows that will be used in the main results.

- (h1) There exist differentiable positive  $T$ -periodic functions  $a_1$  and  $a_2$  and a positive real constant  $\rho$  such that

$$\begin{cases} a_1(t) + \rho = p(t), \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t). \end{cases}$$

- (h2)  $p, q, r, \tau \in C(\mathbb{R}, \mathbb{R}^+)$  are  $T$ -periodic functions with  $\tau(t) \geq \tau^* > 0$  and

$$\int_0^T p(s)ds > \rho T, \quad \int_0^T q(s)ds > 0.$$

- (h3) The functions  $g(t, x)$  and  $f(t, x, y)$  are continuous  $T$ -periodic in  $t$  and continuous in  $x$  and in  $x$  and  $y$ , respectively.

## 2. GREEN'S FUNCTION OF THIRD-ORDER DIFFERENTIAL EQUATION

For  $T > 0$ , let  $P_T$  be the set of all continuous scalar functions  $x$ , periodic in  $t$  of period  $T$ . Then  $(P_T, \|\cdot\|)$  is a Banach space with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0, T]} |x(t)|.$$

We consider

$$(2) \quad x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t),$$

where  $h$  is a continuous  $T$ -periodic function. Obviously, by the condition (h1), (2) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

**Lemma 2.1** ([7]). *If  $y, h \in P_T$ , then  $y$  is a solution of equation*

$$y'(t) + \rho y(t) = h(t),$$

*if only if*

$$(3) \quad y(t) = \int_t^{t+T} G_1(t, s)h(s)ds,$$

where

$$(4) \quad G_1(t, s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}.$$

**Corollary 2.1.** *Green function  $G_1$  satisfies the following properties*

$$\begin{aligned} G_1(t+T, s+T) &= G_1(t, s), & G_1(t, t+T) &= G_1(t, t) \exp(\rho T), \\ G_1(t+T, s) &= G_1(t, s) \exp(-\rho T), & G_1(t, s+T) &= G_1(t, s) \exp(\rho T), \\ \frac{\partial}{\partial t} G_1(t, s) &= -\rho G_1(t, s), \end{aligned}$$

$$\frac{\partial}{\partial s} G_1(t, s) = \rho G_1(t, s),$$

and

$$m_1 \leq G_1(t, s) \leq M_1,$$

where

$$m_1 = \frac{1}{\exp(\rho T) - 1}, \quad M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}.$$

**Lemma 2.2** ([28]). *Suppose that (h1) and (h2) hold and*

$$(5) \quad \frac{R_1 \left[ \exp \left( \int_0^T a_1(v) dv \right) - 1 \right]}{Q_1 T} \geq 1,$$

where

$$R_1 = \max_{t \in [0, T]} \left| \int_t^{t+T} \frac{\exp \left( \int_0^T a_1(v) dv \right)}{\exp \left( \int_0^T a_1(v) dv \right) - 1} a_2(s) ds \right|,$$

$$Q_1 = \left( 1 + \exp \left( \int_0^T a_1(v) dv \right) \right)^2 R_1^2.$$

Then there are continuous  $T$ -periodic functions  $a$  and  $b$  such that

$$b(t) > 0, \quad \int_0^T a(v) dv > 0,$$

and

$$a(t) + b(t) = a_1(t), \quad b'(t) + a(t)b(t) = a_2(t), \quad \text{for } t \in \mathbb{R}.$$

**Lemma 2.3** ([32]). *Suppose the conditions of Lemma 2.2 hold and  $y \in P_T$ . Then the equation*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t),$$

has a  $T$  periodic solution. Moreover, the periodic solution can be expressed by

$$(6) \quad x(t) = \int_t^{t+T} G_2(t, s) y(s) ds,$$

where

$$(7) \quad G_2(t, s) = \frac{\int_t^s \exp \left[ \int_t^v b(u) du + \int_v^s a(u) du \right] dv}{\left[ \exp \left( \int_0^T a(v) dv \right) - 1 \right] \left[ \exp \left( \int_0^T b(v) dv \right) - 1 \right]} + \frac{\int_s^{t+T} \exp \left[ \int_t^v b(u) du + \int_v^{s+T} a(u) du \right] dv}{\left[ \exp \left( \int_0^T a(v) dv \right) - 1 \right] \left[ \exp \left( \int_0^T b(v) dv \right) - 1 \right]}.$$

**Corollary 2.2.** *Green's function  $G_2$  satisfies the following proprieties*

$$\begin{aligned} G_2(t+T, s+T) &= G_2(t, s), \quad G_2(t, t+T) = G_2(t, t), \\ G_2(t+T, s) &= \exp\left(-\int_0^T b(v)dv\right) \\ &\quad \times \left[ G_2(t, s) + \int_t^{t+T} E(t, u) F(u, s) du \right], \\ \frac{\partial}{\partial t} G_2(t, s) &= -b(t)G_2(t, s) + F(t, s), \\ \frac{\partial}{\partial s} G_2(t, s) &= a(t)G_2(t, s) - E(t, s), \end{aligned}$$

where

$$E(t, s) = \frac{\exp\left(\int_t^s b(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right) - 1}, \quad F(t, s) = \frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right) - 1}.$$

**Lemma 2.4** ([28]). *Let  $A = \int_0^T a_1(v)dv$ ,  $B = T^2 \exp\left(\frac{1}{T} \int_0^T \ln(a_2(v))dv\right)$ .*

*If*

$$(8) \quad A^2 \geq 4B,$$

*then*

$$\begin{aligned} \min \left\{ \int_0^T a(v)dv, \int_0^T b(v)dv \right\} &\geq \frac{1}{2} \left( A - \sqrt{A^2 - 4B} \right) = l, \\ \max \left\{ \int_0^T a(v)dv, \int_0^T b(v)dv \right\} &\leq \frac{1}{2} \left( A + \sqrt{A^2 - 4B} \right) = L. \end{aligned}$$

**Corollary 2.3.** *Functions  $G_2$ ,  $E$  and  $F$  satisfy*

$$\begin{aligned} m_2 &\leq G_2(t, s) \leq M_2, \\ E(t, s) &\leq \frac{e^L}{e^l - 1}, \\ F(t, s) &\leq e^L, \end{aligned}$$

where

$$m_2 = \frac{T}{(\exp(L) - 1)^2}, \quad M_2 = \frac{T \exp\left(\int_0^T a_1(v)dv\right)}{(\exp(l) - 1)^2}.$$

**Lemma 2.5** ([15]). *Suppose the conditions of Lemma 2.2 hold and  $h \in P_T$ . Then the equation*

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t),$$

has a  $T$ -periodic solution. Moreover, the periodic solution can be expressed by

$$(9) \quad x(t) = \int_t^{t+T} G(t, s)h(s)ds,$$

where

$$(10) \quad G(t, s) = \int_t^{t+T} G_2(t, \sigma) G_1(\sigma, s) d\sigma.$$

**Corollary 2.4.** *Green's function  $G$  satisfies the following properties*

$$\begin{aligned} G(t+T, s+T) &= G(t, s), \quad G(t, t+T) = G(t, t) \exp(\rho T), \\ \frac{\partial}{\partial t} G(t, s) &= (\exp(-\rho T) - 1) G_1(t, t) G_2(t, s) \\ &\quad - b(t) G(t, s) + \int_t^{t+T} F(t, \sigma) G_1(\sigma, s) d\sigma, \\ \frac{\partial}{\partial s} G(t, s) &= \rho G(t, s), \end{aligned}$$

and

$$m \leq G(t, s) \leq M,$$

where

$$m = \frac{T^2}{(\exp(l) - 1)^2 (\exp(\rho T) - 1)}, \quad M = \frac{T^2 \exp\left(\rho T + \int_0^T a(v) dv\right)}{(\exp(l) - 1)^2 (\exp(\rho T) - 1)}.$$

### 3. MAIN RESULTS

In this section we will study the existence of positive periodic solutions of (1).

**Lemma 3.1.** *Suppose (h1) – (h3) and (5) hold. The function  $x \in P_T$  is a solution of (1) if and only if*

$$(11) \quad \begin{aligned} x(t) &= (\exp(\rho T) - 1) G(t, t) g(t, x(t - \tau(t))) \\ &+ \int_t^{t+T} G(t, s) \{f(s, x(s), x(s - \tau(s))) - \rho g(s, x(s - \tau(s)))\} ds. \end{aligned}$$

*Proof.* Let  $x \in P_T$  be a solution of (1). From Lemma 2.5, we have

$$(12) \quad \begin{aligned} x(t) &= \int_t^{t+T} G(t, s) \left[ f(s, x(s), x(s - \tau(s))) + \frac{\partial}{\partial s} g(s, x(s - \tau(s))) \right] ds \\ &= \int_t^{t+T} G(t, s) f(s, x(s), x(s - \tau(s))) ds \\ &+ \int_t^{t+T} G(t, s) \frac{\partial}{\partial s} g(s, x(s - \tau(s))) ds. \end{aligned}$$

Performing an integration by parts, we get

$$\begin{aligned}
 & \int_t^{t+T} G(t, s) \frac{\partial}{\partial s} g(s, x(s - \tau(s))) ds \\
 &= G(t, s) g(s, x(s - \tau(s))) \Big|_t^{t+T} - \int_t^{t+T} \left[ \frac{\partial}{\partial s} G(t, s) \right] g(s, x(s - \tau(s))) ds \\
 &= (\exp(\rho T) - 1) G(t, t) g(t, x(t - \tau(t))) \\
 (13) \quad & - \rho \int_t^{t+T} G(t, s) g(s, x(s - \tau(s))) ds.
 \end{aligned}$$

We obtain (11) by substituting (13) in (12). Since each step is reversible, the converse follows easily. This completes the proof.  $\square$

Define the mapping  $H : P_T \rightarrow P_T$  by

$$\begin{aligned}
 & (H\varphi)(t) \\
 &= \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - \rho g(s, \varphi(s - \tau(s)))\} ds \\
 (14) \quad & + (\exp(\rho T) - 1) G(t, t) g(t, \varphi(t - \tau(t))).
 \end{aligned}$$

Note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express (14) as

$$(H\varphi)(t) = (H_1\varphi)(t) + (H_2\varphi)(t).$$

where  $H_1, H_2 : P_T \rightarrow P_T$  are given by

$$\begin{aligned}
 & (H_1\varphi)(t) \\
 (15) \quad &= \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - \rho g(s, \varphi(s - \tau(s)))\} ds,
 \end{aligned}$$

and

$$(16) \quad (H_2\varphi)(t) = (\exp(\rho T) - 1) G(t, t) g(t, \varphi(t - \tau(t))).$$

To simplify notations, we introduce the constants

$$(17) \quad \beta = \max_{t \in [0, T]} \{b(t)\}, \quad \delta = \frac{\exp(L)}{\exp(l) - 1}, \quad \gamma = \exp(\rho T) - 1.$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases;  $g(t, x) \geq 0$  and  $g(t, x) \leq 0$  for all  $t \in \mathbb{R}$ . For a non-negative constant  $\mathbf{K}$  and a positive constant  $\mathbf{L}$  we define the set

$$\mathbb{D} = \{\varphi \in P_T : \mathbf{K} \leq \varphi \leq \mathbf{L}\},$$

which is a closed convex and bounded subset of the Banach space  $P_T$ . We assume that the function  $g(t, x)$  is locally Lipschitz continuous in  $x$ . That

is, there exists a positive constant  $k$  such that

$$(18) \quad |g(t, x) - g(t, y)| \leq k \|x - y\|, \text{ for all } t \in [0, T], x, y \in \mathbb{D}.$$

In case  $g(t, x) \geq 0$ , we assume that there exist a nonnegative constant  $k_1$  and a positive constant  $k_2$  such that

$$(19) \quad k_1 x \leq g(t, x) \leq k_2 x, \text{ for all } t \in [0, T], x \in \mathbb{D},$$

$$(20) \quad k_2 M \gamma < 1,$$

and for all  $t \in [0, T], x, y \in \mathbb{D}$

$$(21) \quad \frac{\mathbf{K}(1 - k_1 m \gamma)}{mT} \leq f(t, x, y) - \rho g(t, y) \leq \frac{\mathbf{L}(1 - k_2 M \gamma)}{MT}.$$

**Lemma 3.2.** *Suppose (h1) – (h3), (5), (8) and (19)-(21) hold. Then  $H_1 : \mathbb{D} \rightarrow P_T$  is compact.*

*Proof.* Let  $H_1$  be defined by (15). Obviously,  $H_1 \varphi$  is continuous and it is easy to show that  $(H_1 \varphi)(t + T) = (H_1 \varphi)(t)$ . For  $t \in [0, T]$  and for  $\varphi \in \mathbb{D}$ , we have

$$\begin{aligned} & |(H_1 \varphi)(t)| \\ &= \left| \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - \rho g(s, \varphi(s - \tau(s)))\} ds \right| \\ &\leq MT \frac{\mathbf{L}(1 - k_2 M \gamma)}{MT} = \mathbf{L}(1 - k_2 M \gamma). \end{aligned}$$

Thus from the estimation of  $|(H_1 \varphi)(t)|$  we have

$$\|H_1 \varphi\| \leq \mathbf{L}(1 - k_2 M \gamma).$$

This shows that  $H_1(\mathbb{D})$  is uniformly bounded.

To show that  $H_1(\mathbb{D})$  is equicontinuous, let  $\varphi_n \in \mathbb{D}$ , where  $n$  is a positive integer. Next we calculate  $\frac{d}{dt}(H_1 \varphi_n)(t)$  and show that it is uniformly bounded. By using (h1), (h2) and (h3) we obtain by taking the derivative in (15) that

$$\begin{aligned} & \frac{d}{dt}(H_1 \varphi_n)(t) \\ &= \int_t^{t+T} [(\exp(-\rho T) - 1) G_1(t, t) G_2(t, s) - b(t) G(t, s) \\ &+ \int_t^{t+T} F(t, \sigma) G_1(\sigma, s) d\sigma] \\ &\quad \times [f(s, \varphi_n(s), \varphi_n(s - \tau(s))) - \rho g(s, \varphi_n(s - \tau(s)))] ds. \end{aligned}$$

Consequently, by invoking (17) and (21), we obtain

$$\left| \frac{d}{dt}(H_1 \varphi_n)(t) \right| \leq$$



$$\leq [(1 - \exp(-\rho T)) M_1 M_2 + M\beta + M_1 \delta T] \frac{\mathbf{L}(1 - k_2 M \gamma)}{M} \leq D,$$

for some positive constant  $D$ . Hence the sequence  $(H_1 \varphi_n)$  is equicontinuous. The Ascoli-Arzelà theorem implies that a subsequence  $(H_1 \varphi_{n_k})$  of  $(H_1 \varphi_n)$  converges uniformly to a continuous  $T$ -periodic function. Thus  $H_1$  is continuous and  $H_1(\mathbb{D})$  is contained in a compact subset of  $\mathbb{D}$ .  $\square$

**Lemma 3.3.** *Suppose that (18) holds. If  $H_2$  is given by (16) with*

$$(22) \quad k\gamma M < 1,$$

then  $H_2 : \mathbb{D} \rightarrow P_T$  is a contraction.

*Proof.* Let  $H_2$  be defined by (16). It is easy to show that  $(H_2 \varphi)(t + T) = (H_2 \varphi)(t)$ . Let  $\varphi, \psi \in \mathbb{D}$ , we have

$$\|H_2 \varphi - H_2 \psi\| = \sup_{t \in [0, T]} |(H_2 \varphi)(t) - (H_2 \psi)(t)| \leq k\gamma M \|\varphi - \psi\|.$$

Hence  $H_2 : \mathbb{D} \rightarrow P_T$  is a contraction by (22).  $\square$

**Theorem 3.1.** *Suppose that conditions (h1) – (h3), (5), (8) and (18)-(22) hold. Then equation (1) has a positive  $T$ -periodic solution  $x$  in the subset  $\mathbb{D}$ .*

*Proof.* By Lemma 3.2, the operator  $H_1 : \mathbb{D} \rightarrow P_T$  is compact and continuous. Also, from Lemma 3.3, the operator  $H_2 : \mathbb{D} \rightarrow P_T$  is a contraction. Moreover, if  $\varphi, \psi \in \mathbb{D}$ , we see that

$$\begin{aligned} & (H_2 \psi)(t) + (H_1 \varphi)(t) \\ &= \gamma G(t, t) g(t, \varphi(t - \tau(t))) \\ &+ \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - \rho g(s, \varphi(s - \tau(s)))\} ds \\ &\leq k_2 M \gamma \mathbf{L} + \mathbf{L}(1 - k_2 M \gamma) = \mathbf{L}. \end{aligned}$$

On the other hand

$$\begin{aligned} & (H_2 \psi)(t) + (H_1 \varphi)(t) \\ &= \gamma G(t, t) g(t, \varphi(t - \tau(t))) \\ &+ \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - \rho g(s, \varphi(s - \tau(s)))\} ds \\ &\geq k_1 m \gamma \mathbf{K} + \mathbf{K}(1 - k_1 m \gamma) = \mathbf{K}. \end{aligned}$$

This shows that  $H_2 \psi + H_1 \varphi \in \mathbb{D}$ . Clearly, all the Hypotheses of Theorem 1.1, are satisfied. Thus there exists a fixed point  $x \in \mathbb{D}$  such that  $x = H_1 \psi + H_2 \varphi$ . By Lemma 3.1 this fixed point is a solution of (1) and the proof is complete.  $\square$

In the case  $g(t, x) \leq 0$ , we substitute conditions (19)-(21) with the following conditions respectively. We assume that there exist a negative constant  $k_3$  and a non-positive constant  $k_4$  such that

$$(23) \quad k_3x \leq g(t, x) \leq k_4x, \text{ for all } t \in [0, T], x \in \mathbb{D},$$

$$(24) \quad -k_3M\gamma < 1,$$

and for all  $t \in [0, T], x, y \in \mathbb{D}$

$$(25) \quad \frac{\mathbf{K} - k_3M\gamma\mathbf{L}}{mT} \leq f(t, x, y) - \rho g(t, y) \leq \frac{\mathbf{L} - k_4m\gamma\mathbf{K}}{MT}.$$

**Theorem 3.2.** *Suppose that conditions (h1) – (h3), (5), (8), (18) and (22)-(25) hold. Then equation (1) has a positive  $T$ -periodic solution  $x$  in the subset  $\mathbb{D}$ .*

The proof follows along the lines of Theorem 3.1, and hence we omit it.

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