

On location in a half-plane of zeros of perturbed first order entire functions

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ABSTRACT. We consider the entire functions

$$h(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \quad \text{and} \quad \tilde{h}(z) = \sum_{k=0}^{\infty} \frac{\tilde{a}_k z^k}{k!}$$

($a_0 = \tilde{a}_0 = 1; z, a_k, \tilde{a}_k \in \mathbf{C}, k = 1, 2, \dots$), provided

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty, \quad \sum_{k=0}^{\infty} |\tilde{a}_k|^2 < \infty$$

and all the zeros of $h(z)$ are in a half-plane. We investigate the following problem: how small should be the quantity $q := (\sum_{k=1}^{\infty} |a_k - \tilde{a}_k|^2)^{1/2}$ in order to all the zeros of $\tilde{h}(z)$ lie in the same half-plane?

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Consider the entire functions

$$(1) \quad h(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \quad \text{and} \quad \tilde{h}(z) = \sum_{k=0}^{\infty} \frac{\tilde{a}_k z^k}{k!}$$

($a_0 = \tilde{a}_0 = 1; z, a_k, \tilde{a}_k \in \mathbf{C}, k = 1, 2, \dots$) under the conditions

$$(2) \quad \sum_{k=0}^{\infty} |a_k|^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} |\tilde{a}_k|^2 < \infty.$$

Any function of the type

$$\hat{h}(z) = \sum_{k=0}^{\infty} \frac{b_k z^k}{k!},$$

with $|b_k| \leq \text{const } c_0^k$ and $c_0 \geq 1$, can be reduced to the form (1) with condition (2) if we take $z = w/2c$.

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Let $z_k(f)$ ($k = 1, 2, \dots$) denote the zeros of a function f taken with their multiplicities. We consider the following problem: let

$$(3) \quad \inf_{k=1,2,\dots} \Re z_k(h) > \gamma \quad (\gamma = \text{const} > 0).$$

How small should be the quantity

$$q := \left(\sum_{k=1}^{\infty} |a_k - \tilde{a}_k|^2 \right)^{1/2}$$

in order to provide the inequality

$$(4) \quad \inf_{k=1,2,\dots} \Re z_k(\tilde{h}) > \gamma?$$

The literature on perturbations of the zeros of analytic functions is rather rich. In particular, the results obtained enable us to explore the relations between the zeros of the power series, their partial sums and tails, cf. [5], to estimate the distances between the zeros of entire functions and the zeros of their derivatives, [2, 3, 6, 8]. The variation of the zeros of general analytic functions under perturbations was investigated, in particular, by P. Rosenbloom [17]. He has established the perturbation result that provides the existence of a zero of a perturbed function in a given domain. In the case of entire functions the Rosenbloom's results have been refined in [9] (see also [11]). Of course we cannot survey the whole subject here and refer the reader to the just mentioned papers and books, and references given therein. However, to the best of our knowledge the above pointed problem was not considered in the available literature although it is important, in particular, for localization of the zeros of perturbed functions. Our main tool is the recent norm estimates for solutions of the perturbed Lyapunov equation.

Put

$$\psi(h) := \left(\sum_{k=1}^{\infty} |a_n|^2 + \zeta(2) - 1 \right)^{1/2}, \quad \text{where } \zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \quad (\Re z > 1)$$

is the Riemann zeta function. Below we show that condition (3) implies

$$\sup_k \left| \frac{2\gamma}{z_k(h)} - 1 \right| < 1$$

and therefore

$$r(h, \gamma) := 1 - \sup_k \left| \frac{2\gamma}{z_k(h)} - 1 \right| > 0.$$

Finally, denote

$$\xi(h, \gamma) := \left(\sum_{m=0}^{\infty} \frac{(2\gamma\psi(h))^m}{\sqrt{m!} r^{m+1}(h, \gamma)} \right)^2.$$

In Section 4 we suggest estimates for $\psi(h)$, q and $\xi(h, \gamma)$. Now we are in a position to formulate the main result of this paper.

Theorem 1.1. *Let $h(z)$ and $\tilde{h}(z)$ be defined by (1). Let the conditions (2), (3) and*

$$4\xi(h, \gamma)(\gamma^2 q^2 + \gamma q(1 + 2\gamma\psi(h))) < 1$$

be fulfilled. Then inequality (4) is valid.

The proof of this theorem is presented in the next section.

2. PROOF OF THEOREM 1.1

Let \mathbb{C}^n be the complex n -dimension Euclidean space with a scalar product (\cdot, \cdot) and the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. Denote by $\mathbb{C}^{n \times n}$ the set of $n \times n$ -matrices. For an $A \in \mathbb{C}^{n \times n}$, $\lambda_k(A)$ ($k = 1, \dots, n$) are the eigenvalues taken with the multiplicities, $\sigma(A)$ is the spectrum, $r_s(A) = \max_k |\lambda_k(A)|$ is the spectral radius, A^* is the adjoint one, and $\|A\|$ is the spectral norm: $\|A\|^2 = r_s(A^*A)$; I is the unit $n \times n$ -matrix.

For an integer $n > 1$, let us consider the polynomials

$$f_n(\lambda) = \sum_{k=0}^n \frac{a_k \lambda^{n-k}}{k!} \quad \text{and} \quad \tilde{f}_n(\lambda) = \sum_{k=0}^n \frac{\tilde{a}_k \lambda^{n-k}}{k!}.$$

f_n is the characteristic polynomial of the matrix

$$\begin{pmatrix} -a_1 & -a_2/2! & \cdots & -a_{n-1}/(n-1)! & -a_n/n! \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

As it is shown in [11] Lemma 5.2.1, p. 117, this matrix is similar to the following one

$$F_n = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/n & 0 \end{pmatrix}.$$

So $z_k(f_n) = \lambda_k(F_n)$ ($k = 1, \dots, n$). Similarly, $z_k(\tilde{f}_n) = \lambda_k(\tilde{F}_n)$ ($k = 1, \dots, n$), where

$$\tilde{F}_n = \begin{pmatrix} -\tilde{a}_1 & -\tilde{a}_2 & \cdots & -\tilde{a}_{n-1} & -\tilde{a}_n \\ 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/n & 0 \end{pmatrix}.$$

Put

$$h_n(z) = z^n f_n(1/z) = \sum_{k=0}^n \frac{a_k z^k}{k!} \quad \text{and} \quad \tilde{h}_n(z) = z^n \tilde{f}_n(1/z).$$

Then $z_k(h_n) = \frac{1}{z_k(f_n)} = \frac{1}{\lambda_k(F_n)} = \lambda_k(F_n^{-1})$ and $z_k(\tilde{h}_n) = \lambda_k(\tilde{F}_n^{-1})$.

Due to Hurwitz theorem [15, p. 4] if z_0 is an m -fold zero of $h(z)$, then every sufficient small neighborhood of z_0 contains m zeros counted with their multiplicities of each h_n for all sufficiently large n . Thus from (3) for all sufficiently large n we have

$$(5) \quad \min_k \Re \lambda_k(F_n^{-1}) = \min_k \Re z_k(h_n) > \gamma.$$

Lemma 2.1. *Let condition (5) hold. Then the spectral radius $r_s(2\gamma F_n - I)$ of the matrix $2\gamma F_n - I$ satisfies the inequality*

$$r_s(2\gamma F_n - I) < 1.$$

Proof. Let $\mu = x + iy \in \sigma(F_n^{-1}); x, y \in \mathbb{R}$. Then $x > \gamma$ and

$$\begin{aligned} |2\gamma/\mu - 1|^2 &= |2\gamma - \mu|^2/|\mu|^2 = ((2\gamma - x)^2 + y^2)/|\mu|^2 \\ &= (4\gamma^2 - 4\gamma x + x^2 + y^2)/|\mu|^2 < (x^2 + y^2)/|\mu|^2 < 1. \end{aligned}$$

This proves the lemma. □

For an $A \in \mathbb{C}^{n \times n}$ assume that

$$(6) \quad r_s(A) < 1$$

and put

$$\chi(A) := \sum_{k=0}^{\infty} \|A^k\|^2.$$

Note that

$$\begin{aligned} \int_0^{2\pi} \|(I - Ae^{it})^{-1}h\|^2 dt &= \int_0^{2\pi} \left(\sum_{k=0}^{\infty} A^k e^{itk} h, \sum_{j=0}^{\infty} A^j e^{itj} h \right) dt \\ &= 2\pi \sum_{k=0}^{\infty} \|A^k h\|^2 \quad (h \in \mathbb{C}^n). \end{aligned}$$

Hence, we easily have

$$(7) \quad \chi(A) = \frac{1}{2\pi} \int_0^{2\pi} \|(I - Ae^{it})^{-1}\|^2 dt.$$

Lemma 2.2. *Let $A, \tilde{A} \in \mathbb{C}^{n \times n}$ and condition (6) hold. If, in addition,*

$$\chi(A)(\|A - \tilde{A}\|^2 + 2\|A\|\|A - \tilde{A}\|) < 1,$$

then $r_s(\tilde{A}) < 1$.

Proof. Consider the discrete Lyapunov equation

$$X - A^*XA = I,$$

with given $A \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times n}$ should be found. It can be directly checked that

$$X = \sum_{k=0}^{\infty} (A^*)^k A^k$$

(see also [12, Lemma 2.1] and references therein). With $C = \tilde{A} - A$ we have

$$\begin{aligned} X - \tilde{A}^*X\tilde{A} &= X - (C^* + A^*)X(C + A) \\ &= X - A^*XA - C^*XC - A^*XC - A^*XC \\ &= I - C^*XC - A^*XC - A^*XC. \end{aligned}$$

Obviously, $\|X\| \leq \chi(A)$. Thus the inequalities

$$\|X\|(\|C\|^2 + 2\|A\|\|C\|) \leq \chi(A)(\|C\|^2 + 2\|A\|\|C\|) < 1$$

imply that $X - \tilde{A}^*X\tilde{A}$ is a positive definite operator and therefore by [7, Theorem 6.1] $r_s(\tilde{A}) < 1$, as claimed. \square

Lemma 2.3. *Let the conditions (5), and*

$$(8) \quad 4\chi(2\gamma F_n - I)(\gamma^2\|F_n - \tilde{F}_n\|^2 + \gamma\|2\gamma F_n - I\|\|F_n - \tilde{F}_n\|) < 1$$

be fulfilled. Then

$$\min_k \Re \lambda_k(\tilde{F}_n^{-1}) > \gamma.$$

Proof. Put $B = 2\gamma F_n - I$, $\tilde{B} = 2\gamma \tilde{F}_n - I$. We have $\tilde{B} - B = 2\gamma(\tilde{F}_n - F_n)$. By Lemma 2.1, $r_s(B) = r_s(2\gamma F_n - I) < 1$. So

$$\chi(B) = \chi(2\gamma F_n - I) = \sum_{k=0}^{\infty} \|(2\gamma F_n - I)^k\|^2.$$

By Lemma 2.2, if conditions (5) and (8) hold, then the inequality $r_s(2\gamma \tilde{F}_n - I) < 1$ is valid. So with $\mu = x + iy \in \sigma(\tilde{F}_n^{-1})$ we have

$$\begin{aligned} 1 &> |2\gamma/\mu - 1|^2 = |2\gamma - \mu|^2/|\mu|^2 = ((2\gamma - x)^2 + y^2)/|\mu|^2 \\ &= (4\gamma^2 + x^2 + y^2 - 4\gamma x)/|\mu|^2 = 1 + (4\gamma^2 - 4\gamma x)/|\mu|^2. \end{aligned}$$

Hence,

$$(4\gamma^2 - 4\gamma x)/|\mu|^2 < 0$$

and therefore $x > \gamma$, as claimed.

We need the following quantity:

$$g(A) = [N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2]^{1/2} \quad (A \in \mathbb{C}^{n \times n}),$$

where $N_2(A) = (\text{trace } AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt norm) of A . The following relations are checked in [12, Section 3.1].

$$g^2(A) \leq N_2^2(A) - |\text{Trace } A^2| \quad \text{and} \quad g^2(A) \leq \frac{N_2^2(A - A^*)}{2} = 2N_2^2(A_I),$$

where $A_I = (A - A^*)/2i$. In addition, $g(e^{it}A + zI) = g(A)$ ($z \in \mathbb{C}, t \in \mathbb{R}$). If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \leq g(A_1) + g(A_2)$. Moreover, $g^2(A) \leq N_2^2(A) - n(\det A)^{2/n}$.

According to [12, Theorem 9.5], for any $n \times n$ matrix A ,

$$(9) \quad \|(A - \lambda I)^{-1}\| \leq \sum_{m=0}^{n-1} \frac{g^m(A)}{\sqrt{m!} \rho^{m+1}(A, \lambda)} \quad (\lambda \notin \sigma(A)),$$

where $\rho(A, \lambda) := \min_{k=1, \dots, n} |\lambda - \lambda_k(A)|$.

Since

$$g(2\gamma F_n - I) = g(2\gamma F_n) \leq N_2(2\gamma F_n) = 2\gamma N_2(F_n)$$

and

$$\rho(2\gamma F_n - I, e^{-it}) \geq 1 - r_s(2\gamma F_n - I),$$

provided that $r_s(2\gamma F_n - I) < 1$, according to (9) we have

$$\|(I - (\gamma F_n - I)e^{it})^{-1}\| \leq \sum_{m=0}^{n-1} \frac{N_2^m(2\gamma F_n)}{\sqrt{m!}(1 - r_s(2\gamma F_n - I))^{m+1}}.$$

Hence, (7) implies

$$\begin{aligned} \chi(2\gamma F_n - I) &= \frac{1}{2\pi} \int_0^{2\pi} \|(I - (\gamma F_n - I)e^{it})^{-1}\|^2 dt \\ &\leq \left(\sum_{m=0}^{n-1} \frac{N_2^m(2\gamma F_n)}{\sqrt{m!}(1 - r_s(2\gamma F_n - I))^{m+1}} \right)^2. \end{aligned}$$

Note that

$$N_2^2(F_n) = \sum_{k=1}^n \left(|a_k|^2 + \frac{1}{k^2} \right) - 1 \leq \sum_{k=1}^{\infty} |a_n|^2 + \zeta(2) - 1 = \psi^2(h)$$

and

$$\begin{aligned} r_s(2\gamma F_n - I) &= \max_{k=1, \dots, n} |2\gamma \lambda_k(F_n) - 1| = \max_{k=1, \dots, n} \left| \frac{2\gamma}{z_k(h_n)} - 1 \right| \\ &\leq \sup_{k=1, 2, \dots} \left| \frac{2\gamma}{z_k(h) + \epsilon_n} - 1 \right|, \end{aligned}$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Thus, for sufficiently large n ,

$$\begin{aligned} \chi(2\gamma F_n - I) &\leq \left(\sum_{m=0}^{n-1} \frac{(2\gamma\psi(h))^m}{\sqrt{m!}(1 - \sup_{k=1,2,\dots} |\frac{2\gamma}{z_k(h)+\epsilon_n} - 1|)^{m+1}} \right)^2 \\ &\leq \left(\sum_{m=0}^{n-1} \frac{(2\gamma\psi(h))^m}{\sqrt{m!}(1 - \sup_{k=1,2,\dots} |\frac{2\gamma}{z_k(h)} - 1|)^{m+1}} \right)^2 + \hat{\epsilon}_n \\ &\leq \xi(h, \gamma) + \hat{\epsilon}_n, \end{aligned}$$

where $\hat{\epsilon}_n \geq 0$ and $\hat{\epsilon}_n \rightarrow 0$ as $n \rightarrow \infty$. In addition,

$$\|2\gamma F_n - I\| \leq 1 + 2\gamma\|F_n\| \leq 1 + 2\gamma N_2(F_n) \leq 1 + 2\gamma\psi(h).$$

Moreover,

$$\|F_n - \tilde{F}_n\|^2 \leq \sum_{k=1}^{\infty} |a_k - \tilde{a}_k|^2 = q^2.$$

So, condition (8) is provided by the inequality

$$4(\xi(h, \gamma) + \hat{\epsilon}_n)(\gamma^2 q^2 + \gamma q(1 + 2\gamma\psi(h))) < 1.$$

By Lemma 2.3, for sufficiently large n we have

$$\min_k \Re z_k(\tilde{h}_n) = \min_k \Re \lambda_k(\tilde{F}_n^{-1}) > \gamma.$$

Now letting $n \rightarrow \infty$, we get the required result. \square

3. PERTURBED POLYNOMIALS

In this section we considerably simplify Theorem 1.1 in the case of the polynomials

$$p(z) = \sum_{k=0}^n b_k z^{n-k} \quad \text{and} \quad \tilde{p}(z) = \sum_{k=0}^n \tilde{b}_k z^{n-k}$$

($b_0 = \tilde{b}_0 = 1; b_k, \tilde{b}_k \in \mathbb{C}, k = 1, \dots, n$). The theory of polynomials in spite its long history cf. [1, 16] continues to attract an attention of many mathematicians, for example see [14, 18, 19]. However to the best of our knowledge the above pointed problem has not been considered even for polynomials.

Assume that

$$(10) \quad \hat{\beta}(p) := \min_{k=1,\dots,n} \Re z_k(p) > 0$$

and put

$$\hat{q}_n := \left(\sum_{k=1}^{\infty} |b_k - \tilde{b}_k|^2 \right)^{1/2} \quad \text{and} \quad \eta(p) := \sum_{j,k=0}^{n-1} \frac{(k+j)! \hat{\psi}_n^{k+j}(p)}{2^{k+j} \hat{\beta}^{k+j+1}(p) (k!j!)^{1/2}},$$

where

$$\hat{\psi}_n(p) = \left(n - 1 + \sum_{k=1}^n |b_k|^2 \right)^{1/2}.$$

Theorem 3.1. *Let the conditions (10) and*

$$(11) \quad \hat{q}_n \eta(p) < 1$$

hold. Then

$$(12) \quad \min_{k=1,2,\dots} \Re z_k(\tilde{p}) > 0.$$

Proof. Note that p and \tilde{p} are the characteristic polynomials of the matrices

$$C = \begin{pmatrix} -b_1 & -b_2 & \cdots & -b_{n-1} & -b_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$\tilde{C} = \begin{pmatrix} -\tilde{b}_1 & -\tilde{b}_2 & \cdots & -\tilde{b}_{n-1} & -\tilde{b}_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

respectively. So

$$(13) \quad \beta(C) := \min_k \Re \lambda_k(C) = \hat{\beta}(p) > 0.$$

As is well known [4, Section 1.5], if condition (13) is fulfilled, then the matrix Lyapunov equation

$$C^*Y + YC = 2I,$$

has a selfadjoint solution Y . Due to Lemma 3.1 from [13], under the condition

$$(14) \quad \|Y\| \|\tilde{C} - C\| < 1$$

one has $\min_{k=1,\dots,n} \Re \lambda_k(\tilde{C}) > 0$ and, therefore, (12) is valid. Put

$$\hat{\eta}(C) := \sum_{j,k=0}^{n-1} \frac{(k+j)! g^{k+j}(C)}{2^{k+j} \beta^{k+j+1}(p) (k!j!)^{1/2}}.$$

Due to Lemma 1.9.2 from [10] $\|Y\| \leq \hat{\eta}(C)$. But,

$$g^2(C) \leq N_2^2(C) = n - 1 + \sum_{k=1}^n |b_k|^2 = \hat{\psi}_n^2(p).$$

Thus, $\hat{\eta}(C) = \eta(p)$. Since $\|\tilde{C} - C\| = \hat{q}_n$, condition (14) is provided by condition (11). This proves the theorem. \square

4. ESTIMATES FOR $\xi(h, \gamma)$, q AND $\psi(h)$

By the Schwarz inequality for constants $c \in (0, 1)$ and $a \geq 0$ we have

$$\sum_{k=0}^{\infty} \frac{a^k}{\sqrt{k!}} = \sum_{k=0}^{\infty} \frac{(ac)^k}{c^k \sqrt{k!}} \leq \left(\sum_{j=0}^{\infty} c^{2j} \sum_{k=0}^{\infty} \frac{a^{2k}}{c^{2k} k!} \right)^{1/2} = \frac{1}{(1-c^2)^{1/2}} e^{a^2/(2c^2)}.$$

Thus

$$\xi(h, \gamma) \leq \frac{1}{r^2(h, \gamma)(1-c^2)} \exp \left[\frac{4(\gamma\psi(h))^2}{r^2(h, \gamma)c^2} \right].$$

In particular, taking $c^2 = 1/2$ we obtain

$$(15) \quad \xi(h, \gamma) \leq \frac{2}{r^2(h, \gamma)} \exp \left[\frac{8(\gamma\psi(h))^2}{r^2(h, \gamma)} \right].$$

If

$$|a_k| \leq \text{const } c^k \quad (c \in (0, 1), k = 1, 2, \dots),$$

then for $\Re z \geq 1$ function h admits the Laplace transform

$$H(z) = (\mathcal{L}h)(z) = \int_0^{\infty} e^{-zt} h(t) dt = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}},$$

which can be extended to $|z| \geq 1$. Thus

$$\|H(e^{it})\|_{L^2(0, 2\pi)} := \left[\frac{1}{2\pi} \int_0^{2\pi} |H(e^{it})|^2 dt \right]^{1/2} = \left[\sum_{k=0}^{\infty} |a_k|^2 \right]^{1/2}$$

and, therefore,

$$(16) \quad \psi(h) = (\|H(e^{it})\|_{L^2(0, 2\pi)}^2 + \zeta(2) - 2)^{1/2} \leq \left(\max_{|z|=1} |H(z)|^2 + \zeta(2) - 2 \right)^{1/2}.$$

Since $\zeta(2) \leq 1.645$, we have

$$\psi(h) \leq (\|H(e^{it})\|_{L^2(0, 2\pi)}^2 - 0.355)^{1/2} \leq \left(\max_{|z|=1} |H(z)|^2 - 0.355 \right)^{1/2}.$$

Similarly,

$$(17) \quad q := \|H(e^{it}) - \tilde{H}(e^{it})\|_{L^2(0, 2\pi)} \leq \max_{|z|=1} |H(e^{it}) - \tilde{H}(e^{it})|,$$

where $\tilde{H}(z)$ the Laplace transform to \tilde{h} .

5. EXAMPLE

Consider the functions $h(z) = (1 - z)e^{0.1z}$ and $\tilde{h}(z) = (1 - az)e^{0.1z}$, where a is a positive constant. $h(z)$ has a unique root $z_1(h) = 1$. So condition (3) holds for example, for $\gamma = 0.5$. Clearly, condition (4) holds if $a < 2$. Let us compare this result with Theorem 1.1. To this end note that

$$H(z) = [\mathcal{L}(e^{0.1t} - te^{0.1t})](z) = \frac{1}{z - 0.1} - \frac{1}{(z - 0.1)^2}.$$

Hence, $\max_{|z|=1} |H(z)| \leq 2.346$. Due to (16) $\psi(h) < 2$. Besides,

$$\sup_k \left| \frac{2\gamma}{z_k(h)} - 1 \right| = \left| \frac{2 \cdot 1/2}{1} - 1 \right| = 0,$$

and therefore $r(h, 0.5) = 1$. Hence, by (15)

$$\xi(h, \gamma) \leq 2 \exp [8(0.5 \cdot 2)^2] = 2e^8.$$

In addition,

$$\tilde{H}(z) - H(z) = (1 - a) \frac{1}{(z - 0.1)^2}.$$

Hence, by (17)

$$q \leq |a - 1| \max_{|z|=1} \frac{1}{|z - 0.1|^2} \leq |a - 1| 1.235.$$

So, by Theorem 1.1, inequality (4) holds with $\gamma = 0.5$, provided that

$$\begin{aligned} 8e^8 \left((0.5 \cdot 1.235|a - 1|)^2 + 0.5 \cdot 1.235|a - 1|(1 + 4 \cdot 0.5) \right) = \\ = 8e^8 (0.381|a - 1|^2 + 1.854|a - 1|) < 1. \end{aligned}$$

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